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Ultra g-Bessel Sequences in Hilbert Spaces

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Abstract. In this paper, we introduce ultra g-Bessel sequences and study some properties of this kind of sequences in Hilbert spaces. We also show that every g-frame for a finite dimensional Hilbert space is an ultra g-Bessel sequence.

1. Introduction

The concept of g-frame was introduced by W. Sun [8]. Afterward, several generalizations of g-frames in Hilbert spaces, Banach spaces and Hilbert C^*-modules have been proposed [1, 2, 6]. Some related topics in g-frames have been investigated in [3, 4].

Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces and $\{\mathcal{H}_i\}_{i=1}^{\infty}$ is a sequence of separable Hilbert spaces.

Definition 1.1. We call a sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^{\infty}$, if there exist two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} \|\Lambda_i f\|^2 \leq B\|f\|^2 \tag{1.1}$$

for all $f \in \mathcal{H}$. We call A and B the lower and upper g-frame bounds, respectively. We call $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ a tight g-frame if $A = B$ and Parseval g-frame if $A = B = 1$. A sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is called a g-Bessel sequence if the right hand inequality in (1.1) holds for all $f \in \mathcal{H}$.

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Let $\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)$ for all $i \in \mathbb{N}$. Let us define the set

$$\left(\sum_{i=1}^{\infty} \oplus \mathcal{H}_i\right)_{l_2} = \left\{\{g_i\}_{i=1}^{\infty} : g_i \in \mathcal{H}_i, \sum_{i=1}^{\infty} \|g_i\|^2 < \infty\right\}$$

with this inner product given by $\langle \{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty} \rangle = \sum_{i=1}^{\infty} \langle f_i, g_i \rangle$. It is clear that $(\sum_{i=1}^{\infty} \oplus \mathcal{H}_i)_{l_2}$ is a Hilbert space with respect to the poitwise operations. It is proved in [7], if $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is a g-Bessel sequence for $\mathcal{H}$, then the operator

$$T : \left(\sum_{i=1}^{\infty} \oplus \mathcal{H}_i\right)_{l_2} \to \mathcal{H}, \quad T(\{g_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \Lambda_i^* g_i \tag{1.2}$$

is well defined and bounded and its adjoint is $T^* f = \{\Lambda_i f\}_{i=1}^{\infty}$. Moreover, a sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is a g-frame if and only if the operator T is bounded and onto. The operators T and T^* are called the synthesis and analysis operators of $\{\Lambda_i\}_{i=1}^{\infty}$, respectively. Also in [8] it is proved that if $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is a g-frame for $\mathcal{H}$, then the operator

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{i=1}^{\infty} \Lambda_i^* \Lambda_i f$$

is positive, bounded and invertible. Every $f \in \mathcal{H}$ has an expansion

$$f = \sum_{i=1}^{\infty} S^{-1} \Lambda_i^* \Lambda_i f = \sum_{i=1}^{\infty} \Lambda_i^* \Lambda_i S^{-1} f.$$

The operator S is called the g-frame operator of $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$.

Definition 1.2.[8] A sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is called a g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^{\infty}$, if $\sum_{i=1}^{\infty} \|\Lambda_i f\|^2 = \|f\|^2$ for all $f \in \mathcal{H}$ and

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle = \delta_{ij} \langle g_i, g_j \rangle, \quad g_i \in \mathcal{H}_i, \, g_j \in \mathcal{H}_j, \quad i, j \in \mathbb{N}.$$

In [5], the authors defined an ultra Bessel sequence for a Hilbert space and proved that every frame in a finite dimension Hilbert space is an ultra Bessel sequence. A sequence $\{f_i\}_{i=1}^{\infty}$ in Hilbert space $\mathcal{H}$ is called an ultra Bessel sequence, if

$$\sup_{\|f\|=1} \sum_{i=n}^{\infty} |\langle f, f_i \rangle|^2 \to 0, \quad \text{as} \quad n \to \infty.$$

2. Ultra g-Bessel sequences

In this section, $\|T\|_2$ denotes the Hilbert-Schmidt norm of an operator T.

Definition 2.1. A sequence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is called an ultra g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^{\infty}$, if $\sum_{i=1}^{\infty} \|\Lambda_i f\|^2$ converges uniformly on $\{f \in \mathcal{H} : \|f\| = 1\}$, i.e.,

$$\sup_{\|f\|=1} \sum_{i=n}^{\infty} \|\Lambda_i f\|^2 \to 0, \quad \text{as} \quad n \to \infty. \tag{2.1}$$

It is clear that every ultra g-Bessel sequence is a g-Bessel for $\mathcal{H}$.

Example 2.2. Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ be a g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^{\infty}$. Then

(1) $\{\frac{1}{i}\Lambda_i\}_{i=1}^{\infty}$ is an ultra g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^{\infty}$.

(2) If $g \in \mathcal{H}_n$ such that $\|g\| = 1$, then

$$\sum_{i=n}^{\infty} \|\Lambda_i \Lambda_n^* (\frac{g}{\|\Lambda_n^* g\|})\|^2 = \|\frac{g}{\|\Lambda_n^* g\|}\|^2 = 1.$$

Hence $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is not an ultra g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^{\infty}$.

Proposition 2.3. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ be a sequence of bounded operators such that $\sum_{i=1}^{\infty} \|\Lambda_i\|^2 < \infty$. Then $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is an ultra g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^{\infty}$.*

Proof. For each $n \in \mathbb{N}$, we have

$$\sup_{\|f\|=1} \sum_{i=n}^{\infty} \|\Lambda_i f\|^2 \leq \sum_{i=n}^{\infty} \sup_{\|f\|=1} \|\Lambda_i f\|^2 = \sum_{i=n}^{\infty} \|\Lambda_i\|^2.$$

Example 2.4. Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ be a g-Bessel sequence for $\mathcal{H}$ and $\Lambda_i \neq 0$ for all $i \in \mathbb{N}$, then $\{\frac{1}{i\|\Lambda_i\|}\Lambda_i\}_{i=1}^{\infty}$ is an ultra g-Bessel sequence.

Remark 2.5. Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ be a g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^{\infty}$. Then $\{\frac{1}{\sqrt{i}}\Lambda_i\}_{i=1}^{\infty}$ is an ultra g-Bessel sequence for $\mathcal{H}$, but $\sum_{i=1}^{\infty} \|\frac{1}{\sqrt{i}}\Lambda_i\|^2 = \infty$. This shows that the converse of the Proposition 2.3 is not true.

Proposition 2.6. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ be an ultra g-Bessel sequence for $\mathcal{H}$*

with respect to $\{\mathcal{H}_i\}_{i=1}^{\infty}$ *and* $T : \mathcal{H} \to \mathcal{K}$ *be a non zero bounded operator. Then* $\{\Lambda_i T^*\}_{i=1}^{\infty}$ *is an ultra g-Bessel sequence for* $\mathcal{K}$ *with respect to* $\{\mathcal{H}_i\}_{i=1}^{\infty}$*.*

Proof. Suppose that $T \neq 0$ and $g \in \mathcal{K}$ with $\|g\| = 1$. Then

$$\frac{1}{\|T^*\|^2}\sum_{i=n}^{\infty}\|\Lambda_i T^* g\|^2 = \sum_{i=n}^{\infty}\left\|\Lambda_i\Big(\frac{T^*g}{\|T^*\|}\Big)\right\|^2 \leq \sup_{\|f\|=1}\sum_{i=n}^{\infty}\|\Lambda_i f\|^2.$$

So

$$0 \leq \sup_{\|g\|=1}\sum_{i=n}^{\infty}\|\Lambda_i T^* g\|^2 \leq \|T^*\|^2 \sup_{\|f\|=1}\sum_{i=n}^{\infty}\|\Lambda_i f\|^2.$$

Therefore

$$\sup_{\|g\|=1}\sum_{i=n}^{\infty}\|\Lambda_i T^* g\|^2 \to 0$$

as $n \to \infty$.

Corollary 2.7. *If* $\{\Lambda_i \in B(\mathcal{H},\mathcal{H}_i)\}_{i=1}^{\infty}$ *is an ultra g-Bessel sequence for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i=1}^{\infty}$ *and* T *is the synthesis operator of* $\{\Lambda_i \in B(\mathcal{H},\mathcal{H}_i)\}_{i=1}^{\infty}$*. Then* $\{\Lambda_i T\}_{i=1}^{\infty}$ *is an ultra g-Bessel sequence for* $(\sum_{i=1}^{\infty}\oplus\mathcal{H}_i)_{l_2}$ *with respect to* $\{\mathcal{H}_i\}_{i=1}^{\infty}$*.*

Proposition 2.8. *Let* $\{\Lambda_i \in B(\mathcal{H},\mathcal{H}_i)\}_{i=1}^{\infty}$ *be a sequence of bounded operators and* $\sum_{i=1}^{\infty}\Lambda_i^* g_i$ *converges uniformly on* $(\sum_{i=1}^{\infty}\oplus\mathcal{H}_i)_{l_2}$*. Then* $\{\Lambda_i\}_{i=1}^{\infty}$ *is an ultra g-Bessel sequence for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i=1}^{\infty}$*.*

Proof. Consider the sequence of bounded linear operators

$$T_n : \left(\sum_{i=1}^{\infty}\oplus\mathcal{H}_i\right)_{l_2} \to \mathcal{H}, \qquad T_n\Big(\{g_i\}_{i=1}^{\infty}\Big) = \sum_{i=1}^{n}\Lambda_i^* g_i.$$

Then $T_n \to T$ uniformly (and so pointwise) as $n \to \infty$, where

$$T\Big(\{g_i\}_{i=1}^{\infty}\Big) = \sum_{i=1}^{\infty}\Lambda_i^* g_i.$$

By Banach-Steinhaus Theorem, T is a bounded linear operator from $(\sum_{i=1}^{\infty}\oplus\mathcal{H}_i)_{l_2}$ into $\mathcal{H}$. Therefore $\{\Lambda_i\}_{i=1}^{\infty}$ is a g-Bessel sequence [7]. If $f \in \mathcal{H}$ and $\|f\| = 1$, then

$$\begin{aligned}\sum_{i=n}^{\infty}\|\Lambda_i f\|^2 &= \sum_{i=n}^{\infty}\langle\Lambda_i^*\Lambda_i f, f\rangle = \langle\sum_{i=n}^{\infty}\Lambda_i^*\Lambda_i f, f\rangle \\ &\leq \|\sum_{i=n}^{\infty}\Lambda_i^*\Lambda_i f\| = \|(T_n - T)\{\Lambda_i f\}_{i=1}^{\infty}\|.\end{aligned}$$

Therefore $\{\Lambda_i\}_{i=1}^{\infty}$ is an ultra g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^{\infty}$.

Proposition 2.9. *Suppose that* $\{\Lambda_i \in B(\mathcal{H},\mathcal{H}_i)\}_{i=1}^{\infty}$ *is a sequence of bounded operators. Let* V *be a dense subset of unit sphere of* $\mathcal{H}$ *and*

$$\sup_{f\in V}\sum_{i=n}^{\infty}\|\Lambda_i f\|^2 \to 0, \quad \text{as} \;\; n\to\infty. \tag{2.2}$$

Then $\{\Lambda_i \in B(\mathcal{H},\mathcal{H}_i)\}_{i=1}^{\infty}$ *is an ultra g-Bessel sequence for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i=1}^{\infty}$.

Proof. Let

$$a_n = \sup\Big\{\sum_{i=n}^{\infty}\|\Lambda_i f\|^2 : f\in\mathcal{H}, \;\; \|f\|=1\Big\}.$$

Assume that $\{a_n\}_n$ dose not tend to zero. Then there is $\varepsilon>0$ and a sequence $\{n_k\}$ of positive integers such that $a_{n_k}>\varepsilon$ for each $k\in\mathbb{N}$. Hence for each $k\in\mathbb{N}$ there exists $f^k\in\mathcal{H}$ with $\|f^k\|=1$ such that

$$\sum_{i=n_k}^{\infty}\|\Lambda_i f^k\|^2>\varepsilon. \tag{2.3}$$

Let us consider a sequence $\{f_j^k\}\subseteq V$ such that $f_j^k\to f^k$ as $j\to\infty$. By (2.2), there exists $k_0>0$ such that

$$\sum_{i=n_{k_0}}^{\infty}\|\Lambda_i f_j^{k_0}\|^2<\varepsilon, \quad j\in\mathbb{N}.$$

It follows from (2.3) that there is $l\in\mathbb{N}$ such that $\sum_{i=n_{k_0}}^{l}\|\Lambda_i f^{k_0}\|^2>\varepsilon$. Since

$$\sum_{i=n_{k_0}}^{l}\|\Lambda_i f_j^{k_0}\|^2 \to \sum_{i=n_{k_0}}^{l}\|\Lambda_i f^{k_0}\|^2, \quad \text{as} \;\; j\to\infty,$$

we have $\sum_{i=n_{k_0}}^{\infty}\|\Lambda_i f_j^{k_0}\|^2>\varepsilon$ for sufficiently large j. This is a contradiction.

Proposition 2.10. *Let* $\{\Lambda_i \in B(\mathcal{H},\mathcal{H}_i)\}_{i=1}^{\infty}$ *be an ultra g-Bessel sequence for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i=1}^{\infty}$ *and* $\{\theta_i \in B(\mathcal{H},\mathcal{H}_i)\}_{i=1}^{\infty}$ *be a sequence of bounded operators. If there exists* $M>0$ *such that for any finite subset* $J\subseteq\mathbb{N}$ *we have*

$$\sum_{i\in J}\|\Lambda_i f-\theta_i f\|^2 \le M\sum_{i\in J}\|\Lambda_i f\|^2,$$

then $\{\theta_i \in B(\mathcal{H},\mathcal{H}_i)\}_{i=1}^{\infty}$ *is an ultra g-Bessel sequence for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i=1}^{\infty}$.

Proof. Let $f \in \mathcal{H}$. We have

$$\sum_{i=n}^{\infty} \|\theta_i f\|^2 = \sum_{i=n}^{\infty} \|(\theta_i f + \Lambda_i f - \Lambda_i f)\|^2 \leq 2\sum_{i=n}^{\infty} \|\Lambda_i f - \theta_i f\|^2 + 2\sum_{i=n}^{\infty} \|\Lambda_i f\|^2$$
$$\leq 2(M+1)\sum_{i=n}^{\infty} \|\Lambda_i f\|^2.$$

Therefore $\{\theta_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is an ultra g-Bessel sequence for $\mathcal{H}$.

Proposition 2.11. *Let $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ be an ultra g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^{\infty}$ and $\{\theta_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ be a sequence of bounded operators such that for any $J \subseteq \mathbb{N}$ with $|J| < +\infty$,*

$$\Big\|\sum_{i\in J} \Lambda_i^*\Lambda_i f - \theta_i^*\theta_i f\Big\| \leq \lambda\Big\|\sum_{i\in J}\Lambda_i^*\Lambda_i f\Big\| + \mu\Big\|\sum_{i\in J}\theta_i^*\theta_i f\Big\| + \gamma\Big(\sum_{i\in J}\|\Lambda_i f\|^2\Big)^{\frac{1}{2}},$$

where $\lambda, \gamma \geq 0$ and $0 \leq \mu < 1$. Then $\{\theta_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is an ultra g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^{\infty}$.

Proof. Let $J \subseteq \mathbb{N}$ with $|J| < +\infty$ and $f \in \mathcal{H}$. Then

$$\Big\|\sum_{i\in J}\theta_i^*\theta_i f\Big\| \leq \Big\|\sum_{i\in J}\Lambda_i^*\Lambda_i f - \theta_i^*\theta_i f\Big\| + \Big\|\sum_{i\in J}\Lambda_i^*\Lambda_i f\Big\|$$
$$\leq (\lambda+1)\Big\|\sum_{i\in J}\Lambda_i^*\Lambda_i f\Big\| + \mu\Big\|\sum_{i\in J}\theta_i^*\theta_i f\Big\| + \gamma\Big(\sum_{i\in J}\|\Lambda_i f\|^2\Big)^{\frac{1}{2}}.$$

So

$$\Big\|\sum_{i\in J}\theta_i^*\theta_i f\Big\| \leq \frac{1+\lambda}{1-\mu}\Big\|\sum_{i\in J}\Lambda_i^*\Lambda_i f\Big\| + \frac{\gamma}{1-\mu}\Big(\sum_{i\in J}\|\Lambda_i f\|^2\Big)^{\frac{1}{2}}.$$

Let B be the g-Bessel bound for $\{\Lambda_i\}_{i=1}^{\infty}$, then

$$\Big\|\sum_{i\in J}\Lambda_i^*\Lambda_i f\Big\| \leq \sqrt{B}\Big(\sum_{i\in J}\|\Lambda_i f\|^2\Big)^{\frac{1}{2}}.$$

Hence

$$\Big\|\sum_{i\in J}\theta_i^*\theta_i f\Big\| \leq \Big(\frac{1+\lambda}{1-\mu}\sqrt{B} + \frac{\gamma}{1-\mu}\Big)\Big(\sum_{i\in J}\|\Lambda_i f\|^2\Big)^{\frac{1}{2}}.$$

This implies that

$$\Big\|\sum_{i=n}^{\infty}\theta_i^*\theta_i f\Big\| \leq \Big(\frac{1+\lambda}{1-\mu}\sqrt{B} + \frac{\gamma}{1-\mu}\Big)\Big(\sum_{i=n}^{\infty}\|\Lambda_i f\|^2\Big)^{\frac{1}{2}}$$

for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\sum_{i=n}^{\infty} \|\theta_i f\|^2 =& \langle \sum_{i=n}^{\infty} \theta_i^* \theta_i f, f \rangle \leq \Big\| \sum_{i=n}^{\infty} \theta_i^* \theta_i f \Big\| \\ \leq& \Big(\frac{1+\lambda}{1-\mu} \sqrt{B} + \frac{\gamma}{1-\mu} \Big) \Big(\sum_{i=n}^{\infty} \|\Lambda_i f\|^2 \Big)^{\frac{1}{2}}.\end{aligned}$$

for all $f \in \mathcal{H}$ with $\|f\| = 1$. Hence $\{\theta_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is an ultra g-Bessel sequence for $\mathcal{H}$.

Theorem 2.12. *Suppose that $\mathcal{H}$ is finite dimensional and $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^{\infty}$. Then $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is an ultra g-Bessel sequence and there exists $N \in \mathbb{N}$ such that if $1 \leq n < N$, then $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{n}$ is not a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^{n}$ and if $n \geq N$, then $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{n}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^{n}$.*

Proof. Let A, B be the g-frame bound for $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ and $\{e_j\}_{j=1}^{m}$ be an orthonormal basis for $\mathcal{H}$. Then

$$\sum_{i=1}^{\infty} \|\Lambda_i\|_2^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{m} \|\Lambda_i e_j\|^2 = \sum_{j=1}^{m} \sum_{i=1}^{\infty} \|\Lambda_i e_j\|^2 \leq mB.$$

By Proposition 2.3, $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is an ultra g-Bessel sequence. Let $0 < \varepsilon < A$. Since $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is an ultra g-Bessel sequence, there exists $K > 0$ such that

$$\sum_{i=K+1}^{\infty} \|\Lambda_i f\|^2 < \varepsilon \|f\|^2, \quad f \in \mathcal{H}. \tag{2.4}$$

We have

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} \|\Lambda_i f\|^2 = \sum_{i=1}^{K} \|\Lambda_i f\|^2 + \sum_{i=K+1}^{\infty} \|\Lambda_i f\|^2$$

for all $f \in \mathcal{H}$. It follows from (2.4) that

$$(A - \varepsilon)\|f\|^2 \leq \sum_{i=1}^{K} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

Therefore $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{n}$ is a g-frame for $\mathcal{H}$ for each $n \geq K$. We assume that N is the minimum of the all such K. Then $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{n}$ is not a g-frame for $\mathcal{H}$ for each $n < N$.

Proposition 2.13. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^{\infty}$. Then $S_n \to S$ in $B(\mathcal{H})$ as $n \to \infty$, where S and S_n are the g-frame operator of $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ and*

$\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^n$, *respectively.*

Proof. By Theorem 2.12, there exists $N \in \mathbb{N}$ such that $\{\Lambda_i\}_{i=1}^n$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i=1}^n$ for all $n \geq N$. Let B be the upper g-frame bound, $n \geq N$ and $f \in \mathcal{H}$ with $\|f\| = 1$. Then

$$\begin{aligned}\|S_n(f) - S(f)\| =& \Big\| \sum_{i=n+1}^{\infty} \Lambda_i^* \Lambda_i f \Big\| \\ =& \sup_{\|g\|=1} \Big| \langle \sum_{i=n+1}^{\infty} \Lambda_i^* \Lambda_i f, g \rangle \Big| \\ \leq& \sup_{\|g\|=1} \Big(\sum_{i=n+1}^{\infty} \|\Lambda_i f\|^2 \Big)^{\frac{1}{2}} . \Big(\sum_{i=n+1}^{\infty} \|\Lambda_i g\|^2 \Big)^{\frac{1}{2}} \\ \leq& \sqrt{B} \Big(\sum_{i=n+1}^{\infty} \|\Lambda_i f\|^2 \Big)^{\frac{1}{2}}.\end{aligned}$$

Since $\{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is an ultra g-Bessel for $\mathcal{H}$ (by Theorem 2.12), we get $\|S_n - S\| \to 0$ as $n \to \infty$.

Remark 2.14. Frames (g-frames) in finite dimensional Hilbert spaces play important roles. We proved that every g-frame in a finite dimensional Hilbert space is an ultra g-Bessel sequence. So we can approximate S by S_n (Proposition 2.13).

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