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CHARACTERIZATIONS OF THE GAMMA DISTRIBUTION BY INDEPENDENCE PROPERTY OF RANDOM VARIABLES

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Abstract. Let $\{X_i\,, 1 \leq i \leq n\}$ be a sequence of i.i.d. sequence of positive random variables with common absolutely continuous cumulative distribution function $F(x)$ and probability density function $f(x)$ and $E(X^2) < \infty$. The random variables $X+Y$ and $\frac{(X-Y)^2}{(X+Y)^2}$ are independent if and only if X and Y have gamma distributions. In addition, the random variables S_n and $\dfrac{\sum_{i=1}^{m}(X_i)^2}{(S_n)^2}$ with $S_n = \sum_{i=1}^{n} X_i$ are independent for $1 \leq m < n$ if and only if X_i has gamma distribution for $i = 1, \cdots, n$.

1. Introduction

Let $\{X_i\,, 1 \leq i \leq n\}$ be independent and identically distributed(i.i.d.) non-degenerate and positive random variables with common absolutely continuous cumulative distribution function $F(x)$.

Let X and Y be two independent non-degenerate positive random variables. It is known that X/Y and $X+Y$ are independent if and only if X and Y are gamma distributions with the same scale parameter as used in Lukacs(1955). By using the moment, Findeisen(1978) characterized the gamma distribution. Also, Hwang and Hu(1999) also proved a characterization of the gamma distribution by the independence of the sample mean and the sample coefficient of variation. Recently, Lee and Lim(2009) presented characterizations of gamma distribution with the

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property that the random variables $\frac{X_i X_j}{(\Sigma_{k=1}^n X_k)^2}$ and $\Sigma_{k=1}^n X_k$ are independent for $1 \le i < j \le n$ if and only if X_i has gamma distribution for $i = 1, \cdots, n$.

In this paper, we obtain the characterizations of the gamma distribution by independence property of the quotient of sum and difference of random variables.

2. Results

THEOREM 2.1. *Let X and Y be nondegenerate and positive i.i.d. random variables with common absolutely continuous cumulative distribution function $F(X)$ and $E(X^2) < \infty$. The random variables $X+Y$ and $\frac{(X-Y)^2}{(X+Y)^2}$ are independent if and only if X and Y have gamma distributions.*

THEOREM 2.2. *Let $\{X_i, 1 \le i \le n\}$ be nondegenerate and positive i.i.d. random variables with common absolutely continuous cumulative distribution function $F(X)$ and $E(X^2) < \infty$. The random variables S_n and $\frac{\Sigma_{i=1}^m (X_i)^2}{(S_n)^2}$ are independent for $1 \le m < n$, where $S_n = \Sigma_{i=1}^n X_i$ if and only if X_i has gamma distribution for $i = 1, \cdots, n$.*

3. Proofs

Proof of Theorem 2.1. Since $\frac{(X-Y)^2}{(X+Y)^2}$ is a scale-invariant statistic, $\frac{(X-Y)^2}{(X+Y)^2}$ and $X+Y$ are independent [see Lukacs and Laha(1963)].

We have to prove the reverse. We denote the characteristic functions of $X+Y$, $\frac{(X-Y)^2}{(X+Y)^2}$ and $\left(X+Y, \frac{(X-Y)^2}{(X+Y)^2}\right)$ by $\phi_1(t), \phi_2(s)$ and $\phi(t,s)$, respectively. The independence of $X+Y$ and $\frac{(X-Y)^2}{(X+Y)^2}$ is equivalent to

$$\phi(t,s) = \phi_1(t) \cdot \phi_2(s). \tag{3.1}$$

The left hand side of (3.1) becomes

$$\phi(t,s) = \int_0^\infty \int_0^\infty \exp\left(it(x+y) + \frac{is(x-y)^2}{(x+y)^2}\right) dF(x)dF(y).$$

Also the right hand side of (3.1) becomes

$$\phi_1(t)\cdot\phi_2(s)=\int_0^\infty\int_0^\infty \exp\Big(it(x+y)\Big)dF(x)dF(y)$$
$$\cdot\int_0^\infty\int_0^\infty \exp\left(\frac{is(x-y)^2}{(x+y)^2}\right)dF(x)dF(y).$$

Then (3.1) gives

$$\begin{aligned}&\int_0^\infty\int_0^\infty \exp\left(it(x+y)+\frac{is(x-y)^2}{(x+y)^2}\right)dF(x)dF(y)\\ &=\int_0^\infty\int_0^\infty \exp\Big(it(x+y)\Big)dF(x)dF(y)\\ &\cdot\int_0^\infty\int_0^\infty \exp\left(\frac{is(x-y)^2}{(x+y)^2}\right)dF(x)dF(y).\end{aligned}\tag{3.2}$$

The integrals in (3.2) exist not only for reals t and s but also for complex values $t=u+iv$, $s=u^*+iv^*$, where u and u^* are reals, for which $v=Im(t)\geq 0$, $v^*=Im(s)\geq 0$ and they are analytic for all t, s for $v=Im(t)>0$, $v^*=Im(s)>0$, [see, Lukacs(1955)].

Differentiating (3.2) two times with respect to t and then one time respect to s and setting $s=0$, we get

$$\begin{aligned}&\int_0^\infty\int_0^\infty (x-y)^2\exp\Big(it(x+y)\Big)dF(x)dF(y)\\ &=\theta\int_0^\infty\int_0^\infty (x+y)^2\exp\Big(it(x+y)\Big)dF(x)dF(y)\end{aligned}\tag{3.3}$$

where

$$\theta=E\Big[(\frac{X-Y}{X+Y})^2\Big].$$

The random variable θ is bounded and the moments exists. Then we know that

$$\theta=E\Big[(\frac{X-Y}{X+Y})^2\Big]=E\Big[(1-\frac{2}{1+\frac{X^2+Y^2}{2XY}})\Big].\tag{3.4}$$

Note that, for $x>0$ and $y>0$, $0<2xy\leq x^2+y^2$ and the equality on the right hand side occurs only if $x=y$. By the assumed continuity of $F(x)$, we find $\mathrm{P}(x=y)=0$ and $1<\frac{x^2+y^2}{2xy}<\infty$. It follows $0<\theta<1$ by (3.4) .

Let $\varphi(t)$ be the characteristic function of $F(x)$. Then

$$\varphi'(t) = i\int_0^\infty x\exp\big(itx\big)dF(x),\ \ \varphi''(t) = -\int_0^\infty x^2\exp\big(itx\big)dF(x).$$

Expressing (3.3) as a differential equation for the characteristic function $\varphi(t)$, we get

$$\varphi''(t)\varphi(t) - 2(\varphi'(t))^2 + \varphi(t)\varphi''(t) = \theta[\varphi''(t)\varphi(t) + 2(\varphi'(t))^2 + \varphi(t)\varphi''(t)].$$

That is,

$$\frac{\varphi''(t)}{\varphi'(t)} = \left(\frac{1+\theta}{1-\theta}\right)\frac{\varphi'(t)}{\varphi(t)}, \quad 0 < \theta < 1.$$

After integrating with the initial conditions $\varphi(0) = 1, \varphi'(0) = iE(X)$, we get

$$\varphi'(t) = iE(X)(\varphi(t))^{\frac{1+\theta}{1-\theta}}, \quad \frac{1+\theta}{1-\theta} > 1. \tag{3.5}$$

The solution of this differential equation (3.5) with the above initial conditions is

$$\varphi(t) = (1 - \frac{iE(X)}{\lambda}t)^{-\lambda}, \quad \lambda = \frac{1-\theta}{2\theta} > 0.$$

Consequently, $F(x)$ is a gamma distribution. □

Proof of Theorem 2.2. Since $\frac{\Sigma_{i=1}^m (X_i)^2}{(S_n)^2}$ is a scale-invariant statistic, $\frac{\Sigma_{i=1}^m (X_i)^2}{(S_n)^2}$ and S_n are independent [see Lukacs and Laha(1963)].

We have to prove the reverse. We denote the characteristic functions of $S_n, \frac{\Sigma_{i=1}^m (X_i)^2}{(S_n)^2}$ and $\left(S_n, \frac{\Sigma_{i=1}^m (X_i)^2}{(S_n)^2}\right)$ by $\phi_1(t), \phi_2(s)$ and $\phi(t,s)$, respectively. The independence of S_n and $\frac{\Sigma_{i=1}^m (X_i)^2}{(S_n)^2}$ is equivalent to

$$\phi(t,s) = \phi_1(t)\cdot\phi_2(s). \tag{3.6}$$

The left hand side of (3.6) becomes

$$\phi(t,s) = \int_0^\infty \cdots \int_0^\infty \exp\left[it\big(S_n\big) + \frac{is\big(\Sigma_{i=1}^m (X_i)^2\big)}{\big(S_n\big)^2}\right]dF(x_1)\cdots dF(x_n).$$

Also the right hand side of (3.6) becomes

$$\phi_1(t)\cdot\phi_2(s) = \int_0^\infty \cdots \int_0^\infty \exp\big[it(S_n)\big] dF(x_1)\cdots dF(x_n)$$
$$\cdot \int_0^\infty \cdots \int_0^\infty \exp\left[\frac{is\big(\Sigma_{i=1}^m (X_i)^2\big)}{(S_n)^2}\right] dF(x_1)\cdots dF(x_n).$$

Then (3.6) gives

$$\int_0^\infty \cdots \int_0^\infty \exp\left[it\big(S_n\big) + \frac{is\big(\Sigma_{i=1}^m (X_i)^2\big)}{\big(S_n\big)^2}\right] dF(x_1)\cdots dF(x_n)$$
$$(3.7) \qquad = \int_0^\infty \cdots \int_0^\infty \exp\big[it(S_n)\big] dF(x_1)\cdots dF(x_n)$$
$$\cdot \int_0^\infty \cdots \int_0^\infty \exp\left[\frac{is\big(\Sigma_{i=1}^m (X_i)^2\big)}{\big(S_n\big)^2}\right] dF(x_1)\cdots dF(x_n).$$

The integrals in (3.7) exist not only for reals t and s but also for complex values $t = u + iv$, $s = u^* + iv^*$, where u and u^* are reals, for which $v = Im(t) \geq 0$, $v^* = Im(s) \geq 0$ and they are analytic for all t, s for $v = Im(t) > 0$, $v^* = Im(s) > 0$, [see, Lukacs(1955)].

Differentiating (3.7) two times with respect to t and then one time respect to s and setting $s = 0$, we get

$$(3.8) \qquad \int_0^\infty \cdots \int_0^\infty (\Sigma_{i=1}^m (X_i)^2)\ exp\big[it(S_n)\big] dF(x_1)\cdots dF(x_n)$$
$$= \theta \int_0^\infty \cdots \int_0^\infty (S_n)^2 exp\big[it(S_n)\big] dF(x_1)\cdots dF(x_n)$$

where

$$\theta = E\left[\Big(\frac{\Sigma_{i=1}^m (X_i)^2}{(S_n)^2}\Big)\right].$$

The random variable θ is bounded and the moments exist. Then we know that

$$\theta = E\left[\Big(\frac{(X_2)^2 + \cdots + (X_{m+1})^2}{(S_n)^2}\Big)\right] = E\left[\Big(\frac{(X_1)^2 + (X_3)^2 \cdots + (X_{m+1})^2}{(S_n)^2}\Big)\right]$$
$$= \cdots = E\left[\Big(\frac{(X_{n-m+1})^2 + \cdots + (X_n)^2}{(S_n)^2}\Big)\right]$$

for i.i.d. random variables $X_1, \cdots, X_n$. Then

$$(3.9) \qquad {}_n\mathrm{C_m}\,\theta = \mathrm{E}\left[\Big(\frac{{}_{\mathrm{n-1}}\mathrm{C_{m-1}}\ \Sigma_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{X_i^2}}{(\mathrm{S_n})^2}\Big)\right] = \mathrm{E}\left[\Big(\frac{{}_{\mathrm{n-1}}\mathrm{C_{m-1}}}{1 + \frac{2\Sigma_{1\leq i<j\leq n}\mathrm{X_i X_j}}{\Sigma_{\mathrm{i=1}}^{\mathrm{n}}\mathrm{X_i^2}}}\Big)\right]$$

Note that, for $x_1, \cdots x_n > 0$, the relation $0 < 2\Sigma_{1\le i<j\le n} x_i x_j \le (n-1)(x_1^2 + \cdots + x_n^2)$ holds and the equality on the right hand side occurs only if $x_1 = \cdots = x_n$. By the assumed continuity of $F(x)$ we find $\mathrm{P}(x_1 = \cdots = x_n) = 0$ and $0 < \frac{2\Sigma_{1\le i<j\le n} x_i x_j}{x_1^2+\cdots+x_n^2} < n-1$. It follows that $\frac{m}{n^2} < \theta < \frac{m}{n}$ by (3.9).

Let $\varphi(t)$ be the characteristic function of $F(x)$. Then

$$\varphi'(t) = i\int_0^\infty x\exp[itx]dF(x), \varphi''(t) = -\int_0^\infty x^2\exp[itx]dF(x).$$

Expressing (3.8) as a differential equation for the characteristic function $\varphi(t)$, we get

$$m\varphi''(t)(\varphi(t))^{n-1} = \theta\big[n\varphi''(t)(\varphi(t))^{n-1} + 2\cdot{}_n\mathrm{C}_2(\varphi'(\mathrm{t}))^2(\varphi(\mathrm{t}))^{\mathrm{n}-2}\big].$$

That is,

$$\frac{\varphi''(t)}{\varphi'(t)} = \left(\frac{2\cdot{}_n\mathrm{C}_2\cdot\theta}{m-n\theta}\right)\frac{\varphi'(t)}{\varphi(t)}, \quad \frac{m}{n^2} < \theta < \frac{m}{n}.$$

After integrating with the initial conditions $\varphi(0) = 1, \varphi'(0) = iE(X)$, we get

$$\varphi'(t) = iE(X)(\varphi(t))^{\frac{2\cdot{}_n\mathrm{C}_2\cdot\theta}{m-n\theta}}, \quad \frac{2\cdot{}_n\mathrm{C}_2\cdot\theta}{m-n\theta} > 1. \tag{3.10}$$

The solution of this differential equation (3.10) with the above initial conditions is

$$\varphi(t) = (1 - \frac{iE(X)}{\lambda}t)^{-\lambda}, \quad \lambda = \frac{m-n\theta}{n^2\theta-m} > 0.$$

Consequently, $F(x)$ is a gamma distribution.

□

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