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Approximation of Common Fixed Points of Mean Non-expansive Mapping in Banach Spaces

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Abstract. Let X be a uniformly convex Banach space, and S, T be pair of mean non-expansive mappings. Some necessary and sufficient conditions are given for Ishikawa iterative sequence converge to common fixed points, and we prove that the sequence of Ishikawa iterations associated with S and T converges to the common fixed point of S and T. This generalizes former results proved by Z. Gu and Y. Li [4].

1. Introduction

Let X be a Banach space and S, T be mappings from X to X. In [8] the Ishikawa iteration sequence $\{x_n\}$ of T was defined by

$$y_n = (1-\beta_n)x_n + \beta_n T x_n \tag{1.1}$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \tag{1.2}$$

Where $x_0 \in X, \alpha_n, \beta_n \in [0,1]$.

The pair of mean non-expansive mappings was introduced by Bose in [2].

$$\|Sx - Ty\| \le a\|x-y\| + b\{\|x-Sx\| + \|y-Ty\|\} + c\{\|x-Ty\| + \|y-Sx\|\} \tag{1.3}$$

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for all $x, y \in X, a, b, c \in [0, 1], a + 2b + 2c \leq 1$.

The Ishikawa iteration sequence $\{x_n\}$ of S and T was defined by

$$y_n = (1 - \beta_n)x_n + \beta_n Sx_n \tag{1.4}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n \tag{1.5}$$

Where $x_0 \in X, \alpha_n, \beta_n \in [0, 1]$.

The problems about common fixed point for pair of mappings as an important part of the fixed point theory have been studied by many authors, see [2-4,6-14], for more details. In [2], S. C. Bose defined the pair of mean non-expansive mappings, and proved the existence of the fixed point in Banach spaces. In particular, he proved the following theorem.

Theorem 1.1.[2] *Let X be a uniformly convex Banach space and K a non-empty closed convex subset of X, $S : K \to K$ and $T : K \to K$ are pair of mean non-expansive mappings, and $c \neq 0$, then*
(i) S and T have a common fixed point u.
(ii) Further if $b \neq 0$, then

(a) u is the unique common fixed point and unique as a fixed point of each S and T.

(b) the sequence $\{x_n\}$ defined by $x_1 = Sx_0, x_2 = Tx_1, x_3 = Sx_2 \cdots$ for any $x_0 \in K$, converges strongly to u.

In [14], Z. Gu and Y. Li proved the following theorem.

Theorem 1.2. *Let X be a uniformly convex Banach space, $S : X \to X$ and $T : X \to X$ are pair of mean non-expansive with a nonempty common fixed points set, if $b > 0, 0 < \alpha \leq \alpha_n \leq \frac{1}{2}, 0 \leq \beta_n \leq \beta <1$, then the Ishikawa sequence $\{x_n\}$ converges to the common fixed point of S and T.*

It is our purpose in this paper to give some necessary and sufficient conditions for Ishikawa iterative sequence converge to common fixed points, and consider the iterative scheme, which converges to a common fixed point of the pair of mean non-expansive mappings. Our Theorem 2.1 extends and improves the corresponding results in [14].

To obtain the main results of the paper, we prove the following lemmas.

Lemma 1.3.[13] *Let X be a Banach space. Then X is uniformly convex if and only if for any given number $\rho > 0$, the square norm $\| \cdot \|^2$ of X in uniformly convex on B_ρ, the closed unit ball at the origin with radius ρ; namely, there exist a continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\varphi(\|x-y\|),$$
for all $x, y \in B_\rho, \alpha \in [0,1]$.

Lemma 1.4.[14] *Let* X *be a Banach space,* $S : X \to X$ *and* $T : X \to X$ *are pair of mean non-expansive mappings with a common fixed point, then for any* $x_0 \in X$*, the Ishikawa sequence* $\{x_n\}$ *associated with* S *and* T *is bounded.*

Proof. For a common fixed point p of S and T, we have

$$\begin{aligned} & \|Tx-p\| = \|Tx-Sp\| \\ \leq\ & a\|x-p\| + b\{\|x-Tx\| + \|p-Sp\|\} \\ & +c\{\|x-Sp\| + \|p-Tx\|\} \\ \leq\ & a\|x-p\| + b\{\|x-p\| + \|p-Tx\|\} \\ & +c\{\|x-Sp\| + \|p-Tx\|\} \end{aligned}$$

Let $L = \frac{a+b+c}{1-b-c}$, by $a+2b+2c \leq 1$, it is easy to see that $a+b+c \leq 1-b-c$, thus $0 \leq L \leq 1$, and $\|Tx-p\| \leq L\|x-p\| \leq \|x-p\|$.

Similarly, we have $\|Sx-p\| \leq L\|x-p\| \leq \|x-p\|$.

$$\begin{aligned} \|x_{n+1}-p\| &= \|(1-\alpha_n)x_n + \alpha_n Ty_n - p\| \\ &= \|(1-\alpha_n)(x_n-p) + \alpha_n(Ty_n-p)\| \\ &\leq (1-\alpha_n)\|x_n-p\| + \alpha_n\|Ty_n-p\| \\ &\leq (1-\alpha_n)\|x_n-p\| + \alpha_n L\|y_n-p\| \\ &= (1-\alpha_n)\|x_n-p\| + \alpha_n\|(1-\beta_n)x_n + \beta_n Sx_n - p\| \\ &= (1-\alpha_n)\|x_n-p\| + \alpha_n\|(1-\beta_n)(x_n-p) + \beta_n(Sx_n-p)\| \\ &\leq (1-\alpha_n)\|x_n-p\| + \alpha_n(1-\beta_n)\|x_n-p\| \\ &\quad +\alpha_n\beta_n\|Sx_n-p\| \\ &\leq (1-\alpha_n)\|x_n-p\| + \alpha_n(1-\beta_n)\|x_n-p\| \\ &\quad +\alpha_n\beta_n\|x_n-p\| \\ &= ((1-\alpha_n) + \alpha_n(1-\beta_n) + \alpha_n\beta_n)\|x_n-p\| \\ &= \|x_n-p\| \end{aligned}$$

So

$$\|x_{n+1}-p\| \leq \|x_n-p\| \leq \|x_{n-1}-p\| \leq \dots \leq \|x_0-p\| \tag{1.6}$$

Thus, $\{x_n\}$ is bounded. □

2. Section 2

First, we give some necessary and sufficient conditions for Ishikawa iterative sequence converge to common fixed points.

Theorem 2.1. *Let X be a Banach space, $S : X \to X$ and $T : X \to X$ are pair of mean non-expansive with a nonempty common fixed points set, if $b > 0, c > 0$, then the necessary and sufficient conditions that the Ishikawa sequence $\{x_n\}$ converges to the common fixed point of S and T is:*

$$\lim_{n\to\infty} inf\|x_n - Ty_n\| = 0. \tag{2.1}$$

Proof. For the first step, we will prove the sufficiency.

Let us first prove $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. Since

$$\begin{aligned}
\|x_n - Sx_n\| &\le \|x_n - Ty_n\| + \|Ty_n - Sx_n\| \\
&\le \|x_n - Ty_n\| + a\|x_n - y_n\| + b\{\|x_n - Sx_n\| + \|y_n - Ty_n\|\} \\
&\quad +c\{\|x_n - Ty_n\| + \|y_n - Sx_n\|\} \\
&= (1+c)\|x_n - Ty_n\| + a\|x_n - y_n\| + b\|x_n - Sx_n\| \\
&\quad +b\|y_n - Ty_n\| + c\|y_n - Sx_n\| \\
&= (1+c)\|x_n - Ty_n\| + a\|(1-\beta_n)x_n + \beta_n Sx_n - x_n\| \\
&\quad +b\|x_n - Sx_n\| + b\|(1-\beta_n)x_n + \beta_n Sx_n - Ty_n\| \\
&\quad +c\|(1-\beta_n)x_n + \beta_n Sx_n - Sx_n\| \\
&\le (1+c)\|x_n - Ty_n\| + a\beta_n\|x_n - Sx_n\| \\
&\quad +b\|x_n - Sx_n\| + b\beta_n\|x_n - Sx_n\| + b\|x_n - Ty_n\| \\
&\quad +c(1-\beta_n)\|x_n - Sx_n\| \\
&= (1+b+c)\|x_n - Ty_n\| \\
&\quad +(a\beta_n + b + b\beta_n + c(1-\beta_n))\|x_n - Sx_n\|
\end{aligned}$$

we have

$$(1 - a\beta_n - b - b\beta_n - c(1-\beta_n))\|x_n - Sx_n\| \le (1+b+c)\|x_n - Ty_n\| \tag{2.2}$$

Let $M_1 = 1 - a\beta_n - b - b\beta_n - c(1-\beta_n)$, then

$$\begin{aligned}
M_1 &= 1 - a\beta_n - b - b\beta_n - c + c\beta_n \\
&= 1 - b - c - (a+b-c)\beta_n \\
&\ge a + b + c - (a+b-c)\beta_n \\
&> 0
\end{aligned}$$

so

$$\|x_n - Sx_n\| \le \frac{1+b+c}{M_1}\|x_n - Ty_n\| \tag{2.3}$$

if $\lim_{n\to\infty} inf\|x_n - Ty_n\| = 0$, there exists subsequence $\|x_{n_k} - Ty_{n_k}\| \to 0$ when $n_k \to \infty$, by (2.3) we have $\lim_{n_k\to\infty} \|x_{n_k} - Sx_{n_k}\| = 0$.

Next we will show that sequence $\{Sx_{n_k}\}$ is a Cauchy sequence. For any n_{k_1}, n_{k_2}, we have

$$\begin{aligned}
& \|Sx_{n_{k_1}} - Sx_{n_{k_2}}\| \\
\leq\ & \|Sx_{n_{k_1}} - Ty_{n_{k_2}}\| + \|Sx_{n_{k_2}} - Ty_{n_{k_2}}\| \\
\leq\ & a\|x_{n_{k_1}} - y_{n_{k_2}}\| + b\{\|x_{n_{k_1}} - Sx_{n_{k_1}}\| + \|y_{n_{k_2}} - Ty_{n_{k_2}}\|\} \\
& +c\{\|x_{n_{k_1}} - Ty_{n_{k_2}}\| + \|y_{n_{k_2}} - Sx_{n_{k_1}}\|\} \\
& +\|Sx_{n_{k_2}} - Ty_{n_{k_2}}\| \\
\leq\ & a\{\|x_{n_{k_1}} - Sx_{n_{k_1}}\| + \|Sx_{n_{k_1}} - Sx_{n_{k_2}}\| + \|Sx_{n_{k_2}} - y_{n_{k_2}}\|\} \\
& +b\{\|x_{n_{k_1}} - Sx_{n_{k_1}}\| + \|y_{n_{k_2}} - Ty_{n_{k_2}}\|\} \\
& +c\{\|x_{n_{k_1}} - Sx_{n_{k_1}}\| + \|Sx_{n_{k_1}} - Sx_{n_{k_2}}\| \\
& +\|Sx_{n_{k_2}} - Ty_{n_{k_2}}\| + \|y_{n_{k_2}} - Sx_{n_{k_2}}\| \\
& +\|Sx_{n_{k_2}} - Sx_{n_{k_1}}\|\} + \|Sx_{n_{k_2}} - Ty_{n_{k_2}}\|
\end{aligned}$$

Since $b > 0$, thus obviously we have $1 - a - 2c > 0$. Simplify, then we get

$$\begin{aligned}
\|Sx_{n_{k_1}} - Sx_{n_{k_2}}\| \leq\ & R_1\|x_{n_{k_1}} - Sx_{n_{k_1}}\| + R_2\|y_{n_{k_2}} - Ty_{n_{k_2}}\| \\
& +R_3\|y_{n_{k_2}} - Sx_{n_{k_2}}\| + R_4\|Sx_{n_{k_2}} - Ty_{n_{k_2}}\|
\end{aligned}$$

Where $R_1 = \frac{a+b+c}{1-a-2c} \geq 0$, $R_2 = \frac{b}{1-a-2c} \geq 0$, $R_3 = \frac{a+c}{1-a-2c} \geq 0$, and $R_4 = \frac{1+c}{1-a-2c} \geq 0$

According to Ishikawa iteration

$$\begin{aligned}
y_n &= (1 - \beta_n)x_n + \beta_n Sx_n \\
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n
\end{aligned}$$

It is easy to see that $\|x_{n_k} - y_{n_k}\| = \beta_{n_k}\|x_{n_k} - Sx_{n_k}\|$ and $\|Sx_{n_k} - y_{n_k}\| = (1 - \beta_{n_k})\|x_{n_k} - Sx_{n_k}\|$. Then we consider the sequence $\|Ty_{n_k} - Sx_{n_k}\|$,

$$\begin{aligned}
\|y_{n_k} - Ty_{n_k}\| &= \|(1 - \beta_{n_k})x_{n_k} + \beta_n Sx_{n_k} - Ty_{n_k}\| \\
&\leq (1 - \beta_{n_k})\|x_{n_k} - Ty_{n_k}\| + \beta_{n_k}\|Sx_{n_k} - Ty_{n_k}\|
\end{aligned}$$

From the definition of the mean nonexpansive mappings, we obtain

$$\begin{aligned}\|Ty_{n_k} - Sx_{n_k}\| &\leq a\|x_{n_k} - y_{n_k}\| + b\{\|x_{n_k} - Sx_{n_k}\| + \|y_{n_k} - Ty_{n_k}\|\} \\ &\quad + c\{\|x_{n_k} - Ty_{n_k}\| + \|y_{n_k} - Sx_{n_k}\|\} \\ &\leq a\beta_{n_k}\|x_{n_k} - Sx_{n_k}\| + b\|x_{n_k} - Sx_{n_k}\| \\ &\quad + b(1-\beta_{n_k})\|x_{n_k} - Ty_{n_k}\| + b\beta_{n_k}\|Sx_{n_k} - Ty_{n_k}\| \\ &\quad + c\|x_{n_k} - Ty_{n_k}\| + c(1-\beta_{n_k})\|x_{n_k} - Sx_{n_k}\| \\ &= (a\beta_{n_k} + b + c(1-\beta_{n_k}))\|x_{n_k} - Sx_{n_k}\| \\ &\quad + (b(1-\beta_{n_k}) + c)\|x_{n_k} - Ty_{n_k}\| + b\beta_{n_k}\|Sx_{n_k} - Ty_{n_k}\|\end{aligned}$$

Since

$$\|x_{n_k} - Sx_{n_k}\| \leq \|Sx_{n_k} - Ty_{n_k}\| + \|x_{n_k} - Ty_{n_k}\|$$

we can get the following inequality

$$\begin{aligned}\|Ty_{n_k} - Sx_{n_k}\| &\leq (b(1-\beta_{n_k}) + c + a\beta_{n_k} + b + c(1-\beta_{n_k}))\|x_{n_k} - Ty_{n_k}\| \\ &\quad + (b\beta_{n_k} + a\beta_{n_k} + b + c(1-\beta_{n_k}))\|Sx_{n_k} - Ty_{n_k}\|\end{aligned}$$

Therefor

$$\begin{aligned}&(1 - b - c - (a+b-c)\beta_{n_k})\|Ty_{n_k} - Sx_{n_k}\| \\ &\leq (b(1-\beta_{n_k}) + c + a\beta_{n_k} + b + c(1-\beta_{n_k}))\|x_{n_k} - Ty_{n_k}\|\end{aligned}$$

Since $0\leq \beta_{n_k} \leq 1$, so we can get $1-b-c-(a+b-c)\beta_{n_k} \geq (a+b)(1-\beta_{n_k})+c(1+\beta_{n_k}) > 0$, and it is easy to see that

$$b(1-\beta_{n_k}) + c + a\beta_{n_k} + b + c(1-\beta_{n_k}) > 0$$

According to the condition $\|x_{n_k} - Ty_{n_k}\| \to 0$, thus we get

$$\lim_{n_k\to\infty} \|Sx_{n_k} - Ty_{n_k}\| = 0 \quad \text{and} \quad \lim_{n_k\to\infty} \|y_{n_k} - Ty_{n_k}\| = 0 \tag{2.4}$$

So $\lim_{n_k\to\infty} \|x_{n_k} - y_{n_k}\| = \lim_{n_k\to\infty} \beta_{n_k}\|Sx_{n_k} - x_{n_k}\| = 0$.
We know

$$\|x_{n_k} - Sx_{n_k}\| \to 0, \ \|y_{n_k} - Ty_{n_k}\| \to 0, \ \|Sx_{n_k} - Ty_{n_k}\| \to 0 \tag{2.5}$$

It is easy to see that $\|y_{n_k} - Sx_{n_k}\| \to 0$, in particular, $\|x_{n_{k_1}} - Sx_{n_{k_1}}\| \to 0$, $\|y_{n_{k_2}} - Ty_{n_{k_2}}\| \to 0$, $\|y_{n_{k_2}} - Sx_{n_{k_2}}\| \to 0$, $\|Sx_{n_{k_2}} - Ty_{n_{k_2}}\| \to 0$ thus $\|Sx_{n_{k_1}} - Sx_{n_{k_2}}\| \to 0$, it is also say that $\{Sx_{n_k}\}$ is a Cauchy sequence. We may assume that $p = \lim_{n_k\to\infty} Sx_{n_k}$, so we have $\lim_{n_k\to\infty} x_{n_k} = p$ using (1.4),we can get $\lim_{n\to\infty} x_n = p$ By (1.1), we have

$$\begin{aligned}\|Sx_{n_k} - Tp\| &\leq a\|x_{n_k} - p\| + b\{\|x_{n_k} - Sx_{n_k}\| + \|p - Tp\|\} \\ &\quad + c\{\|x_{n_k} - Tp\| + \|p - Sx_{n_k}\|\}\end{aligned}$$

Let $n_k \to \infty$, we get

$$\|p - Tp\| \leq (b + c)\|p - Tp\|$$

Since $b + c < 1$, that means $\|p - Tp\| = 0$, that is $Tp = p$, p is a fixed point of T. Similarly, we can prove that $Sp = p$. thus, $\{x_n\}$ converges to the common fixed point of S and T.

For the second step, we will prove the necessity, if $\{x_n\}$ converges to the common fixed point of S and T, we assume that $\lim_{n\to\infty} x_n = p$,

$$\begin{aligned}
\|x_{n_k} - Ty_{n_k}\| &\leq \|x_{n_k} - p\| + \|Ty_{n_k} - p\| \\
&\leq \|x_{n_k} - p\| + \|y_{n_k} - p\| \\
&\leq \|x_{n_k} - p\| + \|(1 - \beta_{n_k})x_{n_k} + \beta_{n_k} Sx_{n_k} - p\| \\
&\leq \|x_{n_k} - p\| + (1 - \beta_{n_k})\|x_{n_k} - p\| + \beta_{n_k}\|Sx_{n_k} - p\| \\
&\leq (1 + 1 - \beta_{n_k} + \beta_{n_k})\|x_{n_k} - p\| \\
&= 2\|x_{n_k} - p\|
\end{aligned}$$

Since $\lim_{n\to\infty} x_n = p$, thus $\lim_{n\to\infty} \|x_{n_k} - Ty_{n_k}\| = 0$, that is $\lim_{n\to\infty} inf\|x_n - Ty_n\| = 0$. □

Corollary 2.2. *Let X be a Banach space, $S : X \to X$ and $T : X \to X$ are pair of mean non-expansive with a nonempty common fixed points set, if $b > 0, c > 0$, then the necessary and sufficient conditions that the Ishikawa sequence $\{x_n\}$ converges to the common fixed point of S and T is:*

$$\lim_{n\to\infty} \|x_n - Ty_n\| = 0.$$

Proof. If $\lim_{n\to\infty} \|x_n - Ty_n\| = 0$, then $\lim_{n\to\infty} inf\|x_n - Ty_n\| = 0$. By Theorem 2.1, we get $\lim_{n\to\infty} x_n = p$.

For the converse, we assume that $\lim_{n\to\infty} x_n = p$, then

$$\begin{aligned}
\|x_n - Ty_n\| &\leq \|x_n - p\| + \|Ty_n - p\| \\
&\leq \|x_n - p\| + \|y_n - p\| \\
&\leq \|x_n - p\| + \|(1 - \beta_n)x_n + \beta_n Sx_n - p\| \\
&\leq \|x_n - p\| + (1 - \beta_n)\|x_n - p\| + \beta_n\|Sx_n - p\| \\
&\leq (1 + 1 - \beta_n + \beta_n)\|x_n - p\| \\
&= 2\|x_n - p\|
\end{aligned}$$

Since $\lim_{n\to\infty} x_n = p$, thus $\lim_{n\to\infty} \|x_n - Ty_n\| = 0$. □

The following theorem is a strong convergence theorem for Ishikawa iteration in uniformly convex Banach spaces.

Theorem 2.3. *Let X be a uniformly convex Banach space, $S : X \to X$ and $T : X \to X$ are pair of mean non-expansive with a nonempty common fixed points set, if $b > 0, c > 0$,$0 < \alpha \leq \alpha_n \leq \alpha' < 1$, then the Ishikawa sequence $\{x_n\}$ converges to the common fixed point of S and T.*

Proof. Let p is a common fixed point of S and T, by Lemma 1.3, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \alpha_n)x_n + \alpha_n Ty_n - p\|^2 \\ &\leq \alpha_n \|Ty_n - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 \\ &\quad -\alpha_n(1 - \alpha_n)\varphi(\|Ty_n - x_n\|) \end{aligned}$$

However

$$\begin{aligned} \|y_n - p\| &= \|(1 - \beta_n)x_n + \beta_n Sx_n - p\| \\ &\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|Sx_n - p\| \\ &\leq (1 - \beta_n + \beta_n)\|x_n - p\| \\ &= \|x_n - p\| \end{aligned}$$

Therefore

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \alpha_n(1 - \alpha_n)\varphi(\|Ty_n - x_n\|)$$

We can get

$$\alpha_n(1 - \alpha_n)\varphi(\|Ty_n - x_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2$$

Since $0 < \alpha \leq \alpha_n \leq \alpha' < 1$, thus $\alpha_n(1 - \alpha_n) > \alpha(1 - \alpha') > 0$. By Lemma 1.4, $\|x_n - p\|^2$ is a real decreasing bounded sequence, so $\|x_n - p\|^2$ converges.
Hence for any $\varepsilon > 0$, there exists $n_0 \in N$, such that whenever $n > n_0$, we have

$$\alpha_n(1 - \alpha_n)\varphi(\|Ty_n - x_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 < \varepsilon$$

thus $\varphi(\|Ty_n - x_n\|) \to 0$, as $n \to \infty$, and hence $\|Ty_n - x_n\| \to 0$, by the continuity and strictly increasing nature of φ. By Theorem 2.1, we get that $\{x_n\}$ converges to the common fixed point of S and T, so that the conclusion of the theorem follows. □

Between the pair of mappings, if $S = T$, then
$\|Tx - Ty\| \leq a\|x - y\| + b\{\|x - Tx\| + \|y - Ty\|\} + c\{\|x - Ty\| + \|y - Tx\|\}$
the mapping T is called mean nonexpansive mapping, we obtain the following.

Corollary 2.4. *Let X be a Banach space, $T : X \to X$ is a mean non-expansive with a nonempty fixed points set, if $b > 0, c > 0, 0 < \alpha \leq \alpha_n \leq \alpha' < 1$, then the Ishikawa sequence $\{x_n\}$ converges to the fixed point of T.*

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