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Pure Ideals in Ordered Semigroups

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ABSTRACT. In this paper the concepts of pure ideals, weakly pure ideals and purely prime ideals in ordered semigroups are introduced. We obtain some characterizations of pure ideals and prove that the set of all pure prime ideals is topologized.

1. Introduction and Preliminaries

In [1], Ahsan and Takahashi introduced the notions of pure ideals and purely prime ideals in a semigroup without order. Recently, Bashir and Shabir [3] defined the concepts of pure ideals, weakly pure ideals and purely prime ideals in a ternary semigroup. The authors gave some characterizations of pure ideals and showed that the set of all purely prime ideals of a ternary semigroup is topologized. In this paper, we do in the line of Bashir and Shabir. We introduce the concepts of pure ideals, weakly pure ideals and purely prime ideals on an ordered semigroup. We characterize pure ideals and prove that the set of all purely prime ideals of an ordered semigroup is topologized. Note that the results on semigroups without order become then special cases.

For the rest of this section, we recall some definitions and results used throughout the paper.

A semigroup S with an order relation $\leq$ is called an *ordered semigroup* ([2], [5]) if for $x, y, z \in S$, $x \leq y$ implies $zx \leq zy$ and $xz \leq yz$. An element 0 of S is called a *zero* element of S if $0x = x0 = 0$ for all $x \in S$ and $0 \leq x$ for all $x \in S$. A nonempty subset A of S is called a *subsemigroup* of S if $xy \in A$ for all $x, y \in A$. Note that every subsemigroup of S is an ordered semigroup under the order relation on S. For nonempty subsets A and B of S, let

$$AB = \{xy \mid x \in A, y \in B\}.$$

For $x \in S$, let $Ax = A\{x\}$ and $xA = \{x\}A$. A nonempty subset A of S is a subsemigroup of S if and only if $AA \subseteq A$.

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Let $(S, \cdot, \leq)$ be an ordered semigroup. For a subset A of S, let

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

For $A, B \subseteq S$, we have (1) $A \subseteq (A]$, (2) $A \subseteq B$ implies $(A] \subseteq (B]$ and (3) $((A](B]] = (AB]$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *left (resp. right) ideal* of S if $SA \subseteq A$ (resp. $AS \subseteq A$) and for $x \in A$ and $y \in S$, $y \leq x$ implies $y \in A$. The second condition means that $A = (A]$. A nonempty subset A of S is called a *(two-sided) ideal* of S if A is both a left and a right ideal of S. Note that if A and B are ideals of S then $(AB]$ is an ideal of S. Intersection of ideals of S is an ideal of S if it is nonempty. Union of ideals of S is an ideal of S. Finite intersection of ideals of S is an ideal of S. For a nonempty subset A of S, the intersection of all left (resp. right, two-sided) ideals of S containing A, denoted by $(A)_l$ (resp. $(A)_r$, (A)) is a left (resp. right, two-sided) ideal of S containing A. In [4], it was shown that

$$(A)_l = (A \cup SA], (A)_r = (A \cup AS]$$

and

$$(A) = (A \cup SA \cup AS \cup SAS].$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. An element a of S is said to be *regular* if there exists $x \in S$ such that $a \leq axa$, and S is called a *regular semigroup* if every element of S is regular. Note that S is regular if and only if $a \in (aSa]$ for all $a \in S$.

2. Pure Ideals in Ordered Semigroups

In this section, the notion of a pure ideal in ordered semigroups will be introduced and studied.

Definition 2.1. Let $(S, \cdot, \leq)$ be an ordered semigroup. An ideal A of S is called a *left (resp. right) pure ideal* if for $x \in A$ there exists $y \in A$ such that $x \leq yx$ (resp. $x \leq xy$). One-sided *left (right) pure ideals* can be defined analogously.

Equivalent Definition. $x \in (Ax]$ (resp. $x \in (xA]$).

Theorem 2.2. Let $(S, \cdot, \leq)$ be an ordered semigroup and A an ideal of S. Then A is right pure if and only if $B \cap A = (BA]$ for all right ideals B of S.

Proof. Assume that A is right pure. Let B be a right ideal of S. We have $BA \subseteq BS \subseteq B$. Hence $(BA] \subseteq B$. Since $BA \subseteq SA \subseteq A$, so $(BA] \subseteq A$. Then $(BA] \subseteq B \cap A$. To prove the reverse inclusion, let $x \in B \cap A$. By assumption, there exists $y \in A$ such that $x \leq xy$. Since $xy \in BA$, we obtain $x \in (BA]$. Thus $B \cap A \subseteq (BA]$.

Conversely, suppose that $B \cap A = (BA]$ for all right ideals B of S. Let $x \in A$. Since $(x \cup xS]$ is a right ideal of S and $SA \subseteq A$, we have

$$(x \cup xS] \cap A = ((x \cup xS]A] \subseteq (xA \cup xSA] \subseteq (xA].$$

Since $x \in (x \cup xS] \cap A$, $x \in (xA]$. This proves that A is a right pure ideal. $\square$

Definition 2.3. An ordered semigroup $(S, \cdot, \leq)$ is said to be *right weakly regular* if for any $x \in S$ there exist $y, z \in S$ such that $x \leq xyxz$.

Equivalent Definition. $x \in (xSxS]$.

Note that every regular ordered semigroup is right weakly regular.

Theorem 2.4. Let $(S, \cdot, \leq)$ be an ordered semigroup. The following are equivalent.

(i) S is right weakly regular.

(ii) $(AA] = A$ for all right ideals A of S.

(iii) $B \cap A = (BA]$ for all right ideals B and all ideals A of S.

Proof. (i) $\Rightarrow$ (ii). Assume that S is right weakly regular. Let A be a right ideal of S. We have $AA \subseteq AS \subseteq A$, hence $(AA] \subseteq A$. Let $x \in A$. By assumption, there exist $y, z \in A$ such that $x \leq xyxz$. Since $(xy)(xz) \in AA$, $x \in (AA]$. Then $A \subseteq (AA]$, whence $(AA] = A$.

(ii) $\Rightarrow$ (i). Assume that $(AA] = A$ for all right ideals A of S. Let $x \in S$. Since $(x \cup xS]$ is a right ideal of S, we have

$$\begin{aligned} (x \cup xS] = ((x \cup xS](x \cup xS]] &\subseteq ((x \cup xS)(x \cup xS)] \\ &= (xx \cup xxS \cup xSx \cup xSxS]. \end{aligned}$$

Then $x \in (xx \cup xxS \cup xSx \cup xSxS]$, hence $x \in (xSxS]$. This proves that S is right weakly regular.

(i) $\Rightarrow$ (iii). Assume that S is right weakly regular. Let B and A be a right ideal and an ideal of S, respectively. Since $BA \subseteq BS \subseteq B$, $(BA] \subseteq B$. Similarly, $(BA] \subseteq A$. Then $(BA] \subseteq B \cap A$. Let $x \in B \cap A$. We have $(xSxS] \subseteq (BA]$. By assumption, we get $x \in (xSxS]$, hence $x \in (BA]$. Thus $B \cap A \subseteq (BA]$, whence $B \cap A = (BA]$.

(iii) $\Rightarrow$ (i). Assume that $B \cap A = (BA]$ for all right ideals B and all ideals A of S. To prove that S is right weakly regular, let therefore $x \in S$. We have $(x \cup xS]$ and $(x \cup SxS]$ are right and (two-sided) ideal of S, respectively. Then

$$\begin{aligned} (x \cup xS] \cap (x \cup SxS] &= ((x \cup xS](x \cup SxS]] \\ &\subseteq (xx \cup xSxS \cup xSx \cup xSSxS], \end{aligned}$$

thus $x \in (xx \cup xSxS \cup xSx \cup xSSxS]$. This implies that $x \in (xSxS]$, hence S is right weakly regular. $\square$

Theorem 2.5. An ordered semigroup S is right weakly regular if and only if every ideal of S is right pure.

Proof. This follows from Theorem and Theorem . $\square$

Theorem 2.6. Let $(S, \cdot, \leq)$ be an ordered semigroup with zero 0.

(i) $\{0\}$ is a right pure ideal of S.

(ii) Union of any right pure ideals of S is a right pure ideal of S.

(iii) Finite intersection of right pure ideals of S is a right pure ideal of S.

Proof. (i) This is obvious.

(ii) Let $\{A_i \mid i \in I\}$ be a family of right pure ideals of S. We have $\bigcup_{i\in I} A_i$ is an ideal of S. Let $x \in \bigcup_{i\in I} A_i$. Then $x \in A_j$ for some $j \in I$. Since A_j is right pure, there exists $y \in A_j$ such that $x \leq xy$. We have $y \in A_j \subseteq \bigcup_{i\in I} A_i$, hence $\bigcup_{i\in I} A_i$ is right pure.

(iii) Let $\{A_1, A_2, \ldots, A_n\}$ be a finite family of right pure ideals of S. Then $\bigcap_{i=1}^n A_i$ is an ideal of S. Let $x \in \bigcap_{i=1}^n A_i$. For $k \in \{1, 2, \ldots, n\}$, there exists $y_k \in A_k$ such that $x \leq xy_k$. We have

$$x \leq xy_n \cdots y_2 y_1.$$

Since $y_n \cdots y_2 y_1 \in \bigcap_{i=1}^n A_i$, we conclude that $\bigcap_{i=1}^n A_i$ is right pure. $\square$

Theorem 2.7. Let S be an ordered semigroup with zero 0 and A an ideal of S. Then A contains the largest right pure ideal of S (called the *pure part* of A), denoted by $\mathcal{S}(A)$.

Proof. Since $\{0\}$ is a right pure ideal of S contained in A, it follows that the union of all right pure ideals of S contained in A exists, and hence it is the largest right pure ideal of S contained in A. $\square$

Theorem 2.8. Let $(S, \cdot, \leq)$ be an ordered semigroup with zero 0. Let A, B and $A_i, i \in I$ be ideals of S.

(i) $\mathcal{S}(A \cap B) = \mathcal{S}(A) \cap \mathcal{S}(B)$.

(ii) $\bigcup_{i\in I} \mathcal{S}(A_i) \subseteq \mathcal{S}(\bigcup_{i\in I} A_i)$.

Proof. (i) Since $\mathcal{S}(A) \subseteq A$ and $\mathcal{S}(B) \subseteq B$, we have $\mathcal{S}(A) \cap \mathcal{S}(B) \subseteq A \cap B$. Hence $\mathcal{S}(A) \cap \mathcal{S}(B) \subseteq \mathcal{S}(A \cap B)$. Since $\mathcal{S}(A \cap B) \subseteq A \cap B \subseteq A$, we get $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(A)$. Similarly, $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(B)$. Then $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(A) \cap \mathcal{S}(B)$, whence $\mathcal{S}(A \cap B) = \mathcal{S}(A) \cap \mathcal{S}(B)$.

(ii) Since $\mathcal{S}(A_i) \subseteq A_i$ for all $i \in I$, we have $\bigcup_{i\in I} \mathcal{S}(A_i) \subseteq \bigcup_{i\in I} A_i$. Then $\bigcup_{i\in I} \mathcal{S}(A_i) \subseteq \mathcal{S}(\bigcup_{i\in I} A_i)$. $\square$

Definition 2.9. A right pure ideal A of an ordered semigroup S is said to be *purely maximal* if for any proper right pure ideal B of S, $A \subseteq B$ implies $A = B$.

Definition 2.10. Let A be a proper right pure ideal of an ordered semigroup S. Then A is called *purely prime* if for any right pure ideals B_1, B_2 of S, $B_1 \cap B_2 \subseteq A$ implies $B_1 \subseteq A$ or $B_2 \subseteq A$.

Theorem 2.11. Every purely maximal ideal of an ordered semigroup S is purely prime.

Proof. Let A be a purely maximal ideal of S. Let B and C be right pure ideals of S such that $B \cap C \subseteq A$ and $B \nsubseteq A$. Since $B \cup A$ is a right pure ideal such that $A \subset B \cup A$, so $S = B \cup A$. We have

$$C = C \cap S = C \cap (B \cup A) = (C \cap B) \cup (C \cap A) \subseteq A.$$

Then A is purely prime. $\square$

Theorem 2.12. Let $(S, \cdot, \leq)$ be an ordered semigroup with zero. The pure part of any maximal ideal of S is purely prime.

Proof. Let A be a maxmal ideal of S. To show that $\mathcal{S}(A)$ is purely prime, let B and C be right pure ideals of S such that $B \cap C \subseteq \mathcal{S}(A)$. If $B \subseteq A$, then $B \subseteq \mathcal{S}(A)$. Suppose that $B \nsubseteq A$. We have $B \cup A$ is an ideal of S. By maximality of A, $S = B \cup A$, and hence $C \subseteq A$. Thus $C \subseteq \mathcal{S}(A)$. $\square$

Theorem 2.13. Let $(S, \cdot, \leq)$ be an ordered semigroup and A a right pure ideal of S. If $x \in S \setminus A$, then there exists a purely prime ideal B of S such that $A \subseteq B$ and $x \notin B$.

Proof. Assume that $x \in S \setminus A$. Let

$$P = \{B \mid B \text{ is a right pure ideal of } S,\ A \subseteq B \text{ and } x \notin B\}.$$

We have $P \neq \emptyset$ since $A \in P$. Moreover, P is a partially ordered set under the usual inclusion. Let $\{B_k \mid k \in K\}$ be any totally ordered subset of P. By Theorem , $\bigcup_{k \in K} B_k$ is a right pure ideal. Since $A \subseteq \bigcup_{k \in K} B_k$ and $x \notin \bigcup_{k \in K} B_k$, we obtain $\bigcup_{k \in K} B_k \in P$. By Zorn's lemma, P has a maximal element, say M, such that M is a right pure ideal, $A \subseteq M$ and $x \notin M$. We shall show that M is purely prime. Suppose that A_1 and A_2 are right pure ideals of S such that $A_1 \nsubseteq M$ and $A_2 \nsubseteq M$. Since A_1, A_2 and M are right pure, so $A_1 \cup M$ and $A_2 \cup M$ are right pure ideals containing M. Since $x \in A_1 \cup M$ and $x \notin M$, we have $x \in A_1$. Similarly, $x \in A_2$. Hence $x \in A_1 \cap A_2$. Thus $A_1 \cap A_2 \nsubseteq M$. This shows that M is purely prime. $\square$

Theorem 2.14. Any proper right pure ideal A of an ordered semigroup $(S, \cdot, \leq)$ is the intersection of all the purely prime ideals of S containing A.

Proof. By Theorem , there exist purely prime ideals containing A. Let $\{B_k \mid k \in K\}$ be a family of all purely prime ideals of S containing A. Then $A \subseteq \bigcap_{k\in K} B_k$. Using Theorem , the reverse inclusion follows. $\square$

3. Weakly Pure Ideals in Ordered Semigroups

In this section, we introduce the concept of weakly pure ideal in an ordered semigroup.

Definition 3.1. Let $(S, \cdot, \leq)$ be an ordered semigroup. An ideal A of S is called *left (resp. right) weakly pure* if $A \cap B = (AB]$ (resp. $A \cap B = (BA]$) for all ideals B of S.

In an ordered semigroup, every left (right) pure two-sided ideal is left (right) weakly pure.

Theorem 3.2. Let $(S, \cdot, \leq)$ be an ordered semigroup with zero 0. If A and B are ideals of S, then

$$\begin{aligned} BA^{-1} &= \{s \in S \mid \forall x \in A, xs \in B\} \\ A_{-1}B &= \{s \in S \mid \forall x \in A, sx \in B\} \end{aligned}$$

are ideals of S.

Proof. Clearly, $0 \in BA^{-1}$. Let $u, v \in S$ and $s \in BA^{-1}$. Let $x \in A$. Since $xu \in A$, we have $(xu)s \in B$, and hence $x(usv) = (xu)sv \in B$. Let $a \in BA^{-1}$ and $b \in S$ be such that $b \leq a$. Let $y \in A$, then $yb \leq ya$. Since $ya \in B$, $yb \in B$. This shows that BA^{-1} is an ideal of S. Similarly, $A_{-1}B$ is an ideal of S. $\square$

Theorem 3.3. Let $(S, \cdot, \leq)$ be an ordered semigroup and A an ideal of S. Then A is left (right) weakly pure if and only if $(BA^{-1}) \cap A = A \cap B$ $((A_{-1}B) \cap A = A \cap B)$ for all ideals B of S.

Proof. Assume that A is left weakly pure. Let B be an ideal of S. By Theorem , BA^{-1} is an ideal of S, and thus $A \cap BA^{-1} = (A(BA^{-1})]$. Since $A(BA^{-1}) \subseteq AS \subseteq A$, we have $(A(BA^{-1})] \subseteq (A] = A$. Let $t \in (A(BA^{-1})]$ be such that $t \leq xy$ for some $x \in A, y \in BA^{-1}$. By definition of BA^{-1}, $xy \in B$. Then $t \in B$. This proves that $A \cap BA^{-1} \subseteq A \cap B$. For the reverse inclusion, let $x \in A \cap B$. Since $ax \in B$ for any $a \in A$, we have $x \in BA^{-1}$. We get $x \in BA^{-1} \cap A$, and then $A \cap B \subseteq BA^{-1} \cap A$.

Conversely, assume that $(BA^{-1}) \cap A = A \cap B$ for all ideals B of S. To show that A is left weakly pure, let C be any ideal of S. We shall show that $A \cap C = (AC]$. By assumption, $A \cap C = CA^{-1} \cap A$. Since $AC \subseteq AS \subseteq A$, $(AC] \subseteq A$. Let $t \in (AC]$ such that $t \leq xy$ for some $x \in A, y \in C$ and let $a \in A$. Since $a(xy) = (ax)y \in C$, we obtain $xy \in CA^{-1}$, and so $t \in CA^{-1}$. Then $(AC] \subseteq CA^{-1}$. This proves that $(AC] \subseteq A \cap C$. For the reverse inclusion, we have $C \subseteq (AC]A^{-1}$ because $c \in C, a \in A$ implies $ac \in AC \subseteq (AC]$. Then $A \cap C \subseteq (AC]A^{-1} \cap A = A \cap (AC] \subseteq (AC]$. $\square$

Theorem 3.4. Let $(S, \cdot, \leq)$ be an ordered semigroup. The following are equivalent.

(i) Every ideal is left weakly pure.

(ii) Every ideal A of S, $AA = A$.

(iii) Every ideal is right weakly pure.

Proof. This can be proved similarly as Proposition 4.4 in [3]. □

4. Pure Spectrums of an Ordered Semigroup

Let S be an ordered semigroup such that $SS = S$. The set of all right pure ideals of S and the set of all proper pure prime ideals of S will be denoted by $P(S)$ and $P'(S)$, respectively. For $A \in P(S)$, let

$$I_A = \{J \in P'(S) \mid A \nsubseteq J\} \text{ and } \tau(S) = \{I_A \mid A \in P(S)\}.$$

Theorem 4.1. $\tau(S)$ forms a topology on $P'(S)$.

Proof. Since $\{0\}$ is a right pure ideal of S and $I_{\{0\}} = \emptyset$, we have $\emptyset \in \tau(S)$. Since S is a right pure ideal of S such that $I_S = P'(S)$, we get $P'(S) \in \tau(S)$. Let $\{I_{A_\alpha} \mid \alpha \in \Lambda\} \subseteq \tau(S)$. We have $\bigcup_{\alpha\in\Lambda} I_{A_\alpha} = I_{\bigcup_{\alpha\in\Lambda} A_\alpha}$, whence $\bigcup_{\alpha\in\Lambda} I_{A_\alpha} \in \tau(S)$. Let $I_{A_1}, I_{A_2} \in \tau(S)$. We shall show that $I_{A_1} \cap I_{A_2} = I_{A_1\cap A_2}$, therefore let $J \in I_{A_1} \cap I_{A_2}$. We have $J \in P'(S)$, $A_1 \nsubseteq J$ and $A_2 \nsubseteq J$. Suppose that $A_1 \cap A_2 \subseteq J$. Since J is pure prime, $A_1 \subseteq J$ or $A_2 \subseteq J$. This is a contradiction. Then $J \in I_{A_1\cap A_2}$, hence $I_{A_1} \cap I_{A_2} \subseteq I_{A_1\cap A_2}$. For the reverse inclusion, let $J \in I_{A_1\cap A_2}$. Since $A_1 \cap A_2 \nsubseteq J$, $A_1 \nsubseteq J$ and $A_2 \nsubseteq J$. This implies that $J \in I_{A_1} \cap I_{A_2}$, thus $I_{A_1\cap A_2} \subseteq I_{A_1} \cap I_{A_2}$. Then $\tau(S)$ forms a topology on $P'(S)$. □

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