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THE HYERS-ULAM STABILITY OF CUBIC FUNCTRIONAL EQUATIONS IN FUZZY BANACH SPACES

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Abstract. In this paper, we consider the following cubic functional equation

$f(3x + y) + f(3x - y) = f(x + 2y) + 2f(x - y) + 2f(3x) - 3f(x) - 6f(y)$

and prove the generalized Hyers-Ulam stability for it in fuzzy normed spaces.

1. Introduction

In 1940, Ulam [13] proposed the following stability problem :

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exists a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ such that $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In 1941, Hyers [5] answered this problem under the assumption that the groups are Banach spaces. Aoki [1] and Rassias [11] generalized the result of Hyers. Rassias [11] solved the generalized Hyers-Ulam stability of the functional inequality

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p)$$

for some $\epsilon \ge 0$, a real number p with $p < 1$ and all $x, y \in X$, where $f : X \longrightarrow Y$ is a function between Banach spaces. The paper of Rassias [11] has provided a lot of influence in the development of what we call *the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Rassias theorem was obtained by Găvruta [4] by replacing the

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unbounded Cauchy difference by a general control function in the spirit of Rassis approach. In 2003, Bag and Samanta [2] modified the definition of Cheng and Mordeson [3] by removing a regular condition.

In this paper, we consider the fuzzy version stability problem in the fuzzy normed linear space setting. The concept of fuzzy norm on a linear space was introduced by Katsaras [6] in 1984, which was later on studied, following Cheng and Mordeson [3], to give a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [7].

Rassias [10], Park and Jung [9] introduced the following cubic functional equations

$$f(x+2y)+3f(x)=3f(x+y)+f(x-y)+6f(y) \tag{1}$$

and

$$f(3x+y)+f(3x-y)=3f(x+y)+3f(x-y)+48f(x) \tag{2}$$

and investigated its general solution and the generalized Hyers-Ulam-Rassias stability respectively. It is easy to see that the function $f(x)=ax^3$ is a solution of the functional equation (1) and (2), which explains why they are called *a cubic functional equation.*

In this paper, we consider the the following functional equation

$$f(3x+y)+f(3x-y)=f(x+2y)+2f(x-y)+51f(x)-6f(y) \tag{3}$$

which is the difference of (1) and (2) and

$$f(3x+y)+f(3x-y)=f(x+2y)+2f(x-y)+2f(3x)-3f(x)-6f(y). \tag{4}$$

Moreover we prove the generalized Hyers-Ulam stability for (4) in fuzzy normed spaces. Mirmostafaee and Moslehian [8] proved the stability of a cubic functional equation in fuzzy normed spaces.

Definition 1. Let X be a linear space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ is called *a fuzzy norm on* X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N1) $N(x,\ t)=0$ for $t \leq 0$;
(N2) $x=0$ if and only if $N(x,\ t)=1$ for all $t>0$;
(N3) $N=(cx,\ t)=N(x,\frac{t}{|c|})$ if $c \neq 0$;
(N4) $N=(x+y,\ s+t) \geq min\{N(x,s),\ N(y,t)\}$;
(N5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,\ t)=1$;
(N6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X,N) is called *a fuzzy normed space.*

Definition 2. Let (X,N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *convergent* if there exists an $x \in X$ such that $\lim_{t\to\infty} N(x_n-x,t)=1$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* X and one denotes it by $N-\lim_{t\to\infty} x_n = x$.

Definition 3. Let (X, N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *Cauchy* if for any $\epsilon > 0$, there is an $m \in \mathbb{N}$ such that for any $n \geq m$ and any positive integer p, $N(x_{n+p} - x_n, t) > 1 - \epsilon$ for all $t > 0$.

It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space.*

Throughtout this paper, X is a linear space, (Y, N) is a fuzzy Banach space, and (Z, N') is a fuzzy normed space.

2. Solutions of (4)

In this section, we investigate solutions of (3) and (4) between X and Y. And then, in Corollary 2.2, it can be concluded that $f : X \longrightarrow Y$ satisfies (3) if and only if f satisfies (4). We start with the following theorem.

Theorem 2.1. *Let $f : X \longrightarrow Y$ be a mapping. Then f satisfies (4) if and only if f is cubic.*

Proof. Clearly, $f(0) = 0$. Letting $x = 0$ and $y = x$ in (4), we have

$$7f(x) - f(-x) - f(2x) = 0 \tag{5}$$

for all $x \in X$ and letting $x = 0$ and $y = -x$ in (4), we have

$$7f(-x) - f(x) - f(-2x) = 0 \tag{6}$$

for all $x \in X$. Letting $y = x$ in (4), we have

$$f(4x) + f(2x) - 3f(3x) + 9f(x) = 0 \tag{7}$$

for all $x \in X$. Letting $y = -x$ in (4), we have

$$f(4x) - f(2x) - 2f(3x) + 3f(x) + 5f(-x) = 0 \tag{8}$$

for all $x \in X$ and letting $x = 0$ and $y = 2x$ in (4), we have

$$7f(2x) - f(-2x) - f(4x) = 0 \tag{9}$$

for all $x \in X$. Calulating $\{(6) + 2 \times (7) - 3 \times (8) - 8 \times (5) - (9)\}$, we have

$$f(2x) = 2^3 f(x) \tag{10}$$

for all $x \in X$. By (7) and (10), we have

$$f(3x) = 3^3 f(x) \tag{11}$$

for all $x \in X$. By (5) and (10),

$$f(x) = -f(-x) \tag{12}$$

for all $x \in X$. Replacing y by $3y$ in (4), we have

$$\begin{aligned}(13)\qquad &27f(x+y)+27f(x-y)\\ &=f(x+6y)+2f(x-3y)+2f(3x)-3f(x)-6f(3y)\end{aligned}$$

for all $x, y \in X$. Interchanging x and y in (13), by (12), we have

$$\begin{aligned}(14)\qquad &27f(x+y)-27f(x-y)\\ &=f(6x+y)-2f(3x-y)+2f(3y)-3f(y)-6f(3x)\end{aligned}$$

for all $x, y \in X$. Relpacing y by $2y$ in (14), by (10), we have

$$\begin{aligned}(15)\qquad &27f(x+2y)-27f(x-2y)\\ &=8f(3x+y)-2f(3x-2y)+16f(3y)-24f(y)-6f(3x)\end{aligned}$$

for all $x, y \in X$. Relpacing y by $-y$ in (4), by (12), we have

$$\begin{aligned}(16)\qquad &f(3x+y)+f(3x-y)\\ &=f(x-2y)+2f(x+y)+2f(3x)-3f(x)+6f(y)\end{aligned}$$

for all $x, y \in X$. By (4) and (16), we have

$$f(x+2y)-f(x-2y)-2f(x+y)+2f(x-y)-12f(y)=0 \qquad (17)$$

for all $x, y \in X$. Letting $y=-y$ in (15), by (12), we have

$$\begin{aligned}(18)\qquad &27f(x-2y)-27f(x+2y)\\ &=8f(3x-y)-2f(3x+2y)-16f(3y)+24f(y)-6f(3x)\end{aligned}$$

for all $x, y \in X$. By (15) and (18), we have

$$8[f(3x+y)+f(3x-y)]-2[f(3x+2y)+f(3x-2y)]-12f(3x)=0 \qquad (19)$$

for all $x, y \in X$. Replacing $3x$ by x in (19), by (11), we have

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)$$

for all $x, y \in X$ and so f is a cubic, quadratic and additive mapping([14]). Since $f(2x)=2^3f(x)$ for all $x \in X$, f is cubic. The converse is trivial. □

We remark that if f satisfies (4), then $f(3x)=27f(x)$ for all $x \in X$ and that if f satisfies (4), then f satisfies (3). Similarly, if f satisfies (3), then f satisfies (4). Hence we have the following corollary

Corollary 2.2. *Let $f : X \longrightarrow Y$ be a mapping. Then the following are equivalent*
(1) f satisfies (3),
(2) f satisfies (4), and
(3) f is cubic.

3. The Generalized Hyers-Ulam stability for (4)

In this section, we prove the generalized Hyers-Ulam stability of functional equation (4) in fuzzy normed spaces.

For any mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$Df(x,y) = f(3x+y)+f(3x-y)-f(x+2y)-2f(x-y)-2f(3x)+3f(x)+6f(y)$$

for all $x, y \in X$.

Theorem 3.1. *Let $\phi : X^2 \longrightarrow Z$ be a function and r a real number such that $0 < |r| < 2^3$*

$$N'(\phi(2x,2y),\ \ t) \geq N'(r\phi(x,y),\ \ t) \tag{20}$$

for all $x, y \in X$ all $t > 0$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(Df(x,y),\ \ t) \geq N'(\phi(x,y),\ \ t) \tag{21}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ satisfying (4) and the inequality

$$N(f(x) - C(x),\ \ t) \geq \Phi(x, 6(2^3 - |r|)t) \tag{22}$$

holds for all $x \in X$ and all $t > 0$, where

$$\Phi(x,t) = \min\{N'(\phi(0,-x), \frac{t}{15}), N'(\phi(x,x), \frac{t}{15}),$$
$$N'(\phi(x,-x), \frac{t}{15}), N'(\phi(0,x), \frac{t}{15}), N'(\phi(0,2x), \frac{t}{15})\}.$$

Proof. By (20) and (N3), we have

$$N'(\phi(2^n x, 2^n y),\ \ t) \geq N'(r^n\phi(x,y),\ \ t) = N'(\phi(x,y),\ \ \frac{t}{|r|^n}) \tag{23}$$

for all $x, y \in X$ and all $t > 0$ and so by (23), we have

$$N'(\phi(2^n x, 2^n y),\ \ |r|^n t) \geq N'(\phi(x,y),\ \ t) \tag{24}$$

for all $x, y \in X$ and all $t > 0$. By (21), we have

$$\begin{aligned}
&N(6f(2x) - 48f(x), t)\\
&= N(Df(0,-x) + 2Df(x,x) - 3Df(x,-x) - 8Df(0,x) - Df(0,2x), t)\\
&\geq \min\{N(Df(0,-x), \frac{t}{15}), N(2Df(x,x), \frac{2t}{15}), N(3Df(x,-x), \frac{3t}{15}),\\
&\quad N(8Df(0,x), \frac{8t}{15}), N(Df(0,2x), \frac{t}{15})\} \geq \Phi(x,t)
\end{aligned}$$

for all $x \in X$ and all $t > 0$. Hence, by (21) and (N3), we have

$$N(f(x) - \frac{f(2x)}{2^3},\ \frac{t}{6 \times 2^3}) \geq \Phi(x,t) \tag{25}$$

for all $x \in X$ and all $t > 0$. By (24), (25), and (N3), we have

$$N(\frac{f(2^n x)}{2^{3n}} - \frac{f(2^{n+1}x)}{2^{3(n+1)}},\ \frac{|r|^n t}{6 \times 2^{3(n+1)}}) \geq \Phi(2^n x, |r|^n t) \geq \Phi(x,t) \tag{26}$$

for all $x \in X$, all $t > 0$ and all positive integer n. Hence by (26) and (N4), for any $x \in X$, we have

$$\begin{aligned} & N(f(x) - \frac{f(2^n x)}{2^{3n}},\ \sum_{i=0}^{n-1} \frac{|r|^i t}{6 \times 2^{3(i+1)}}) \\ & = N(\sum_{i=0}^{n-1}[\frac{f(2^i x)}{2^{3i}} - \frac{f(2^{i+1}x)}{2^{3(i+1)}}],\ \sum_{i=0}^{n-1} \frac{|r|^i t}{6 \times 2^{3(i+1)}}) \\ & \geq min\{N(\frac{f(2^i x)}{2^{3i}} - \frac{f(2^{i+1}x)}{2^{3(i+1)}},\ \frac{|r|^i t}{6 \times 2^{3(i+1)}}) \mid 0 \leq i \leq n-1\} \geq \Phi(x,t) \end{aligned} \tag{27}$$

for all $x \in X$, all $t > 0$ and all positive integer n. So for any $x \in X$, , we have

$$\begin{aligned} & N(\frac{f(2^m x)}{2^{3m}} - \frac{f(2^{m+p}x)}{2^{3(m+p)}},\ \sum_{i=m}^{m+p-1} \frac{|r|^i t}{6 \times 2^{3(i+1)}}) \\ & = N(\sum_{i=m}^{m+p-1} [\frac{f(2^i x)}{2^{3i}} - \frac{f(2^{i+1}x)}{2^{3(i+1)}}],\ \sum_{i=m}^{m+p-1} \frac{|r|^i t}{6 \times 2^{3(i+1)}}) \\ & \geq \Phi(x,t) \end{aligned} \tag{28}$$

for all $x \in X$, all $t > 0$, all non-negative integer m and all positive integer p. Thus, by (28) and (N3), for any $x \in X$, we have

$$N(\frac{f(2^m x)}{2^{3m}} - \frac{f(2^{m+p}x)}{2^{3(m+p)}},\ t) \geq \Phi(x, \frac{t}{\sum_{i=m}^{m+p-1} \frac{|r|^i}{6 \times 2^{3(i+1)}}}) \tag{29}$$

for all $x \in X$, all $t > 0$, all non-negative integer m and all positive integer p. Since $\sum_{i=0}^{\infty} \frac{|r|^i}{2^{3(i+1)}}$ is convergent, $\lim_{m\to\infty} \frac{t}{\sum_{i=m}^{m+p-1} \frac{|r|^i}{2^{3(i+1)}}} = \infty$. Since $\lim_{t\to\infty} \Phi(x,t) = 1$, $\{\frac{f(2^m x)}{2^{3m}}\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, there is a mapping $C : X \longrightarrow Y$ defined by

$$\begin{aligned} & C(x) = N - \lim_{n\to\infty} \frac{f(2^n x)}{2^{3n}}\ or \\ & \lim_{n\to\infty} N(\frac{f(2^n x)}{2^{3n}} - C(x),\ t) = 1,\ t > 0 \end{aligned} \tag{30}$$

for all $x \in X$. Moreover by (29), we have

$$N(f(x) - \frac{f(2^n x)}{2^{3n}},\ t) \geq \Phi(x, \frac{t}{\sum_{i=0}^{n-1} \frac{|r|^i}{6\times 2^{3(i+1)}}}) \tag{31}$$

for all $x \in X$, all $t > 0$ and all positive integer n. Let ϵ be a real number with $0 < \epsilon < 1$. Then, by (30), (31), and (N4), we have

$$\begin{aligned} &N(f(x) - C(x),\ t) \\ &\geq min\{N(f(x) - \frac{f(2^n x)}{2^{3n}},\ (1-\epsilon)t), N(\frac{f(2^n x)}{2^{3n}} - C(x),\ \epsilon t)\} \\ &\geq \Phi(x, \frac{(1-\epsilon)t}{\sum_{i=0}^{n-1} \frac{|r|^i}{6\times 2^{3(i+1)}})}) \geq \Phi(x, 6(1-\epsilon)(2^3 - |r|)t) \end{aligned} \tag{32}$$

for sufficiently large positive integer n, all $x \in X$, and all $t > 0$. Since $N(x, \cdot)$ is continuous on $\mathbb{R}$, we get

$$N(f(x) - C(x),\ t) \geq \Phi(x, 6(2^3 - |r|)t) \tag{33}$$

for all $x \in X$ and all $t > 0$ and so we have (22). By (21) and (N3), we have

$$N(\frac{Df(2^n x, 2^n y)}{2^{3n}},\ t) \geq N'(\phi(2^n x, 2^n y), \geq N'(\phi(x,y),\ \frac{2^{3n}}{|r|^n} t) \tag{34}$$

for all $x, y \in X$ and all $t > 0$. Since $\lim_{n\to\infty} N'(\phi(x,y),\ \frac{2^{3n}}{|r|^n} t) = 1$, by (30), (34), and (N4), we have

$$\begin{aligned} &N(DC(x,y),\ t) \\ &\geq min\{N(DC(x,y) - \frac{Df(2^n x, 2^n y)}{2^{3n}},\ \frac{t}{2}),\ N(\frac{Df(2^n x, 2^n y)}{2^{3n}},\ \frac{t}{2})\} \\ &\geq N(\frac{Df(2^n x, 2^n y)}{2^{3n}},\ \frac{t}{2}) \geq N'(\phi(x,y),\ \frac{2^{3n}}{2|r|^n} t) \end{aligned} \tag{35}$$

for sufficiently large n, all $x, y \in X$ and all $t > 0$. Since $\lim_{n\to\infty} N'(\phi(x,y),\ \frac{2^{3n}}{|r|^n} t) = 1$, $N(DC(x,y),\ t) = 1$ for all $t > 0$ and so, by (N2), $DC(x,y) = 0$ for all $x, y \in X$. By Theorem 20, C is cubic.

To prove the uniquness of C, let $C_1 : X \longrightarrow Y$ be another cubic mapping satisfying (22). Then for any $x \in X$ and any positive integer n, $C_1(2^n x) = 2^{3n} C_1(x)$ and so by (31),

$$\begin{aligned} &N(C(x) - C_1(x),\ t) \\ &\geq min\{N(\frac{C(2^n x)}{2^{3n}} - \frac{f(2^n x)}{2^{3n}},\ \frac{t}{2}), N(\frac{C_1(2^n x)}{2^{3n}} - \frac{f(2^n x)}{2^{3n}},\ \frac{t}{2})\} \\ &\geq \Phi(2^n x, 3(2^3 - |r|)2^{3n} t) \geq \Phi(x, \frac{3(2^3 - |r|)2^{3n} t}{|r|^n}) \end{aligned} \tag{36}$$

holds for all $x \in X$, all positive integer n, and all $t > 0$. Since $|r| < 2^3$, $\lim_{n\to\infty} \Phi(x, \frac{2^{3n}(2^3-|r|)t}{|r|^n}) = 1$ and so $C(x) = C_1(x)$ for all $x \in X$. □

We remark that if $f(0) \neq 0$ in Theorem 3.1, the inequality (22) can be replaced by

$$N(f(x) - f(0) - C(x),\ \ t) \geq \Phi(x, 6(2^3 - |r|)t)$$

holds for all $x \in X$ and all $t > 0$.

Related with Theorem 3.1, we can also have the following theorem. And the proof is similar to that of Theorem 3.1.

Theorem 3.2. *Let $\phi : X^2 \longrightarrow Z$ be a function and r a real number such that $2^3 < |r|$ and*

$$N'(\phi(\frac{x}{2}, \frac{y}{2}),\ \ t) \geq N'(\frac{1}{r}\phi(x, y),\ \ t)$$

for all $x, y \in X$ all $t > 0$. Let $f : X \longrightarrow Y$ be a mapping satisfying $f(0) = 0$ and (21). Then there exists a unique cubic mapping $C : X \longrightarrow Y$ satisfying (4) and the inequality

$$N(f(x) - C(x),\ \ t) \geq \Phi(x, 6(|r| - 2^3)t)$$

holds for all $x \in X$ and all $t > 0$.

As an example of $\phi(x, y)$ in Theorem 3.1 and Theorem 3.2, we can take $\phi(x, y) = \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$ which is appeared in [12]. Then we can formulate the following corollary

Corollary 3.3. *Let X be a normed space and Y a Banach space. Let $f : X \longrightarrow Y$ be a mapping such that*

$$\|Df(x, y)\| \leq \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p}) \tag{37}$$

for all $x, y \in X$ and a fixed real number p with $0 < p < \frac{3}{2}$ or $\frac{3}{2} < p$. Then there is a unique cubic mapping $C : X \longrightarrow Y$ such that

$$\|f(x) - C(x)\| \leq \begin{cases} \frac{15\|x\|^{2p}}{2(2^3-2^{2p})}, & \textit{if } 0 < p \leq \log_4 3 \\ \frac{5\times 2^{2p}\|x\|^{2p}}{2(2^3-2^{2p})}, & \textit{if } \log_4 3 < p < \frac{3}{2} \\ \frac{5\times 2^{2p}\|x\|^{2p}}{2(2^{2p}-2^3)}, & \textit{if } \frac{3}{2} < p \end{cases}$$

for all $x \in X$.

Proof. Define a fuzzy norm on $\mathbb{R}$ by

$$N_{\mathbb{R}}(x, t) = \begin{cases} \frac{t}{t+|x|}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

for all $x \in \mathbb{R}$ and all $t > 0$. Similary we can define a fuzzy norm N_Y on Y. Then (Y, N_Y) is a fuzzy Banach space. Let $\phi(x,y) = \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$. Then by(37), f satisies the following inequality

$$N_Y(Df(x,y),t) \geq N_{\mathbb{R}}(\phi(x,y),t)$$

for all $x, y \in X$ and all $t > 0$. Note that $N_{\mathbb{R}}(\phi(2x,2y),t)) = N_{\mathbb{R}}(2^{2p}\phi(x,y),t)$ for all $x, y \in X$ and all $t > 0$ and that

$$\begin{aligned}
&\Phi(x, 6(2^3 - 2^{2p})t) \\
&\geq \min\{N_{\mathbb{R}}(3\|x\|^{2p}, \frac{6(2^3-2^{2p})t}{15}), N_{\mathbb{R}}(2^{2p}\|x\|^{2p}, \frac{6(2^3-2^{2p})t}{15})\} \\
&\geq \begin{cases} N_{\mathbb{R}}(\|x\|^{2p}, \frac{2(2^3-2^{2p})t}{15}), & \text{if } 0 < p \leq \log_4 3 \\ N_{\mathbb{R}}(\|x\|^{2p}, \frac{2(2^3-2^{2p})t}{5\times 2^{2p}}), & \text{if } \log_4 3 < p < \frac{3}{2} \\ N_{\mathbb{R}}(\|x\|^{2p}, \frac{2(2^{2p}-2^3)t}{5\times 2^{2p}}), & \text{if } \frac{3}{2} < p \end{cases}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. By Theorem 3.1, there is a unique cubic mapping $C : X \longrightarrow Y$ such that

$$N_Y(f(x) - C(x),\ \ t) \geq \begin{cases} N_{\mathbb{R}}(\|x\|^{2p}, \frac{2(2^3-2^{2p})t}{15}), & \text{if } 0 < p \leq \log_4 3 \\ N_{\mathbb{R}}(\|x\|^{2p}, \frac{2(2^3-2^{2p})t}{5\times 2^{2p}}), & \text{if } \log_4 3 < p < \frac{3}{2} \\ N_{\mathbb{R}}(\|x\|^{2p}, \frac{2(2^{2p}-2^3)t}{5\times 2^{2p}}), & \text{if } \frac{3}{2} < p \end{cases}$$

for all $x \in X$ and all $t > 0$. Hence we have the result. □

We remark that the functional equation (4) is not stable for $p = \frac{3}{2}$ in Corollary 3.3. The following example shows that the (37) is not stable for $p = \frac{3}{2}$.

Example 1. Let $t : \mathbb{R} \longrightarrow \mathbb{R}$ be a mapping defined by

$$t(x) = \begin{cases} x^3, & \text{if } |x| < 1 \\ 1, & \text{ortherwise} \end{cases}$$

and define a mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = \sum_{n=0}^{\infty} \frac{t(2^n x)}{8^n}$. We will show that f satisfies the functional inequality

$$|Df(x,y)| \leq \frac{2^{10}}{7}(|x|^{\frac{3}{2}}|y|^{\frac{3}{2}} + |x|^3 + |y|^3) \tag{38}$$

for all $x, y \in \mathbb{R}$, but there do not exist a cubic mapping $C : \mathbb{R} \longrightarrow \mathbb{R}$ and a positive constant K such that

$$|C(x) - f(x)| \leq K|x|^3 \tag{39}$$

for all $x \in \mathbb{R}$.

Proof. Note that $|f(x)| \leq \frac{8}{7}$ for all $x \in \mathbb{R}$.

First, suppose that $\frac{1}{8} \leq |x|^{\frac{3}{2}}|y|^{\frac{3}{2}}+|x|^3+|y|^3$. Then $|Df(x,y)| \leq \frac{2^{10}}{7}(|x|^{\frac{3}{2}}|y|^{\frac{3}{2}}+|x|^3+|y|^3)$ for all $x, y \in \mathbb{R}$.

Now suppose that $\frac{1}{8} > |x|^{\frac{3}{2}}|y|^{\frac{3}{2}}+|x|^3+|y|^3$. Then there is a non-negative integer m such that

$$\frac{1}{2^{3m+4}} \leq |x|^{\frac{3}{2}}|y|^{\frac{3}{2}}+|x|^3+|y|^3 < \frac{1}{2^{3m+3}}$$

and so $2^m|x| < \frac{1}{2}$, $2^m|y| < \frac{1}{2}$. Hence we have

$$\{2^{m+1}(2x \pm y),\ 2^{m+1}(x \pm y),\ 2^{m+1}x,\ 2^{m+1}y\} \subseteq (-1,\ 1)$$

and so for any $n = 0, 1, 2, \cdots, m+1$, $Dt(2^n x, 2^n y) = 0$ for all $x, y \in X$. Thus

$$\begin{aligned} Df(x,y) &\leq \sum_{n=0}^{\infty} \frac{1}{8^n} Dt(2^n x, 2^n y) \leq \sum_{n=m+2}^{\infty} \frac{1}{8^n} Dt(2^n x, 2^n y) \\ &\leq \frac{2^5}{7 \times 2^{3m+4}} \leq \frac{2^5}{7}(|x|^{\frac{3}{2}}|y|^{\frac{3}{2}}+|x|^3+|y|^3). \end{aligned}$$

Thus f satisfies (38).

Suppose that there exist a cubic mapping $C : \mathbb{R} \longrightarrow \mathbb{R}$ and a positive constant K with (39). Since $|f(x)| \leq \frac{8}{7}$,

$$-K|x|^3 - \frac{8}{7} \leq C(x) \leq K|x|^3 + \frac{8}{7}$$

for all $x \in X$ and since C is cubic,

$$-K|x|^3 - \frac{8}{7n^3} \leq C(x) \leq K|x|^3 + \frac{8}{7n^3}$$

for all $x \in X$ and all positive integer n. Hence we have $|C(x)| \leq K|x|^3$ for all $x \in X$ and so, by (39), we have $|f(x)| \leq 2K|x|^3$ for all $x \in X$. Take a positive integer l such that $l > 2K$, and pick $x \in \mathbb{R}$ with $0 < 2^l x < 1$. Then

$$f(x) = \sum_{n=0}^{\infty} \frac{t(2^n x)}{8^n} > \sum_{n=0}^{l-1} \frac{t(2^n x)}{8^n} = \sum_{n=0}^{l-1} x^3 = lx^3 > 2Kx^3$$

which is a cotradiction. □

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