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A New Family of q-analogue of Genocchi Numbers and Polynomials of Higher Order

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Abstract. In the present paper, we introduce the new generalization of q-Genocchi polynomials and numbers of higher order. Also, we give some interesting identities. Finally, by applying q-Mellin transformation to the generating function for q-Genocchi polynomials of higher order put we define novel q-Hurwitz-Zeta type function which is an interpolation for this polynomials at negative integers.

1. Introduction

Recently, the Bernoulli, Euler and Genocchi polynomials and their families have been studied by many mathematicians for a long time (for details, see [1-22]). These polynomials possess a number of interesting properties not only in Analytic numbers theory and Complex analysis, but also in Mathematical physics, q-analysis, p-adic analysis and other areas of mathematics.

Throughout this work, we assume that $q \in \mathbb{C}$ with $|q| < 1$. The q-integer of x is defined by $[x]_q = \frac{1-q^x}{1-q}$ and we note that $\lim_{q\to 1} [x]_q = x$. The q-derivative is also defined by F. H. Jackson as follows:

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(x) - f(qx)}{(1-q)x}. \tag{1.1}$$

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Taking $f(x) = x^n$ in (1.1), it becomes as follows:

$$D_q x^n = \frac{x^n - (qx)^n}{(1-q)x} = [n]_q x^{n-1} \text{ and } \left(\frac{d}{d_q x}\right)^n f(x) = [n]_q!$$

where $[n]_q! = [n]_q [n-1]_q \cdots [1]_q$. Now, we give definitions of two kinds of q-exponential functions as follows:

For any $z \in \mathbb{C}$ with $|z| < 1$,

$$e_q(z) = \sum_{l=0}^{\infty} \frac{z^l}{[l]_q!} \text{ and } E_q(z) = \sum_{l=0}^{\infty} q^{\binom{l}{2}} \frac{z^l}{[l]_q!}. \tag{1.2}$$

By (1.2), it is not difficult to show that $[l]_{\frac{1}{q}}! = q^{-\binom{l}{2}} [l]_q!$. Then, we have the following

$$e_{\frac{1}{q}}(z) = E_q(z). \tag{1.3}$$

For the q-commuting variables x and y such that $yx = qxy$, we know that

$$e_q(x+y) = e_q(x) e_q(y). \tag{1.4}$$

The q-integral was defined by Jackson as follows:

$$\int_0^x f(\xi) d_q \xi = (1-q) x \sum_{l=0}^{\infty} f\left(q^l x\right) q^l \tag{1.5}$$

provided that the series on the right hand side converges absolutely.

In particular, if $f(\xi) = \xi^n$, then we have

$$\int_0^x \xi^n d_q \xi = \frac{1}{[n+1]_q} x^{n+1}. \tag{1.6}$$

The definitions of q-integral and q-derivative imply the following formula:

$$D_q \left(\int_0^x f(t) d_q t\right) = f(x) \tag{1.7}$$

and

$$D_q (f(x) g(x)) = f(x) D_q (g(x)) + g(qx) D_q (f(x)). \tag{1.8}$$

For more information of Eqs. (1-8), one can refer to [23-27].

The ordinary Euler polynomials are defined via the following generating function:

$$e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t+1} e^{xt}, \ |t| < \pi$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$ (see [1-22]).

In [16], the new q-generalization of Euler polynomials was introduced by T. Kim as follows:

$$\sum_{j=0}^{\infty} \frac{z^j}{[j]_q!} E_{j,q}(x) = \frac{[2]_q}{e_q(z)+1} e_q(xz).$$

By using the above generating function, Kim gave some interesting properties for new q-generalization of Euler numbers and polynomials. We note that these polynomials are useful to study in the theory of special functions. By the same motivation, in the next section, we shall introduce the generating function of q-Genocchi numbers and polynomials of higher order. Additionally, we shall give their applications.

2. On The New q-Genocchi Numbers and Polynomials of Higher Order

In this section, we introduce the generating function for q-Genocchi numbers of higher order due to Kim's method in [16]. Thus, we now start as follows:

$$S_q(z:\alpha) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} G_{n,q}^{(\alpha)} = \left(\frac{[2]_q z}{e_q(z)+1}\right)^{\alpha}. \tag{1.1}$$

Here $G_{n,q}^{(\alpha)}$ are called the q-Genocchi numbers of higher order. By using q-derivative operator, we compute as follows:
(1.2)

$$S_q\left(\frac{d}{d_q x}:\alpha\right) x^k = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} G_{n,q}^{(\alpha)} \left(\frac{d}{d_q x}\right)^n x^k = \sum_{n=0}^{k} \binom{k}{n}_q G_{n,q}^{(\alpha)} x^{k-n} = G_{k,q}^{(\alpha)}(x).$$

where $G_{k,q}^{(\alpha)}(x)$ are called q-Genocchi polynomials of higher order and

$$\binom{k}{n}_q = \frac{[k]_q [k-1]_q \cdots [k-n+1]_q}{[n]_q!}.$$

Similarly, by (1.2), we develop as follows:

$$\begin{aligned} S_q\left(\frac{d}{d_q x}:\alpha\right) e_q(tx) &= \sum_{j=0}^{\infty} \frac{G_{j,q}^{(\alpha)}}{[j]_q!} \left(\frac{d}{d_q x}\right)^j \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} t^k \\ &= \sum_{j=0}^{\infty} \frac{t^j}{[j]_q!} G_{j,q}^{(\alpha)}(x) \\ &= S_q(x,t:\alpha). \end{aligned}$$

From this point of view, we can also consider the q-Genocchi polynomials of higher order in the form:

$$\sum_{n=0}^{\infty}\frac{z^n}{[n]_q!}G_{n,q}^{(\alpha)}(x)=\left(\frac{[2]_q z}{e_q(z)+1}\right)^{\alpha}e_q(zx). \tag{1.3}$$

As $q\to 1$ and $\alpha=1$ in Eq. (1.3), we easily reach the following

$$\lim_{q\to 1}G_{n,q}^{(1)}(x)=G_n(x)$$

in which $G_n(x)$ are known ordinary Genocchi polynomials (for details, see [14], [18], [26], [2], [4], [5]).

By (1.1) and (1.3), we readily see that

$$\sum_{j=0}^{\infty}\frac{z^j}{[j]_q!}G_{j,q}^{(\alpha)}(x)=\sum_{j=0}^{\infty}\left(\sum_{n=0}^{j}\binom{j}{n}_q x^{j-n}G_{n,q}^{(\alpha)}\right)\frac{z^j}{[j]_q!}.$$

Matching the coefficients of $\frac{z^j}{[j]_q!}$ in the both sides of the above equation, then we obtain the following theorem.

Theorem 2.1. For any $j\in\mathbb{N}$, we have

$$G_{j,q}^{(\alpha)}(x)=\sum_{n=0}^{j}\binom{j}{n}_q x^{j-n}G_{n,q}^{(\alpha)}.$$

By applying q-derivative operator to (1.3), then we see that

$$\sum_{n=1}^{\infty}\frac{z^n}{[n]_q!}\left\{\frac{d}{d_qx}G_{n,q}^{(\alpha)}(x)\right\}=z\sum_{n=0}^{\infty}\frac{z^n}{[n]_q!}G_{n,q}^{(\alpha)}(x).$$

By comparing the coefficients of z^n on the both sides of the above equation, we arrive at the following theorem.

Theorem 2.2. For any $n\in\mathbb{N}^*=\mathbb{N}\cup\{0\}$, we have

$$\frac{d}{d_qx}G_{n,q}^{(\alpha)}(x)=[n]_q\,G_{n-1,q}^{(\alpha)}(x).$$

For q-commuting variables x and y ($yx=qxy$), we note that

$$\begin{aligned}\sum_{l=0}^{\infty}\frac{z^l}{[l]_q!}G_{l,q}^{(\alpha)}(x+y) &= \left(\frac{[2]_q z}{e_q(z)+1}\right)^{\alpha}e_q(z(x+y))\\ &= e_q(zy)\left(e_q(zx)\left(\frac{[2]_q z}{e_q(z)+1}\right)^{\alpha}\right)\\ &= \sum_{l=0}^{\infty}\left(\sum_{j=0}^{l}\binom{l}{j}_q y^{l-j}G_{j,q}^{(\alpha)}(x)\right)\frac{z^l}{[l]_q!}.\end{aligned}$$

As a result, we procure the following theorem.

Theorem 2.3. For any $n \in \mathbb{N}^*$, we have

$$G_{n,q}^{(\alpha)}(x+y) = \sum_{j=0}^{n} \binom{n}{j}_q y^{n-j} G_{n,q}^{(\alpha)}(x).$$

By expression (1.3), we compute as follows:

$$\begin{aligned} (1.4) \qquad & \sum_{l=0}^{\infty} \frac{z^l}{[l]_q!} G_{l,q}^{(\alpha+\beta)}(x) \\ = & \left[\left(\frac{[2]_q z}{e_q(z)+1} \right)^{\alpha} \right] \left[\left(\frac{[2]_q z}{e_q(z)+1} \right)^{\beta} e_q(zx) \right] \\ = & \left[\sum_{j=0}^{\infty} \frac{z^j}{[j]_q!} G_{j,q}^{(\alpha)} \right] \left[\sum_{k=0}^{\infty} \frac{z^k}{[k]_q!} G_{k,q}^{(\beta)}(x) \right] \end{aligned}$$

by using Cauchy product on the above equation, we derive that

$$(1.5) \qquad \sum_{l=0}^{\infty} \frac{z^l}{[l]_q!} \left(\sum_{n=0}^{l} \binom{l}{n}_q G_{n,q}^{(\alpha)} G_{l-n,q}^{(\beta)}(x) \right).$$

Comparing the coefficients of Eqs. (1.4) and (1.5), we get the following theorem.

Theorem 2.4. For any $n \in \mathbb{N}^*$, we have

$$G_{n,q}^{(\alpha+\beta)}(x) = \sum_{k=0}^{n} \binom{n}{k}_q G_{k,q}^{(\alpha)} G_{n-k,q}^{(\beta)}(x).$$

Jackson defined the q-analogue of the Gamma function by

$$(1.6) \qquad \Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}, \; x \neq 0, -1, -2, \cdots.$$

which have the following properties:

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \; \Gamma_q(1) = 1 \text{ and } \lim_{q \to 1^-} \Gamma_q(x) = \Gamma(x), \; \Re(x) > 0.$$

It has the q-integral representation as follows:

$$(1.7) \qquad \Gamma_q(s) = \int_0^{\frac{1}{1-q}} t^{s-1} E_q(-qt) \, d_q t.$$

When $\frac{\log(1-q)}{\log q} \in \mathbb{Z}$, becomes

$$\Gamma_q(s) = \int_0^\infty t^{s-1} E_q(-qt)\, d_q t. \tag{1.8}$$

The q-Mellin transformation of a suitable function f on $\mathbb{R}_{q,+}$ is defined by

$$M_q(f)(s) = \int_0^\infty t^{s-1} f(t)\, d_q t \tag{1.9}$$

(for details of Eqs. (14-17), see [20], [10], [15]).

In [27], the novel q-differential operator was defined by Rubin as follows:

$$\partial_q(f)(x) = \frac{f\left(q^{-1}x\right) + f\left(-q^{-1}x\right) - f(qx) + f(-qx) - 2f(-x)}{2(1-q)x}. \tag{1.10}$$

By (1.10), we note that

$$\lim_{q\to 1} \partial_q(f)(x) = f'(x). \tag{1.11}$$

By applying Rubin's q-differential operator to the generating function of q-Genocchi numbers and polynomials of higher order, we compute as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \partial_q G_{n,q}^{(\alpha)}(x) &= \partial_q \left\{ \left(\frac{[2]_q z}{e_q(z)+1} \right)^{\alpha} e_q(xz) \right\} \\ = \sum_{n=0}^{\infty} &\left(\frac{1}{2(1-q)} \sum_{l=0}^{n} \binom{n}{l}_q \left\{ \begin{array}{c} q^{-l} + (-1)^l q^{-l} - q^l \\ + (-1)^l q^l + 2(-1)^l \end{array} \right\} x^{l-1} G_{n-l,q}^{(\alpha)} \right) \frac{z^n}{[n]_q!}.\end{aligned}$$

By comparing the coefficients of $\frac{z^n}{[n]_q!}$ on the both sides of the above equation. Then, we state the following theorem.

Theorem 2.5. Let $T_q(l) = q^{-l} + (-1)^l q^{-l} - q^l + (-1)^l q^l + 2(-1)^l$, then we get

$$\partial_q G_{n,q}^{(\alpha)}(x) = \frac{1}{2(1-q)} \sum_{l=0}^{n} \binom{n}{l}_q T_q(l)\, x^{l-1} G_{n-l,q}^{(\alpha)}. \tag{1.12}$$

By (1.12), we readily derive the following

$$\begin{aligned}\partial_q G_{n,q}^{(\alpha)}(x) &= \frac{1}{(1-q)} \sum_{l=0}^{\left[\frac{n}{2}\right]} \binom{n}{2l}_q \left\{ q^{-2l} + 1 \right\} G_{n-2l,q}^{(\alpha)} \\ &+ \frac{1}{(q-1)} \sum_{l=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2l+1}_q \left\{ q^{2l+1} + 1 \right\} G_{n-1-2l,q}^{(\alpha)}.\end{aligned}$$

Here [.] is Gauss' symbol. Consequently, we state the following theorem.

Theorem 2.6. For any $n \in \mathbb{N}^*$, we have

$$\partial_q G_{n,q}^{(\alpha)}(x) = \frac{1}{(1-q)} \sum_{l=0}^{\left[\frac{n}{2}\right]} \binom{n}{2l}_q \left\{q^{-2l}+1\right\} G_{n-2l,q}^{(\alpha)}$$
$$+\frac{1}{(q-1)} \sum_{l=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2l+1}_q \left\{q^{2l+1}+1\right\} G_{n-1-2l,q}^{(\alpha)}.$$

From (1.12) and Theorem 2.6, we conclude as follows:

Corollary 2.7. The following identity

$$\begin{aligned} & \frac{1}{2(1-q)} \sum_{l=0}^{n} \binom{n}{l}_q T_q(l)\, x^{l-1} G_{n-l,q}^{(\alpha)} \\ = \;& \frac{1}{(1-q)} \sum_{l=0}^{\left[\frac{n}{2}\right]} \binom{n}{2l}_q \left\{q^{-2l}+1\right\} G_{n-2l,q}^{(\alpha)} \\ + \;& \frac{1}{(q-1)} \sum_{l=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2l+1}_q \left\{q^{2l+1}+1\right\} G_{n-1-2l,q}^{(\alpha)} \end{aligned}$$

is true.

By (1.12), we have the following corollary.

Corollary 2.8. For any $n \in \mathbb{N}^*$, we get

$$\lim_{q\to 1} \partial_q G_{n,q}^{(\alpha)}(x) = n G_{n-1}^{(\alpha)}(x)$$

where $G_n^{(\alpha)}(x)$ are known Genocchi polynomials of higher order defined by the following generating function:

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x) \frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt}.$$

We now give a q-analogue of D. Miliĉic's Lemma (see [22, page 1, Lemma 1.2.1]).

Lemma 2.9. Let $a_{n,q}$, $n \in \mathbb{N}^*$ be complex numbers such that $\sum_{n=0}^{\infty} |a_{n,q}|$ converges. Let

$$\lambda = \left\{-n \mid n \in \mathbb{N}^* \text{ and } a_{n,q} \neq 0\right\}.$$

Then,

$$g_q(z) = \sum_{n=0}^{\infty} \frac{a_{n,q}}{[z+n]_q}$$

converges absolutely for $z \in \mathbb{C} - \lambda$ and uniformly on bounded subsets of $\mathbb{C} - \lambda$. The q-function is a meromorphic function on complex plane with simple poles at the points in λ and Re $s(g_q, -n)$ for any $-n \in \lambda$.

Proof. By using similar method in lecture notes of D. Miliĉic's in [22], the proof of Lemma can be easily done. So, we omit it.

We now want to indicate that Γ extends to a meromorphic function by using Lemma 2.9. That is, we discover the following

$$\Gamma_q(z) = \int_0^\infty t^{z-1} E_q(-qt)\, d_q t = \int_0^1 t^{z-1} E_q(-qt)\, d_q t + \int_1^\infty t^{z-1} E_q(-qt)\, d_q t.$$

Then, the second integral converges for any complex z and represents an entire function. On the other hand, the q-exponential function is entire, and we have

$$\begin{aligned} \int_0^1 t^{z-1} E_q(-qt)\, d_q t &= \int_0^1 t^{z-1} \left\{ \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}}}{[j]_q!} t^j \right\} d_q t \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}}}{[j]_q!} \left\{ \int_0^1 t^{z+j-1} d_q t \right\} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}}}{[j]_q!} \frac{1}{[z+j]_q} \end{aligned}$$

for any $z \in \mathbb{C}$. Now also, we can write as follows:

$$\Gamma_q(z) = \int_1^\infty t^{z-1} E_q(-qt)\, d_q t + \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j+1}{2}}}{[j]_q!} \frac{1}{[z+j]_q}$$

for any z in the right half plane. From Lemma 2.9, the right hand-side of the above identity defines a meromorphic function on the complex plane with simple poles at $z = -j$, $j \in \mathbb{N}^*$. Then, we have the following theorem.

Theorem 2.10. For any $j \in \mathbb{N}^*$, we derive the following

$$Res(\Gamma_q, -j) = \frac{(-1)^j q^{\frac{j(j+1)}{2}}}{[j]_q!}. \tag{1.13}$$

As $q \to 1$ into (1.13), we easily derive that

$$\lim_{q \to 1} Res(\Gamma_q, -j) = \frac{(-1)^j}{j!}$$

which is residue of Euler's Gamma function (see [25]).

Now also, by applying q-Mellin Transformation to generating function of q-Genocchi polynomials of higher order, then we compute as follows:

For q-commuting variables x and y ($yx = qxy$),

$$\Im_q (z, x : \alpha) = \frac{1}{\Gamma_{\frac{1}{q}}(z)} \int_0^\infty t^{z-\alpha-1} \{(-1)^\alpha S_q (x, -t : \alpha)\} d_{\frac{1}{q}} t$$

$$= \sum_{l_1,l_2,\ldots,l_\alpha=0}^{\infty} (-1)^{l_1+l_2+\ldots+l_\alpha} \left\{ \frac{1}{\Gamma_{\frac{1}{q}}(z)} \int_0^\infty t^{z-1} E_{\frac{1}{q}} \left(-\frac{t}{q} \left(qx + q \sum_{k=1}^{\alpha} l_k \right) \right) d_{\frac{1}{q}} t \right\}$$

$$= [2]_q^\alpha \sum_{l_1,l_2,\ldots,l_\alpha=0}^{\infty} \frac{q^{-z} (-1)^{l_1+l_2+\ldots+l_\alpha}}{(l_1 + l_2 + \ldots + l_\alpha + x)^z}.$$

So, we now introduce definition of q-Hurwitz-Zeta type function as follows:

Definition 2.11. For any $z \in \mathbb{C}$, we define

$$\Im_q (z, x : \alpha) = [2]_q^\alpha \sum_{l_1,l_2,\ldots,l_\alpha=0}^{\infty} \frac{q^{-z} (-1)^{l_1+l_2+\ldots+l_\alpha}}{(l_1 + l_2 + \ldots + l_\alpha + x)^z}.$$

By the above definition, we derive interpolation functions for q-Genocchi polynomials of higher order at negative integers.

Theorem 2.12. The following equality holds true:

$$\Im_q (-n, x : \alpha) = \frac{q^{-n} G_{n+\alpha,q}^{(\alpha)}(x)}{[\alpha]_q! \binom{n+\alpha}{\alpha}_q}.$$

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