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Weakly Semicommutative Rings and Strongly Regular Rings

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Abstract. A ring R is called weakly semicommutative ring if for any $a, b \in R^* = R \setminus \{0\}$ with $ab = 0$, there exists $n \geq 1$ such that either $a^n \neq 0$ and $a^nRb = 0$ or $b^n \neq 0$ and $aRb^n = 0$. In this paper, many properties of weakly semicommutative rings are introduced, some known results are extended. Especially, we show that a ring R is a strongly regular ring if and only if R is a left $SF-$ring and weakly semicommutative ring.

1. Introduction

All rings considered in this paper are associative rings with identity, and all modules are unital. Let R be a ring, write $R^* = R \setminus \{0\}$ and $E(R)$ and $N(R)$ denote the set of all idempotents and the set of all nilpotents of R, respectively. For any nonempty subset X of R, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the set of right annihilators of X and the set of left annihilators of X, respectively. Especially, if $X = a$, we write $l(X) = l(a)$ and $r(X) = r(a)$.

A ring R is called (von Neumann) regular ring if for every $a \in R$ there exists $b \in R$ such that $a = aba$. A ring R is strong regular if for every $a \in R$ there exists $b \in R$ such that $a = a^2b$. A ring R is called reduced if R has no nonzero nilpotent elements. It is well known that R is a strongly regular ring if and only if R is a reduced regular ring. A ring R is called left (resp., right) quasi-duo ring if every maximal left (resp., right) ideal of R is an ideal. A ring R is called $MELT$ (resp.,

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$MERT$) ring if every essentian maximal left (resp., right) ideal of R is an ideal. According to Ramamurthi (1975), a ring R is called a left (resp., right) $SF-$ring if each simple left (resp., right) $R-$module is flat. It is known that regular rings are left and right $SF-$rings. Ramamurthi (1975) initiated the study of $SF-$rings and the question whether an $SF-$ring is necessarily regular. For several years, $SF-$rings have been studied by many authors and the regularity of $SF-$rings which satisfy certain additional conditions is showed (cf. Ramamurthi, 1975; Rege, 1986; Yue Chi Ming, 1980, 1982, 1988; Zhang and Du, 1992, 1993; Zhang, 1994, 1998; Zhou and Wang, 2004a, 2004b; Zhou, 2007). But the question remains open. Yue Chi Ming (1988) proved the strong regularity of right $SF-$rings whose complement left ideals are ideals, and he proposed the following question: Is R strong regular if R is a left $SF-$rings whose complement left ideals are ideals? Zhang and Du (1992) affirmatively answered the question. Zhou and Wang (2004a) proved that if R is a right $SF-$rings whose all essential maximal right ideals are $GW-$ideals, then R is a regular ring. Zhang (1998) proved that if R is an $MELT$ and right $SF-$rings, then R is a regular ring. Zhou (2007) proved that if R is a left $SF-$rings whose complement left (right) ideals are $W-$ideals, then R is a strongly regular ring. Following Zhou and Wang (2004a), a left ideal L of a ring R is called $GW-$ideal, if for any $a \in L$, there exists a positive integer n such that $a^nR \subseteq L$. Clearly, ideal is $GW-$ideal, but the converse is not true, in general, by Zhou and Wang (2004a, Example 1.2).

According to Zhou (2007), a left ideal L of a ring R is called a weak ideal ($W-$ideal), if for any $0 \neq a \in L$, there exists $n \geq 1$ such that $a^n \neq 0$ and $a^nR \subseteq L$. A right ideal K of a ring R is defined similarly to be a weak ideal. Clearly, ideals are $W-$ideals and $W-$ideals are $GW-$ideals, but the converse are not true, in general, by Zhou (2007).

According to Cohn (1999), a ring R is called symmetric if $abc = 0$ implies $acb = 0$ for $a, b, c \in R$, and R is said to be reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$, and R is said to be semicommutative if $ab = 0$ implies $aRb = 0$. Clearly, reduced $\Longrightarrow$symmetric $\Longrightarrow$ reversible $\Longrightarrow$ semicommutative .

A ring R is called weakly semicommutative ring if for any $a, b \in R^*$ and $ab = 0$, there exists $n \geq 1$ such that either $a^n \neq 0$ and $a^nRb = 0$ or $b^n \neq 0$ and $aRb^n = 0$.

Clearly, semicommutative rings are weak semicommutative.

The first purpose of this paper is to study the properties of weakly semicommutative rings, the next purpose of this paper is to give a new characterization of strongly regular rings in terms of left $SF-$rings and weakly semicommutative rings. Finally some known results in Rege(1986) can be extended.

According to Hwang (2007), a ring R is called NCI if $N(R) = 0$ or there exists a nonzero ideal of R contained in $N(R)$. Clearly, NI rings (that is, $N(R)$ forms an ideal of R) are NCI, but the converse is not true, in general, by Hwang (2007).

Following Wei and Chen (2007), left $R-$module M is called $Wnil-$injective if for any $0 \neq a \in N(R)$, there exists $n \geq 1$ such that $a^n \neq 0$ and every left $R-$homomorphism Ra^n to M extends to R. Evidently, $YJ-$injective modules (c.f., Kim, Nam and Kim (1999)) are $Wnil-$injective, but the converse is not true, in general, by Wei and Chen (2007).

2. Main Results

We begin with the following theorem.

Theorem 1. (1) *Weakly semicommutative rings are Abelian.*

(2) *Weakly semicommutative rings are NCI.*

(3) *Let R be a weakly semicommutative ring. If every simple singular left $R-$module is $Wnil-$injective, then R is a reduced ring.*

Proof. (1) Let R be a weakly semicommutative ring and $e \in E(R)$. We can assume that $0 \neq e \neq 1$. Since R is a weakly semicommutative ring and $e(1-e) = 0$, there exists $n \geq 1$ such that either $e^n \neq 0$ and $e^n R(1-e) = 0$ or $(1-e)^n \neq 0$ and $eR(1-e)^n = 0$. Therefore we obtain $eR(1-e) = 0$, which implies R is an Abelian ring.

(2) If $N(R) \neq 0$, then there exists $0 \neq a \in N(R)$. Let $n \geq 1$ such that $a^n = 0$ and $a^{n-1} \neq 0$. Since R is a weakly semicommutative ring and $a^{n-1}a = 0$, there exists $m \geq 1$ such that either $(a^{n-1})^m \neq 0$ and $(a^{n-1})^m Ra = 0$ or $a^m \neq 0$ and $a^{n-1}Ra^m = 0$. If $(a^{n-1})^m \neq 0$ and $(a^{n-1})^m Ra = 0$, then $m = 1$, so $a^{n-1}Ra = 0$, this gives $a^{n-1}Ra^{n-1} = 0$. If $a^m \neq 0$ and $a^{n-1}Ra^m = 0$, then $m \leq n-1$, so $a^{n-1}Ra^{n-1} = 0$. Therefore $Ra^{n-1}R$ is a nonzero nilpotent ideal of R, so R is a NCI ring.

(3) Let $a^2 = 0$. If $a \neq 0$, then there exists a maximal left ideal M of R such that $l(a) \subseteq M$. If M is not essential in ${}_RR$, then $M = l(e)$ for some $e \in E(R)$. Thus $ae = 0$ because $a \in l(a) \subseteq M$. By (1), R is an Abelian ring, so $ea = 0$. This gives $e \in l(a) \subseteq l(e)$, a contradiction. Hence M is an essential left ideal of R, so R/M is a simple singular left $R-$module. By hypothesis, R/M is a $Wnil-$injective left $R-$module. Let $f : Ra \longrightarrow R/M$ defined by $f(ra) = r+M$. Then f is a well defined left $R-$homomorphism, so there exists a left $R-$homomorphism $g : R \longrightarrow R/M$ such that $g(a) = f(a)$. Hence there exists $c \in R$ such that $1 + M = f(a) = g(a) = ag(1) = ac + M$. Since R is a weakly semicommutative ring, $aRa = 0$. Thus $ac \in l(a) \subseteq M$. This leads to $1 \in M$, which is a contradiction. Hence $a = 0$. □

A ring R is called directly finite if $ab = 1$ implies $ba = 1$ for $a, b \in R$. It is well known that Abelian rings are directly finite. Hence weakly semicommutative rings are directly finite by Theorem 1. According to Hwang (2007), NCI rings need not be directly finite. Hence NCI rings need not be weakly semicommutative. It is well known that NI rings are directly finite, so, we ask:

Is NI ring weakly semicommutative?

Regretfully, the answer is "*no*". For example, let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Since $N(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is an ideal of R, R is a NI ring. Clearly, $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$ are idempotents in R. Since $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} = 0$

and $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \neq 0$, R is not weakly semicommutative.

A ring R is called left WNV if every simple singular left $R-$module is $Wnil-$injective. Clearly, left $V-$rings and reduced rings are left WNV. Since reduced rings are reversible and reversible rings are semicommutative, by Theorem 1, we have the following corollary.

Corollary 2. *The following conditions are equivalent for a ring R:*

(1) *R is a reduced ring;*

(2) *R is a reversible ring and left WNV ring;*

(3) *R is a semicommutative ring and left WNV ring;*

(4) *R is a weakly semicommutative ring and left WNV ring.*

A ring R is called biregular if for every $a \in R$, RaR is generated by a central idempotent of R. A ring R is called weakly regular if for any $a \in R$, $a \in RaRa \cap aRaR$. Clearly, biregular rings are weak regular, but the converse is not true, in general. Certainly, reduced weakly regular rings are biregular. Kim, Nam and Kim (1999, Theorem 4) proved that if R is a left semicommutative ring whose every simple singular left module is $YJ-$injective, then R is a reduced weakly regular ring. Hence, by Corollary 2, we have the following corollary.

Corollary 3. *Let R be a weakly semicommutative ring. If every simple singular left $R-$module is $YJ-$injective, then R is a reduced biregular ring.*

Wei (2007, Theorem 16) proved that a ring R is a strongly regular ring if and only if R is a semicommutative $MELT$ ring whose simple singular left modules are $YJ-$injective. Hence, by Corollary 2, we have the following corollary.

Corollary 4. *A ring R is a strongly regular ring if and only if R is a $MELT$ weakly semicommutative ring whose every simple singular left module is $YJ-$injective.*

It is well known that a ring R is a reduced ring if and only if R is a semiprime ring and semicommutative ring. On the other hand, semiprime weakly semicommutative rings are reversible (In fact, if $ab = 0$, then $(ba)^2 = 0$. If R is a weakly semicommutative ring, then $baRba = 0$. Since R is a semiprime ring, $ba = 0$). so we have the following theorem.

Theorem 5. *The following conditions are equivalent for a ring R:*

(1) *R is a reduced ring;*

(2) *R is a semiprime weakly semicommutative ring.*

A ring R is called left $Kasch$ if every simple left $R-$module is isomorphic to a minimal left ideal of R. Recently, in their paper Lam and Dugas (2005, Proposition 4.11), Lam and Dugas showed that left $Kasch$ semicommutative ring is left quasi-duo. We can generalize the result as follows.

Theorem 6. *Let R be a left Kasch weakly semicommutative ring. Then R is left quasi-duo ring.*

Proof. Let M be a maximal left ideal of R. Since R is a left $Kasch$ ring, $M = l(k)$ for some $k \in R$. Clearly, $Rk \cong R/l(k) = R/M$, so Rk is a minimal left ideal of R. For any $a \in R^*$, we have $Ma \subseteq M$. If not, then there exists $a \in R^*$ such that $Ma \subsetneqq M$, so $Ma + M = R$ and $a \notin M$. Let $xa + y = 1$ for some $x, y \in M$. Since $ax \in M$, $axk = 0$. If $ax = 0$, then $a = a1 = axa + ay = ay \in M$, which is a contradiction. Hence $ax \neq 0$. Since R is a weakly semicommutative ring, there exists $n \geq 1$ such that either $(ax)^n \neq 0$ and $(ax)^n Rk = 0$ or $k^n \neq 0$ and $axRk^n = 0$. If $k^n \neq 0$ and $(ax)Rk^n = 0$, then $Rk^n = Rk$ because Rk is a minimal left ideal of R. Hence $axRk = 0$, so $axak = 0$ and $axa \in l(k) = M$, which implies $a = a1 = axa + ay \in M$, a contradiction. If $(ax)^n \neq 0$ and $(ax)^n Rk = 0$, then $(ax)^n ak = 0$ and so $(ax)^n a \in M$. Therefore $(ax)^{n-1}a = (ax)^{n-1}a1 = (ax)^{n-1}axa + (ax)^{n-1}ay = (ax)^n a + (ax)^{n-1}ay \in M$. Repeating the process mentioned above, we obtain $axa \in M$, so $a \in M$, which is a contradiction. Therefore $Ma \subseteq M$ for any $a \in R$, so M is an ideal of R and then R is a left quasi-duo ring. □

Evidently, the class of weakly semicommutative rings is closed under subrings. But we do not know whether the class of weakly semicommutative rings is closed under direct products. Although we know that it is a known result that the class of semicommutative rings is closed under subrings and direct products.

Theorem 7. *Let R be a ring and Δ be a multiplicatively closed subset of R consisting of central regular elements. Then R is a weakly semicommutative ring if and only if $\Delta^{-1}R$ is a weakly semicommutative ring.*

Proof. The sufficiency is clear.

Now let $\alpha\beta = 0$ with $\alpha = u^{-1}a, \beta = v^{-1}b \in (\Delta^{-1}R)^*, u, v \in \Delta$ and $a, b \in R$. Since Δ is contained in the center of R, we have $0 = \alpha\beta = u^{-1}av^{-1}b = (u^{-1}v^{-1})ab = (uv)^{-1}ab$ and $ab = 0$. Clearly, $a, b \in R^*$. Since R is a weakly semicommutative ring, there exists $n \geq 1$ such that either $a^n \neq 0$ and $a^n Rb = 0$ or $b^n \neq 0$ and $aRb^n = 0$. Hence, either $\alpha^n \neq 0$ and $\alpha^n(\Delta^{-1}R)\beta = (u^{-1})^n v^{-1}\Delta^{-1}a^n Rb = 0$ or $\beta^n \neq 0$ and $\alpha(\Delta^{-1}R)\beta^n = u^{-1}\Delta^{-1}(v^{-1})^n aRb^n = 0$. Hence $\Delta^{-1}R$ is a weakly semicommutative ring. □

The ring of Laurent polynomials in x, coefficients in a ring R, consists of all formal sums $\Sigma_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers; denote it by $R[x; x^{-1}]$.

Corollary 8. *For a ring R, $R[x]$ is a weakly semicommutative ring if and only if $R[x; x^{-1}]$ is a weakly semicommutative ring.*

Proof. It suffices to establish necessity. Let $\Delta = \{1, x, x^2, \cdots, x^n, \cdots\}$. Then, clearly, Δ is a multiplicatively closed subset of $R[x]$. Since $R[x; x^{-1}] = \Delta^{-1}R[x]$ and Δ is contained in the center of $R[x]$, it follows that $R[x; x^{-1}]$ is a weakly semicommutative ring by Theorem 7. □

Theorem 9. *The finite subdirect product of weakly semicommutative rings is a* *weakly semicommutative ring.*

Proof. Let $\{L_i | i = 1, 2, \cdots, n\}$ be ideals of R such that $\cap_{i=1}^n L_i = 0$ and all R/L_i be weakly semicommutative rings. Let $a, b \in R^*$ with $ab = 0$. For any $i \in \{1, 2, \cdots, n\}$, if one of a, b belongs to L_i, then $aRb \subseteq L_i$. If $a \notin L_i$ and $b \notin L_i$, then $\bar{a}, \bar{b} \in (\bar{R})^*$ and $\bar{a}\bar{b} = 0$, where $\bar{R} = R/L_i$. Since R/L_i is a weakly semicommutative ring, there exists $n_i \geq 1$ such that either $\bar{a}^{n_i} \neq 0$ and $\bar{a}^{n_i}\bar{R}\bar{b} = 0$ or $\bar{b}^{n_i} \neq 0$ and $\bar{a}\bar{R}\bar{b}^{n_i} = 0$. So either $a^{n_i} \neq 0$ and $a^{n_i}Rb \subseteq L_i$ or $b^{n_i} \neq 0$ and $aRb^{n_i} \subseteq L_i$. In any case, there exists $m_i \geq 1$ (in fact either $m_i = 1$ or $m_i = n_i$) such that either $a^{m_i} \neq 0$ and $a^{m_i}Rb \subseteq L_i$ or $b^{m_i} \neq 0$ and $aRb^{m_i} \subseteq L_i$. Let $m = max\{m_i | i = 1, 2, \cdots, n\}$. Then, clearly, either $a^m \neq 0$ and $a^m Rb \subseteq \cap_{i=1}^n L_i$ or $b^m \neq 0$ and $aRb^m \subseteq \cap_{i=1}^n L_i$, that is $a^m Rb = 0$ or $aRb^m = 0$. Hence R is a weakly semicommutative ring. □

It is well known that a ring R is a semicommutative ring if and only if R is an Abelian ring and for any idempotent e of R, eRe and $(1-e)R(1-e)$ are all semicommutative rings.

Corollary 10. *Let R be a ring and $e^2 = e \in R$. Then R is a weakly semicommutative ring if and only if R is an Abelian ring and eRe and $(1-e)R(1-e)$ are weakly semicommutative rings.*

Proof. It suffices to establish sufficiency. Since R is an Abelian ring, we have ring isomorphism: $R/eRe \cong (1-e)R(1-e)$ and $R/(1-e)R(1-e) \cong eRe$. By hypothesis, R/eRe and $R/(1-e)R(1-e)$ are weakly semicommutative rings. Since $eRe \cap (1-e)R(1-e) = 0$, R is a subdirect product of R/eRe and $R/(1-e)R(1-e)$. By Theorem 9, R is a weakly semicommutative ring. □

Rege (1986, Remark 3.13) pointed out that if R is a reduced left (right) $SF-$ring, then R is a strongly regular ring. We can extend this result to weakly semicommutative rings.

Theorem 11. *If R is a left $SF-$ring and weakly semicommutative ring, then R is a strongly regular ring.*

Proof. Assume that $a \in R$. If $a = 0$, then a is a von Neumann regular element. If $a \neq 0$, then we claim that $Ra + \cup_{i=1}^{\infty} r(a^i R) = R$. If not, there exists a maximal left ideal M of R containing $Ra + \cup_{i=1}^{\infty} r(a^i R)$, so R/M is a simple left $R-$module, by hypothesis, R/M is flat left $R-$module. Since $a \in M$, $a = ab$ for some $b \in M$. Clearly, $b \neq 1$. Since R is a weakly semicommutative ring and $a(1-b) = 0$, there exists $n \geq 1$ such that either $a^n \neq 0$ and $a^n R(1-b) = 0$ or $(1-b)^n \neq 0$ and $aR(1-b)^n = 0$. If $a^n \neq 0$ and $a^n R(1-b) = 0$, then $1 - b \in r(a^n R) \subseteq M$, so $1 = (1-b) + b \in M$, which is a contradiction. If $(1-b)^n \neq 0$ and $aR(1-b)^n = 0$, then $(1-b)^n \in r(aR) \subseteq M$, so $1 = (1-(1-b)^n) + (1-b)^n = (1 + (1-b) + (1-b)^2 + \cdots + (1-b)^{n-1})b + (1-b)^n \in M$, which is a contradiction. Thus $Ra + \cup_{i=1}^{\infty} r(a^i R) = R$. Let $1 = ca + x$ where $c \in R$ and $x \in \cup_{i=1}^{\infty} r(a^i R)$. Let $x \in r(a^m R)$. Then $a^m x = 0$, so $a^m = a^m 1 = a^m ca + a^m x = a^m ca$. If $m = 1$, then $a = aca$, so a is a von Neumann regular element. If $m \geq 2$, then let $b = a^{m-1} - a^{m-1}ca$. By computing, we have $b^2 = 0$. If $b = 0$, then $a^{m-1} = a^{m-1}ca$. If $b \neq 0$, then we claim that $Rb + r(bR) = R$. If not, there exists a maximal left ideal L of R containing $Rb+r(bR)$. Certainly, R/L is a flat left $R-$module, so $b = bd$ for some $d \in L$. Since there exists $t \geq 1$ such that either $b^t \neq 0$ and $b^tR(1-d) = 0$ or $(1-d)^t \neq 0$ and $bR(1-d)^t = 0$. If $b^t \neq 0$ and $b^tR(1-d) = 0$, then $t = 1$, so $bR(1-d) = 0$, which implies $1 - d \in r(bR) \subseteq L$. This is impossible because $d \in L$. If $(1-d)^t \neq 0$ and $bR(1-d)^t = 0$, then $(1-d)^t \in r(bR) \subseteq L$. This also gives $1 \in L$, a contradiction. Hence $Rb + r(bR) = R$, this leads to $b = bub$ for some $u \in R$, so, by computing, we obtain that $a^{m-1} = a^{m-1}(c + (1-ca)u(a^{m-2} - a^{m-1}c))a$. In any case, we obtain $v \in R$ (in fact, either $v = c$ or $v = c + (1-ca)u(a^{m-2} - a^{m-1}c)$) satisfying $a^{m-1} = a^{m-1}va$. Repeating the process mentioned above, we can obtain $w \in R$ such that $a = awa$, so a is a von Neumann regular element. Therefore R is a von neumann regular ring. By Theorem 1(1), R is an Ableian ring, so R is a strongly regular ring. □

Corollary 12. *The following conditions are equivalent for a ring R:*

(1) *R is a strongly regular ring;*
(2) *R is a left $SF-$ring and semicommutative ring;*
(3) *R is a left $SF-$ring and reversible ring;*
(4) *R is a left $SF-$ ring and symmetric ring;*
(5) *R is a left $SF-$ring and reduced ring.*

Since regular rings are left $SF-$rings, by Theorem 11, we have the following corollary.

Corollary 13. *The following conditions are equivalent for a ring R:*

(1) *R is a strongly regular ring;*
(2) *R is a regular ring and weakly semicommutative ring.*

According to Wei and Chen (2007), a ring R is called $n-$regular if for each $a \in N(R)$, $a \in aRa$. Clearly, reduced rings are $n-$regular. Wei and Chen (2008, Theorem 2.7) points out that a ring R is reduced if and only if R is $n-$regular and Abelian. Hence by Theorem 1, we have the following corollary.

Corollary 14. *The following conditions are equivalent for a ring R:*

(1) *R is a reduced ring;*
(2) *R is a $n-$regular ring and weakly semicommutative ring.*

Observing Theorem 1 and Corollary 14, we obtain the following theorem.

Theorem 15. *The following conditions are equivalent for a ring R:*

(1) *R is a reduced ring;*
(2) *R is a $n-$regular ring and NCI ring.*

Proof. (1) $\Longrightarrow$ (2) is trivial.

(2) $\Longrightarrow$ (1) If $N(R) \neq 0$, then there exists a nonzero ideal I of R contained in $N(R)$ because R is NCI. Let $0 \neq a \in I$. Then $a = aba$ for some $b \in R$ by

hypothesis. Since $ba \in E(R)$ and $ba \in I \subseteq N(R)$, $ba = 0$. Hence $a = aba = 0$, which is a contradiction. Hence $N(R) = 0$ and so R is a reduced ring. □

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