KYUNGPOOK Math. J. 54(2014), 113-122
http://dx.doi.org/10.5666/KMJ.2014.54.1.113

Some Properties of S-metric Spaces and Fixed Point Results

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Abstract. In this paper, we introduce S-metric spaces and give their some properties. Also we present a common fixed point theorem for multivalued maps on complete S-metric spaces. The single valued case and an illustrative example are given.

1. Introduction

In the present paper, we introduce the concept of S-metric spaces and give some properties of them. Then a common fixed point theorem for two multivalued mappings on complete S-metric spaces is given. In addition, we give an illustrative example for the single valued case.

We begin with the following definition.

Definition 1.1. Let X be a nonempty set. An *S-metric* on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

1. $S(x, y, z) \geq 0$,
2. $S(x, y, z) = 0$ if and only if $x = y = z$,

* Corresponding Author.
Received June 11, 2012; accepted May 5, 2013.
2010 Mathematics Subject Classification: 54H25, 47H10.
Key words and phrases: S-metric space, fixed point.

3. $S(x,y,z) \le S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called an *S-metric space*.

Immediate examples of such S-metric spaces are:

1. Let $X = \mathbb{R}^n$ and $||\cdot||$ a norm on X, then $S(x,y,z) = ||y+z-2x|| + ||y-z||$ is an S-metric on X.
2. Let $X = \mathbb{R}^n$ and $||\cdot||$ a norm on X, then $S(x,y,z) = ||x-z|| + ||y-z||$ is an S-metric on X.
3. Let X be a nonempty set, d is ordinary metric on X, then $S(x,y,z) = d(x,z) + d(y,z)$ is an S-metric on X.

Lemma 1.2. *In an S-metric space, we have $S(x,x,y) = S(y,y,x)$.*

Proof. By third condition of S-metric, we have

$$S(x,x,y) \le S(x,x,x) + S(x,x,x) + S(y,y,x) = S(y,y,x) \tag{1.1}$$

and similarly

$$S(y,y,x) \le S(y,y,y) + S(y,y,y) + S(x,x,y) = S(x,x,y). \tag{1.2}$$

Hence by (1.1) and (1.2), we get $S(x,x,y) = S(y,y,x)$.

Definition 1.3. Let (X,S) be an S-metric space. For $r > 0$ and $x \in X$ we define the open ball $B_S(x,r)$ and closed ball $B_S[x,r]$ with center x and radius r as follows respectively:

$$B_S(x,r) = \{y \in X : S(y,y,x) < r\},$$
$$B_S[x,r] = \{y \in X : S(y,y,x) \le r\}.$$

Example 1.4. Let $X = \mathbb{R}$. Denote $S(x,y,z) = |y+z-2x| + |y-z|$ for all $x,y,z \in \mathbb{R}$. Thus

$$\begin{aligned} B_S(1,2) &= \{y \in \mathbb{R} : S(y,y,1) < 2\} = \{y \in \mathbb{R} : |y-1| < 1\} \\ &= \{y \in \mathbb{R} : 0 < y < 2\} = (0,2). \end{aligned}$$

Definition 1.5. Let (X,S) be an S-metric space and $A \subset X$.

1. If for every $x \in A$ there exists $r > 0$ such that $B_S(x,r) \subset A$, then the subset A is called open subset of X.
2. Subset A of X is said to be *S-bounded* if there exists $r > 0$ such that $S(x,x,y) < r$ for all $x,y \in A$.
3. A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\forall\ n \ge n_0 \Longrightarrow S(x_n,x_n,x) < \varepsilon$$

and we denote by $\lim_{n\to\infty} x_n = x$.

4. Sequence $\{x_n\}$ in X is called a *Cauchy sequence* if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$.
5. The S-metric space (X, S) is said to be *complete* if every Cauchy sequence is convergent.
6. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X (induced by the S-metric S).

Lemma 1.6. *Let (X, S) be an S-metric space. If $r > 0$ and $x \in X$, then the ball $B_S(x, r)$ is open subset of X.*

Proof. Let $y \in B_S(x, r)$, hence $S(y, y, x) < r$. If set $\delta = S(x, x, y)$ and $r' = \frac{r-\delta}{2}$ then we prove that $B_S(y, r') \subseteq B_S(x, r)$. Let $z \in B_S(y, r')$, then $S(z, z, y) < r'$. By third condition of S-metric we have

$$S(z, z, x) \leq S(z, z, y) + S(z, z, y) + S(x, x, y) < 2r' + \delta = r$$

Hence $B_S(y, r') \subseteq B_S(x, r)$. That is the ball $B_S(x, r)$ is a open subset of X.

Lemma 1.7. *Let (X, S) be an S-metric space. If sequence $\{x_n\}$ in X converges to x, then x is unique.*

Proof. Let $\{x_n\}$ converges to x and y, then for each $\varepsilon > 0$ there exist $n_1, n_2 \in \mathbb{N}$ such that

$$\forall\ n \geq n_1 \Longrightarrow S(x_n, x_n, x) < \frac{\varepsilon}{4}$$

and

$$\forall\ n \geq n_2 \Longrightarrow S(x_n, x_n, y) < \frac{\varepsilon}{2}.$$

If set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ by third condition S-metric we have:

$$S(x, x, y) \leq 2S(x, x, x_n) + S(y, y, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $S(x, x, y) = 0$ so $x = y$.

Lemma 1.8. *Let (X, S) be an S-metric space. If sequence $\{x_n\}$ in X is converges to x, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Since $\lim_{n\to\infty} x_n = x$ then for each $\varepsilon > 0$ there exists $n_1, n_2 \in \mathbb{N}$ such that

$$n \geq n_1 \Rightarrow S(x_n, x_n, x) < \frac{\varepsilon}{4}$$

and

$$m \geq n_2 \Rightarrow S(x_m, x_m, x) < \frac{\varepsilon}{2}.$$

If set $n_0 = \max\{n_1, n_2\}$, then for every $n, m \geq n_0$ by third condition of S-metric we have:

$$S(x_n, x_n, x_m) \leq 2S(x_n, x_n, x) + S(x_m, x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence.

Lemma 1.9. *Let (X, S) be an S- metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then*

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

Proof. Since $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then for each $\varepsilon > 0$ there exist $n_1, n_2 \in \mathbb{N}$ such that

$$\forall\ n \geq n_1 \Rightarrow S(x_n, x_n, x) < \frac{\varepsilon}{4}$$

and

$$\forall\ n \geq n_2 \Rightarrow S(y_n, y_n, y) < \frac{\varepsilon}{4}.$$

If set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ by third condition of S-metric we have:

$$\begin{aligned} S(x_n, x_n, y_n) &\leq 2S(x_n, x_n, x) + S(y_n, y_n, x) \\ &\leq 2S(x_n, x_n, x) + 2S(y_n, y_n, y) + S(x, x, y) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + S(x, x, y) = \varepsilon + S(x, x, y). \end{aligned}$$

Hence we have:

$$S(x_n, x_n, y_n) - S(x, x, y) < \varepsilon. \tag{1.3}$$

On the other hand, we have

$$\begin{aligned} S(x, x, y) &\leq 2S(x, x, x_n) + S(y, y, x_n) \\ &\leq 2S(x, x, x_n) + 2S(y, y, y_n) + S(x_n, x_n, y_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + S(x_n, x_n, y_n) = \varepsilon + S(x_n, x_n, y_n), \end{aligned}$$

that is

$$S(x, x, y) - S(x_n, x_n, y_n) < \varepsilon. \tag{1.4}$$

Therefore by relations (1.3) and (1.4) we have $|S(x_n, x_n, y_n) - S(x, x, y)| < \varepsilon$, that is

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

Let (X, S) be an S-metric space, $C(X)$ denotes the family of all nonempty closed subsets of X. For A and B two nonempty subsets of X we define;

$$dist(x, A) = \inf_{a\in A}\{S(x, x, a)\}$$

and

$$\mathcal{S}(A, A, B) = \sup_{a\in A,\ b\in B}\{S(a, a, b)\}.$$

By the definition of $dist(x, A)$, it is clear that $dist(x, A) = 0 \Leftrightarrow x \in \overline{A}$.

2. Implicit Relations

Implicit relations on metric spaces have been used in many articles. For examples, [1], [2], [3], [4], [5], [6], [7], [8]. Let $\mathbb{R}_+$ be the set of nonnegative real numbers and let $\mathcal{T}$ be the set of all functions $T : \mathbb{R}^6_+ \to \mathbb{R}$ satisfying the following conditions:

$T_0:$ $T(\liminf_{n\to\infty} p_n) \leq \liminf_{n\to\infty} T(p_n)$ for any $p_n \in \mathbb{R}^6_+$, where $\liminf_{n\to\infty} p_n$ means component-wise lim inf.

$T_1:$ $T(t_1, ..., t_6)$ is nonincreasing in $t_2, ..., t_6$.

$T_2:$ there exists a continuous strictly increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\phi(t) < t$ for $t > 0$ and $\varepsilon > 0$ such that the inequalities

$$u \leq w + \varepsilon$$

and

$$T(w, v, v, u, 2u + v, 0) \leq 0 \quad \text{or} \quad T(w, v, u, v, 0, 2u + v) \leq 0$$

implies $w \leq \phi(v)$.

$T_3:$ $T(w, 0, v, 0, 0, v) \leq 0$ and $T(w, 0, 0, v, v, 0) \leq 0$ implies $w \leq \phi(v)$, where ϕ is the function in T_2.

Example 2.1. $T(t_1, ..., t_6) = t_1 - f(\max\{t_2, t_3, t_4, \frac{1}{3}(t_5 + t_6)\})$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ continuous strictly increasing function with $f(t) < t$ for $t > 0$.

T_0 and T_1 : Obviously.

T_2 : Let $u > 0$, then choose $\varepsilon > 0$ so that $f(u)+\varepsilon < u$ (this is possible since $f(u) < u$). Now let $u \leq w + \varepsilon$ and $T(w, v, v, u, 2u + v, 0) = w - f(\max\{u, v\}) \leq 0$. If $u \geq v$, then $u \leq w+\varepsilon \leq f(u)+\varepsilon < u$, a contradiction. Thus $u < v$ and $w \leq f(v)$. Similarly, $u \leq w+\varepsilon$ and $T(w, v, u, v, 0, 2u + v) \leq 0$ imply $w \leq f(v)$. If $u = 0$, then $w \leq f(v)$. Thus T_2 is satisfied with $\phi = f$.

T_3 : $T(w, 0, v, 0, 0, v) = T(w, 0, 0, v, v, 0) = w - f(v) \leq 0 \Rightarrow w \leq f(v) = \phi(v)$.

3. Fixed Point Theory

Our main result for fixed point theory of this work as follows.

Theorem 3.1. *Let (X, S) be a complete S-metric space, $x_0 \in X, r > 0$ with $F, G : B_S[x_0, r] \to C(X)$. Suppose, for all $x, y \in B_S[x_0, r]$ sets Fx, Gy are bounded and*

$$T(\mathcal{S}(Fx, Fx, Gy), S(x, x, y), dist(x, Fx), dist(y, Gy), dist(x, Gy), dist(y, Fx)) \leq 0 \tag{3.1}$$

where $T \in \mathcal{T}$. Also assume the following conditions are satisfied:

$$dist(x_0, Fx_0) < \frac{r - \phi(r)}{2} \tag{3.2}$$

and

$$\sum_{i=1}^{\infty} \phi^i \left(\frac{r - \phi(r)}{2}\right) \leq \frac{\phi(r)}{2} \tag{3.3}$$

where ϕ is the function in T_2. Then there exists $x \in B_S[x_0, r]$ with $x \in Fx$ and $x \in Gx$.

Proof. From (3.2) we can choose $x_1 \in Fx_0$ with

$$S(x_0, x_0, x_1) < \frac{r - \phi(r)}{2} \tag{3.4}$$

Hence $S(x_1,x_1,x_0) < r$ so $x_1 \in B_S[x_0,r]$. Since ϕ is strictly increasing by (3.4) we can choose $\varepsilon > 0$ such that

$$\phi(S(x_0,x_0,x_1)) + \varepsilon < \phi\left(\frac{r-\phi(r)}{2}\right). \tag{3.5}$$

On the other hand, for this ε there is $x_2 \in Gx_1$ so that

$$S(x_1,x_1,x_2) \leq dist(x_1,Gx_1) + \varepsilon \leq \mathcal{S}(Fx_0,Fx_0,Gx_1) + \varepsilon. \tag{3.6}$$

Now since $x_0, x_1 \in B_S[x_0,r]$ we can use the inequality (3.1) to obtain

$$T(\mathcal{S}(Fx_0,Fx_0,Gx_1), S(x_0,x_0,x_1), dist(x_0,Fx_0), dist(x_1,Gx_1),$$
$$dist(x_0,Gx_1), dist(x_1,Fx_0)) \leq 0.$$

From T_1 we have

$$T(\mathcal{S}(Fx_0,Fx_0,Gx_1), S(x_0,x_0,x_1), S(x_0,x_0,x_1), S(x_1,x_1,x_2), S(x_0,x_0,x_2), 0) \leq 0,$$

that is

$$T(w,v,v,u,2u+v,0) \leq 0,$$

where $w = \mathcal{S}(Fx_0,Fx_0,Gx_1), v = S(x_0,x_0,x_1)$ and $u = S(x_1,x_1,x_2)$. Therefore, from T_2,

$$\mathcal{S}(Fx_0,Fx_0,Gx_1) \leq \phi(S(x_0,x_0,x_1))$$

and (3.6) yields

$$S(x_1,x_1,x_2) \leq \phi(S(x_0,x_0,x_1)) + \varepsilon.$$

Thus from (3.5) we have:

$$S(x_1,x_1,x_2) < \phi\left(\frac{r-\phi(r)}{2}\right). \tag{3.7}$$

Now by (3.3), (3.4), (3.7) and third condition of S-metric have:

$$\begin{aligned} S(x_2,x_2,x_0) = S(x_0,x_0,x_2) &\leq 2S(x_0,x_0,x_1) + S(x_1,x_1,x_2) \\ &< r - \phi(r) + \phi\left(\frac{r-\phi(r)}{2}\right) \\ &< r - \phi(r) + 2\sum_{i=1}^{\infty} \phi^i\left(\frac{r-\phi(r)}{2}\right) \leq r \end{aligned}$$

so $x_2 \in B_S[x_0,r]$. Again by (3.7) and strictly increasing ϕ there is $\delta > 0$ so that

$$\phi(S(x_1,x_1,x_2)) + \delta < \phi^2\left(\frac{r-\phi(r)}{2}\right), \tag{3.8}$$

also for this $\delta > 0$ there is $x_3 \in Fx_2$ so that

$$S(x_2,x_2,x_3) \leq dist(x_2,Fx_2) + \delta \leq \mathcal{S}(Gx_1,Gx_1,Fx_2) + \delta. \tag{3.9}$$

As above, since $x_1, x_2 \in B_S[x_0,r]$ we can use the inequality (3.1) to obtain

$$T(\mathcal{S}(Fx_2,Fx_2,Gx_1), S(x_2,x_2,x_1), dist(x_2,Fx_2),$$

$$dist(x_1, Gx_1), dist(x_2, Gx_1), dist(x_1, Fx_2)) \leq 0$$

and so from T_1 we have

$$T(\mathcal{S}(Fx_2, Fx_2, Gx_1), S(x_2, x_2, x_1), S(x_2, x_2, x_3), S(x_1, x_1, x_2), 0, S(x_1, x_1, x_3)) \leq 0$$

that is

$$T(w, v, u, v, 0, 2u + v) \leq 0,$$

where $w = \mathcal{S}(Fx_2, Fx_2, Gx_1), v = S(x_1, x_1, x_2)$ and $u = S(x_2, x_2, x_3)$. Therefore from T_2,

$$w \leq \phi(v)$$

that is

$$\mathcal{S}(Fx_2, Fx_2, Gx_1) \leq \phi(S(x_1, x_1, x_2))$$

and so (3.9) gives

$$S(x_2, x_2, x_3) \leq \phi(S(x_1, x_1, x_2)) + \delta.$$

Thus from (3.8) we have

$$S(x_2, x_2, x_3) < \phi^2 \left(\frac{r - \phi(r)}{2} \right). \tag{3.10}$$

Now (3.3), (3.4), (3.7), (3.10) and third condition of S-metric implies:

$$\begin{aligned} S(x_3, x_3, x_0) = S(x_0, x_0, x_3) &\leq 2S(x_0, x_0, x_1) + 2S(x_1, x_1, x_2) + S(x_2, x_2, x_3) \\ &< r - \phi(r) + 2\phi \left(\frac{r - \phi(r)}{2} \right) + \phi^2 \left(\frac{r - \phi(r)}{2} \right) \\ &\leq r - \phi(r) + 2 \sum_{i=1}^{\infty} \phi^i \left(\frac{r - \phi(r)}{2} \right) \leq r \end{aligned}$$

Thus $x_3 \in B_S[x_0, r]$.

Continuing this way we can obtain a sequence $\{x_n\} \subseteq B_S[x_0, r]$ such that $x_{2n+2} \in Gx_{2n+1}$ and $x_{2n+1} \in Fx_{2n}$ for $n \geq 0$ and

$$S(x_n, x_n, x_{n+1}) < \phi^n \left(\frac{r - \phi(r)}{2} \right).$$

Next we show that $\{x_n\}$ is a Cauchy sequence. Notice by (3.3) and above inequality for each $n, m \in \mathbb{N}$ with $m > n$ we have:

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2 \sum_{i=n}^{m-2} S(x_i, x_i, x_{i+1}) + S(x_{m-1}, x_{m-1}, x_m) \\ &\leq 2 \sum_{i=n}^{m-1} S(x_i, x_i, x_{i+1}) < 2 \sum_{i=n}^{m-1} \phi^i \left(\frac{r - \phi(r)}{2} \right) \\ &\leq 2 \sum_{i=n}^{\infty} \phi^i \left(\frac{r - \phi(r)}{2} \right) \end{aligned}$$

so (3.3) guarantees that $\{x_n\}$ is a Cauchy sequence. Thus there exists $x \in B_S[x_0, r]$ with $x_n \to x$. It remains to show $x \in Fx$ and $x \in Gx$. For n even (since $x_n, x \in B_S[x_0, r]$) we can use the inequality (3.1), we have

$$T(\mathcal{S}(Fx, Fx, Gx_{n-1}), S(x, x, x_{n-1}), dist(x, Fx), dist(x_{n-1}, Gx_{n-1}),$$

$$dist(x, Gx_{n-1}), dist(x_{n-1}, Fx)) \leq 0.$$

Now taking limit inferior as $n \to \infty$ (using T_0) we have (notice $dist(x, Gx_{n-1}) \leq S(x, x, x_n) \to 0$, and also $dist(x_{n-1}, Gx_{n-1}) \leq S(x_{n-1}, x_{n-1}, x_n) \to 0$)

$$T(\liminf_{n\to\infty} \mathcal{S}(Fx, Fx, Gx_{n-1}), 0, dist(x, Fx), 0, 0, dist(x, Fx)) \leq 0.$$

From T_3 we have

$$\liminf_{n\to\infty} \mathcal{S}(Fx, Fx, Gx_{n-1}) \leq \phi(dist(x, Fx)).$$

Now

$$dist(x, Fx) \leq 2S(x, x, x_n) + dist(x_n, Fx) \leq 2S(x, x, x_n) + \mathcal{S}(Gx_{n-1}, Gx_{n-1}, Fx)$$

and so

$$dist(x, Fx) \leq 0 + \liminf_{n\to\infty} \mathcal{S}(Fx, Fx, Gx_{n-1}) \leq \phi(dist(x, Fx)).$$

Thus $dist(x, Fx) = 0$ since $\phi(t) < t$ for $t > 0$, so $x \in \overline{Fx} = Fx$.

For n odd ,

$$dist(x, Gx) \leq S(x, x, x_n) + dist(x_n, Gx) \leq S(x, x, x_n) + \mathcal{S}(Fx_{n-1}, Fx_{n-1}, Gx),$$

and as above we obtain $dist(x, Gx) = 0$, so $x \in Gx$.

Now we give some corollaries.

Corollary 3.2. *Let (X, S) be a complete S-metric space, $x_0 \in X, r > 0$ with $F, G : B_S[x_0, r] \to C(X)$. Suppose, for all $x, y \in B_S[x_0, r]$ sets Fx, Gy are bounded and*

$$\mathcal{S}(Fx, Fx, Gy) \leq k \max\{S(x, x, y), dist(x, Fx), dist(y, Gy), \frac{dist(x, Gy)}{3}, \frac{dist(y, Fx)}{3}\}$$

where $0 < k < 1$. Also assume the following condition is satisfied:

$$dist(x_0, Fx_0) < \frac{1-k}{2} r.$$

Then there exists $x \in B_S[x_0, r]$ with $x \in Fx$ and $x \in Gx$.

Proof. By Theorem 3.1 , it is enough to set $T(t_1, t_2, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, \frac{t_5}{3}, \frac{t_6}{3}\}$. In this case, $\phi(t) = kt$ and

$$\sum_{i=1}^{\infty} \phi^i \left(\frac{r - \phi(r)}{2}\right) = \frac{kr}{2} = \frac{\phi(r)}{2}.$$

Corollary 3.3. *Let (X, S) be a complete S-metric space, $x_0 \in X, r > 0$ with $F, G : B_S[x_0, r] \to X$. Suppose for all $x, y \in B_S[x_0, r]$,*

$$S(Fx, Fx, Gy) \leq k \max\{S(x, x, y), S(x, x, Fx), S(y, y, Gy), \frac{S(x, x, Gy)}{3}, \frac{S(y, y, Fx)}{3}\}$$

where $0 < k < 1$. Also assume the following condition is satisfied:

$$S(x_0, x_0, Fx_0) < \frac{1-k}{2} r.$$

Then there exists a unique $x \in B_S[x_0, r]$ with $Fx = Gx = x$.

Proof. By Corollary 3.2 , there exists an $x \in X$ such that $Fx = Gx = x$. It is enough prove that x is unique.

Let y be another common fixed point of F and G, that is $y = Fy = Gy$, then we have

$$\begin{aligned} S(x,x,y) = S(Fx,Fx,Gy) &\leq k\max\{S(x,x,y), S(x,x,x), S(y,y,y)\} \\ &= kS(x,x,y), \end{aligned}$$

which is a contradiction. Therefore F and G have a unique common fixed point in $B_S[x_0, r]$.

Corollary 3.4. *Let (X, S) be a complete S-metric space, $x_0 \in X, r > 0$ with $F : B_S[x_0, r] \to X$. Suppose for all $x, y \in B_S[x_0, r]$,*

$$S(Fx,Fx,Fy) \leq k\max\{S(x,x,y), S(x,x,Fx), S(y,y,Fy), \frac{S(x,x,Fy)}{3}, \frac{S(y,y,Fx)}{3}\}$$

where $0 < k < 1$. Also assume the following condition is satisfied:

$$S(x_0, x_0, Fx_0) < \frac{1-k}{2}r.$$

Then there exists a unique $x \in B_S[x_0, r]$ with $Fx = x$.

Now we give an example.

Example 3.5. Let $X = \mathbb{R}$ and $S(x, y, z) = |x - z| + |y - z|$ .Then (X, S) is a complete S-metric space. Let $x_0 = 1$ and $r = 6$, then

$$\begin{aligned} B_S[x_0, r] &= B_S[1, 6] \\ &= \{y \in X : S(y, y, x) \leq 6\} \\ &= [-2, 4]. \end{aligned}$$

Now let $F : B_S[x_0, r] \to X$, $Fx = \frac{x}{2}$ and let $k = \frac{1}{2}$, then

$$S(x_0, x_0, Fx_0) = S(1, 1, \frac{1}{2}) = 1 < \frac{3}{2} = \frac{1-k}{2}r.$$

Also, for all $x, y \in B_S[x_0, r]$, we have

$$\begin{aligned} S(Fx,Fx,Fy) &= 2\,|Fx - Fy| \\ &= |x - y| \\ &= \frac{1}{2}(2\,|x-y|) \\ &= \frac{1}{2}S(x,x,y) \\ &\leq \frac{1}{2}\max\{S(x,x,y), S(x,x,Fx), S(y,y,Fy), \frac{S(x,x,Fy)}{3}, \frac{S(y,y,Fx)}{3}\}. \end{aligned}$$

Therefore all conditions of Corollary 3.4 are satisfied, thus F has a unique fixed point in $B_S[x_0, r] = [-2, 4]$.

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