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GLOBAL EXISTENCE OF WEAK SOLUTIONS FOR A KELLER-SEGEL-FLUID MODEL WITH NONLINEAR DIFFUSION

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Abstract. We consider the Cauchy problem for a Keller-Segel-fluid model with degenerate diffusion for cell density, which is mathematically formulated as a porus medium type of Keller-Segel equations coupled to viscous incompressible fluid equations. We establish the global-in-time existence of weak solutions and bounded weak solutions depending on some conditions of parameters such as chemotactic sensitivity and consumption rate of oxygen for certain range of diffusive exponents of cell density in two and three dimensions.

1. Introduction

In this paper, we consider a mathematical model that is originated with dynamics of swimming bacteria, so called *Bacillus subtilis*, which live in fluid and consume oxygen. To be more precise, we study the Cauchy problem on the coupled Keller-Segel-Navier-Stokes equations in $\mathbb{R}^d \times (0,T)$ with $0 < T \le \infty$ and $d = 2,3$:

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n^{1+\alpha} - \nabla \cdot (\chi(c) n \nabla c), \\ \partial_t c + u \cdot \nabla c = \Delta c - \kappa(c) n, \\ \partial_t u + \tau (u \cdot \nabla) u + \nabla p = \Delta u - n \nabla \phi, \qquad \operatorname{div} u = 0, \end{cases} \tag{1}$$

where n, c, u and p are the cell density, oxygen concentration, velocity field and pressure of the fluid, respectively. Here τ is a constant such that the case that $\tau = 1$ corresponds to the Navier-Stokes equations and if $\tau = 0$, the fluid equations becomes the Stokes system. Under our consideration, the Navier-Stokes system is studied for two dimensions and the Stokes system is considered for three dimensions. The functions $\chi : \mathbb{R} \to \mathbb{R}$ and $\kappa : \mathbb{R} \to \mathbb{R}$ represent the chemotactic sensitivity and consumption rate of oxygen. To

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describe the fluid motions, Boussinesq approximation is used to denote the effect due to heavy bacteria. The time-independent function $\phi = \phi(x)$ denotes the potential function produced by different physical mechanisms, e.g., the gravitational force or centrifugal force. Thus, $\phi(x) = ax_d$ is one example of gravity force, and $\phi(x) = \phi(|x|)$ is an example of centrifugal force. In this paper, we assume that ϕ is smooth in spatial variable, independent of time variable and $\|\nabla\phi\|_{L^\infty(\mathbb{R}^d)} < \infty$.

The chemotaxis-fluid system above has been proposed in [15] by Tuval et al. for the case $\alpha = 0$ and numerical simulation of plumes was performed in [2] in two dimensions. For the case $\chi(c) \equiv \chi$, Lorz [10] showed the local existence of solutions for the chemotaxis-Stokes system in two dimensional bounded domain with the mixed boundary conditions and chemotaxis-Navier-Stokes system in three dimensional bounded domain with boundary conditions $\partial_\nu n = \partial_\nu c = u = 0$. One can refer to, for example, [3], [9], [17] and [1] for further results in case that $\alpha = 0$.

Our main purpose of this paper is to establish existence of weak and bounded weak solutions for the system (1), when the equation of n is of porous medium type, i.e., $\alpha > 0$. In [4], Francesco, Lorz and Markowich showed the global existence of a bounded solution to (1) on a bounded domain in $\mathbb{R}^2$ with the boundary conditions $\partial_\nu n^{1+\alpha} = \partial_\nu c = u = 0$, when $\alpha \in (1/2, 1]$. Tao and Winkler [13] extended the result to the case $\alpha > 0$ on a bounded domain in $\mathbb{R}^2$, in case that the fluid equation is the Stokes system. In [9], Liu and Lorz proved the global existence of a weak solution in $\mathbb{R}^3$ when $\alpha = 1/3$. Some special case, i.e., $\chi = 1$ and $\kappa(c) = c$, was studied in [14] and it was shown that if $\alpha > 1/7$, bounded weak solutions exist in $\mathbb{R}^3$, when fluid equation is the Stokes system. Recently, Vorotnikov [16] proved existence of weak solutions in the case that a bacterial growth term is added to (1).

Here we make some comments on Keller-Segel model of porus medium type, which is given as

$$n_t = \Delta n^{1+\alpha} - \nabla\cdot(\chi n\nabla c), \quad \tau c_t = \Delta c - \beta c + \gamma n, \tag{2}$$

where χ is the sensitivity, β decay rate of c and γ is production rate of n. The system (2) consists of a system of the dynamics of cell density n and the concentration of chemical attractant substance c. In case that $\alpha = 0$, we refer to Patlak-Keller-Segel equation, which was suggested by Patlak [11] and Keller-Segel [6, 7] and it has been extensively studied by many authors and we do not list relevant references here. For the case of porus medium equation, in [5] Ishida and Yokota showed that if $\alpha > 1/3$, bounded weak solutions of (2) exist in dimension three (more general type of equations were considered in general domain in [5] and see also Sugiyama and Kunii [12] for the case of parabolic-elliptic case).

As mentioned earlier, our aim is to obtain global existence of weak and bounded weak solutions for the system (1). We start with defining the notion of weak solutions.

Definition 1.1. Let $d = 2$ or 3, $\alpha > 0$ and $0 < T < \infty$. A triple (n, c, u) is said to be a *weak solution* of the system (1) if the followings are satisfied:

(i) n and c are non-negative functions and u is a vector function defined in $\mathbb{R}^d \times (0, T)$ such that

$$n(1 + |x| + |\log n|) \in L^\infty(0, T; L^1(\mathbb{R}^d)), \quad \nabla n^{\frac{1+\alpha}{2}} \in L^2(0, T; L^2(\mathbb{R}^d)),$$
$$c \in L^\infty(0, T; H^1(\mathbb{R}^d)) \cap L^2(0, T; H^2(\mathbb{R}^d)), \quad c \in L^\infty(\mathbb{R}^d \times [0, T)),$$
$$u \in L^\infty(0, T; L^2(\mathbb{R}^d)) \cap L^2(0, T; H^1(\mathbb{R}^d).$$

(ii) (n, c, u) satisfies the equation (1) in the sense of distributions, namely,

$$\int_0^T \int_{\mathbb{R}^d} \left(-n\varphi_t + \nabla n^{1+\alpha} \cdot \nabla\varphi + nu \cdot \nabla\varphi + n\chi(c)\nabla c \cdot \nabla\varphi\right) dxdt = \int_{\mathbb{R}^d} n_0\varphi(\cdot, 0)dx,$$
$$\int_0^T \int_{\mathbb{R}^d} (-c\varphi_t + \nabla c \cdot \nabla\varphi + cu \cdot \nabla\varphi + n\kappa(c)\varphi)\, dxdt = \int_{\mathbb{R}^d} c_0\varphi(\cdot, 0)dx,$$
$$\int_0^T \int_{\mathbb{R}^d} (-u \cdot \psi_t + \nabla u \cdot \nabla\psi - (\tau(u \cdot \nabla)u) \cdot \psi + n\nabla\phi \cdot \psi)\, dxdt = \int_{\mathbb{R}^d} u_0 \cdot \psi(\cdot, 0)dx$$

for all test functions $\varphi \in C_0^\infty\left(\mathbb{R}^d \times [0, T)\right)$ and $\psi \in C_0^\infty\left(\mathbb{R}^d \times [0, T), \mathbb{R}^d\right)$ with $\nabla \cdot \psi = 0$.

The chemotactic sensitivity χ and consumption rate of oxygen κ are L^∞_{loc} functions defined on $[0, \infty)$ with the following hypothesis:

(A) $\kappa(\cdot) \geq 0$ and $\kappa(0) = 0$.

We further assume additional hypotheses of the following in accordance with the range of $\alpha > 0$.

(B1) $\chi \in C^1$ with $\chi'(\cdot) \geq 0$.
(B2) $\chi \in C^1$ with $\chi'(\cdot) \geq \chi_0$ for some constant $\chi_0 > 0$.
(B3) $\kappa \in C^1$ with $\kappa'(\cdot) \geq \kappa_0$ for some constant $\kappa_0 > 0$.

Here we present two different types of assumptions on chemotatic sensitivity χ and consumption rate κ together with the range of α in dimension three. The first one is related to weak solutions.

Assumption 1.2. Let $d = 3$. We assume that one of the following holds:

(i) $\alpha > \frac{1}{3}$ and χ satisfies (B1).
(ii) $\alpha > \frac{1}{6}$ and χ satisfies (B2).
(iii) $\frac{1}{6} < \alpha < 1$ and κ satisfies (B3).
(iv) $\alpha > \frac{1}{6}$, χ satisfies (B1) and κ satisfies (B3).

Another assumption is prepared for bounded weak solutions (see Definition 1.6).

Assumption 1.3. Let $d = 3$. We assume that one of the following holds:

(i) $\alpha \geq \frac{1}{2}$ and χ satisfies (B1).
(ii) $\alpha \geq \frac{1}{4}$ and χ satisfies (B2).

(iii) $\frac{1}{4} \le \alpha < 1$ and κ satisfies (B3).
(iv) $\alpha \ge \frac{1}{4}$, χ satisfy (B1) and κ satisfy (B3).

Remark 1.4. We expect that the restriction, $\alpha < 1$, can be removed in Assumption 1.2(iii) and Assumption 1.3(iii). It is, however, unclear at this moment and remains to be a future work.

The first main result of this paper is the existence of global-in-time weak solution of the system (1) in spatial dimension three. More precisely, the result reads as follows:

Theorem 1.5. *Let $d = 3$ and $\tau = 0$. Suppose that κ satisfies the hypothesis* (A) *and the initial datum (n_0, c_0, u_0) satisfies*

$$n_0(1+|x|+|\log n_0|) \in L^1(\mathbb{R}^d),\ c_0 \in L^\infty(\mathbb{R}^d)\cap H^1(\mathbb{R}^d),\ u_0 \in H^1(\mathbb{R}^d). \tag{3}$$

Assume further that Assumption 1.2 holds. Then, there exists a weak solution (n, c, u) for the system (1). *Moreover, if the initial mass $\|n_0\|_{L^1(\mathbb{R}^d)}$ is sufficiently small, then the limiting case $\alpha = 1/3$ or $\alpha = 1/6$ can be included in the hypothesis* (i) $\sim$ (iv) *of Assumption 1.2.*

Next we introduce another notion of weak solutions, so called *bounded weak solutions*, which show a bit higher regularity of solutions compared to ones defined in Definition 1.1. To be more precise, the notion of bounded weak solutions is defined as follows:

Definition 1.6. Let $d = 2, 3$, $\alpha > 0$ and $T > 0$. A triple (n, c, u) is said to be a *bounded weak solution* of the system (1) if (n, c, u) is a weak solution in Definition 1.1 and furthermore satisfies the following: For any $p \in [1, \infty]$ and $q \in [2, \infty)$

(i) $n \in L^\infty(0, T; L^p(\mathbb{R}^d))$, $\quad \nabla n^{\frac{\alpha+q}{2}} \in L^2(0, T; L^2(\mathbb{R}^d))$.
(ii) $c \in L^q(0, T; W^{2,q}(\mathbb{R}^d))$, $\quad c_t \in L^q(0, T; L^q(\mathbb{R}^d))$.
(iii) $u \in L^\infty(0, T; W^{1,q}(\mathbb{R}^d))$, $\quad u_t, \Delta u \in L^q(0, T; L^q(\mathbb{R}^d))$.

If the range of α in the hypothesis (i)$\sim$(iv) in Theorem 1.5 is a little restrictive and initial data are a bit more regular, we can construct bounded weak solutions. More precisely, our second result reads as follows:

Theorem 1.7. *Let $d = 3$, $\tau = 0$. Suppose that κ satisfies the hypothesis* (A) *and the initial datum (n_0, c_0, u_0) satisfies* (3) *as well as*

$$n_0 \in L^\infty(\mathbb{R}^d), \quad c_0 \in W^{1,q}(\mathbb{R}^d), \quad u_0 \in W^{1,q}(\mathbb{R}^d) \qquad q < \infty. \tag{4}$$

Assume further that Assumption 1.3 holds. Then, there exists a bounded weak solution (n, c, u) for the system (1).

In Tables 1 and 2, we summarize our results in Theorem 1.5 and Theorem 1.7 regarding the range of α depending on the hypothesis (i) $\sim$ (iv).

We remark that Tao and Winkler established existence of bounded weak solutions in [14] in smoothly bounded convex domains for $\alpha > 1/7$ but as

TABLE 1. Conditions for the existence of weak solutions (3D case)

κ (A)	χ' (B1)	χ' (B2)	κ' (B3)	α
○	○	○	○	$\alpha > \frac{1}{6}$
○	○	○	×	
○	○	×	○	
○	○	×	×	$\alpha > \frac{1}{3}$
○	×	×	○	$\frac{1}{6} < \alpha < 1$
○	×	×	×	unknown
	○ : valid		× : invalid	

TABLE 2. Conditions for the existence of bounded weak solutions (3D case)

κ (A)	χ' (B1)	χ' (B2)	κ' (B3)	α
○	○	○	○	$\alpha \geq \frac{1}{4}$
○	○	○	×	
○	○	×	○	
○	○	×	×	$\alpha \geq \frac{1}{2}$
○	×	×	○	$\frac{1}{4} \leq \alpha < 1$
○	×	×	×	unknown
	○ : valid		× : invalid	

mentioned earlier, it is a kind of special case, namely $\chi = 1$ and $\kappa(c) = c$. In our case χ and κ are a bit more general and we do not know whether or not our restriction $\alpha \geq 1/4$ can be extended to $1/7$.

So far, we have considered the three dimensional case. If the spatial domain is two dimensional plane, then the fluid equations can be extended to the Navier-Stokes equations, i.e. $\tau = 1$ and the existence of bounded weak solutions can be proved under a weaker assumption on α. To be more precise, we obtain the third main result:

Theorem 1.8. *Let $d = 2$ and $\tau = 1$. Suppose that κ satisfies the hypothesis* (A) *and the initial datum (n_0, c_0, u_0) satisfies* (4) *as well as* (3). *Assume further that one of the following holds*:

(i) *χ satisfies* (B1) *and $\alpha > 0$.*
(ii) *κ satisfies* (B3) *and $0 < \alpha < 1$.*

Then, there exists a bounded weak solution (n, c, u) for the system (1). *Moreover, if the initial mass $\|n_0\|_{L^1(\mathbb{R}^d)}$ is sufficiently small, then the limiting case $\alpha = 0$ can be included in the hypothesis* (i) *and* (ii).

We remark that, as previously stated, global existence of bounded solution was obtained in [13] regarding the case $\alpha > 0$ on a bounded domain in $\mathbb{R}^2$, in case that the fluid equation is the Stokes system. Compared to [13], our result is for the case of whole space with different assumptions on χ and κ and, in addition, we treat the case of the Navier-Stokes equations in $\mathbb{R}^2$.

This paper is organized as follows. In Section 2, some estimates of solutions of an approximate system of (1) are given. Section 3 is devoted to present the proofs of the main results.

2. Some estimates of solutions of a regularized system

Throughout this section, we study the solutions of the approximate problem of (1) given by

$$\begin{cases} \partial_t n_\varrho + u_\varrho \cdot \nabla n_\varrho = \Delta(n_\varrho + \varrho)^{1+\alpha} - \nabla \cdot (\chi(c_\varrho) n_\varrho \nabla c_\varrho)\,, \\ \partial_t c_\varrho + u_\varrho \cdot \nabla c_\varrho = \Delta c_\varrho - \kappa(c_\varrho) n_\varrho, \\ \partial_t u_\varrho + \tau(u_\varrho \cdot \nabla) u_\varrho + \nabla p_\varrho = \Delta u_\varrho - n_\varrho \nabla\phi, \qquad \operatorname{div} u_\varrho = 0, \end{cases} \tag{5}$$

in $\mathbb{R}^d \times (0,T)$ with smooth initial data $(n_{0\varrho}, c_{0\varrho}, u_{0\varrho})$, where $\varrho \in (0,1)$.

It is known that, due to the standard theory of existence and regularity as done in [4] and [13], there exists a classical solution of the equation (5) locally in time for each $\varrho \in (0,1)$. The main objective of this section is to derive appropriate uniform estimates, independent of ϱ, of the solutions. The estimates are crucially used in Section 3 to extend the above local solution to any given time interval $(0,T)$ and to construct the weak solutions and the bounded weak solutions of the equation (1).

We start with some notations. Let Ω be an open domain in $\mathbb{R}^d$ with $d = 2,3$. For $1 \le q \le \infty$, we denote by $W^{k,q}(\Omega)$ the usual Sobolev spaces, namely $W^{k,q}(\Omega) = \{f \in L^q(\Omega) : D^\alpha f \in L^q(\Omega), 0 \le |\alpha| \le k\}$. The set of q-th power Lebesgue integrable functions on Ω is denoted by $L^q(\Omega)$. In what follows, for simplicity, $\|\cdot\|_p$ denotes $\|\cdot\|_{L^p(\mathbb{R}^d)}$ for $1 \le p \le \infty$, unless there is any confusion to be expected. We also denote by $W^{-k,q'}(\Omega)$ dual space of $W^{k,q}_0(\Omega)$, where q and q' are Hölder conjugates. The letter $C = C(*,\dots,*)$ is used to represent a generic constant, depending on $*,\dots,*$, which may change from line to line.

2.1. Uniform estimates for weak solutions

In the following lemma, we give an estimate of solutions of (5) under the hypothesis (A) and the Assumption 1.2. The estimate is used in Section 3 to construct the weak solution of the equation (1). For the sake of simplicity, throughout Section 2.1 $\sim$ 2.3, we denote n_ϱ, c_ϱ and u_ϱ by n, c and u. Also, we define functionals $E(t)$ and $D(t)$ as follows:

$$E(t) := \int_{\mathbb{R}^d} n(t)\,(\log n(t) + 2\langle x\rangle)\,dx + \|\nabla c(t)\|_2^2 + \|\omega(t)\|_2^2 \tag{6}$$

and

$$D(t) := \|\nabla n(t)^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla^2 c(t)\|_2^2 + \|\nabla\omega(t)\|_2^2, \tag{7}$$

where $\langle x\rangle = (1+|x|^2)^{\frac12}$ and $\omega := \nabla \times u$.

Lemma 2.1. *Let $d = 3$, $\tau = 0$ and $T > 0$ be given. Suppose that (n, c, u) is a classical solution for the system $(5)_\varrho$, $\varrho \in (0, 1)$ with the smooth initial datum $(n_{0\varrho}, c_{0\varrho}, u_{0\varrho})$ satisfies the initial condition (3). Assume further that κ satisfies the hypothesis (A) and the Assumption 1.2 holds. Then, for any $0 < t \le T$, $E(t)$ and $D(t)$ defined in (6)-(7) satisfy*

$$\frac{d}{dt}E(t) + D(t) < CE(t). \tag{8}$$

Moreover, it satisfies the following energy inequality

$$\begin{aligned} &\int_{R^3} \left(n(|\log n| + 2\langle x\rangle) + \frac{|\nabla c|^2}{2} + \frac{|\omega|^2}{2} \right) dx \\ &+ \int_0^T \|\nabla \omega\|_{L^2}^2 + \|\nabla n^{\frac{1+\alpha}{2}}\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 dt \le C, \end{aligned} \tag{9}$$

with $C = C(T, \|c_{0\varrho}\|_{L^\infty},\ \|\langle x\rangle n_{0\varrho}\|_{L^1},\ \|\nabla c_{0\varrho}\|_{L^2},\ \|n_{0\varrho}|\ln n_{0\varrho}|\|_{L^1},\ \|\nabla\phi\|_{L^\infty},$ $\|\phi\|_{L^\infty})$. In addition, if the initial mass $\|n_{0\varrho}\|_{L^1(\mathbb{R}^d)}$ is sufficiently small, then the limiting case $\alpha = 1/3$ or $\alpha = 1/6$ can be included in the above hypothesis (i) $\sim$ (iv) *of Assumption 1.2.*

Proof. We first consider the case (i) in the Assumption 1.2. We observe, due to maximum principle, that $\|c\|_{L^\infty(Q_T)} \le \|c_0\|_{L^\infty(\mathbb{R}^d)}$, where $Q_T = \mathbb{R}^d \times [0, T)$. We also note that the total mass of n is preserved, i.e., $\|n(t)\|_{L^1(\mathbb{R}^d)} \equiv \|n_0\|_{L^1(\mathbb{R}^d)}$. Multiplying $(5)_1$ with $(1 + \log n)$ and integrating it parts, we have

$$\frac{d}{dt}\int_{\mathbb{R}^3} n \log n dx + \int_{\mathbb{R}^3} \nabla \log n \cdot \nabla (n + \varrho)^{1+\alpha} dx = \int_{\mathbb{R}^3} \nabla n \cdot (\chi(c)\nabla c)\, dx.$$

Since $\nabla \log n \cdot \nabla n = 4|\nabla n^{1/2}|^2$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla \log n \cdot \nabla (n + \varrho)^{1+\alpha} dx &= \int_{\mathbb{R}^3} \nabla \log n \cdot (1 + \alpha)(n + \varrho)^\alpha \nabla n\ dx \\ &\ge \int_{\mathbb{R}^3} \nabla \log n \cdot (1 + \alpha) n^\alpha \nabla n\ dx \\ &= \frac{4}{1 + \alpha} \left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2. \end{aligned}$$

Thus we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^3} n \log n dx + \frac{4}{1+\alpha}\left\| \nabla n^{\frac{1+\alpha}{2}} \right\|_2^2 &\le \int_{\mathbb{R}^3} \nabla n \cdot (\chi(c)\nabla c)\, dx \\ &= -\int_{\mathbb{R}^3} n\chi'(c)\,|\nabla c|^2\, dx - \int_{\mathbb{R}^3} n\chi(c)\Delta c\ dx. \end{aligned}$$

Note that using Young's inequality, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3} n \log n + \frac{4}{1+\alpha}\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 < \sup(|\chi(c)|)\left(\epsilon_1 \|\Delta c\|_2^2 + C(\epsilon_1)\|n\|_2^2\right), \tag{10}$$

where $\epsilon_1 > 0$ is a small number, which will be specified later. On the other hands, since $6/(2+3\alpha) < 2$ via $\alpha > 1/3$, it follows from the energy conservation of $n(x,t)$, Sobolev embedding and the Young's inequality that

$$\|n\|_2^2 \le C\|n\|_1^{\frac{1+3\alpha}{2+3\alpha}}\|\nabla n^{\frac{1+\alpha}{2}}\|_2^{\frac{6}{2+3\alpha}} \le \|n_0\|_1^{\frac{1+3\alpha}{2+3\alpha}}\left(C(\epsilon_2)+\epsilon_2\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2\right), \tag{11}$$

where $\epsilon_2 > 0$ is a small number depending on ϵ_1. Combining the above two results, we conclude that

$$\frac{d}{dt}\int_{\mathbb{R}^3} n\log n dx + \frac{4}{1+\alpha}\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 < C + \epsilon_1\|\Delta c\|_2^2 + C(\epsilon_1)\epsilon_2\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2. \tag{12}$$

Multiplying $(5)_2$ with $-\Delta c$ and integrating gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla c\|_2^2 + \|\Delta c\|_2^2 &= -\int_{\mathbb{R}^3}\nabla c\cdot\nabla(u\cdot\nabla c)dx + \int_{\mathbb{R}^3}\Delta c\kappa(c)n\ dx\\ &\le -\int_{\mathbb{R}^3}\nabla c\cdot\nabla(u\cdot\nabla c)dx + \sup(\kappa(c))\int_{\mathbb{R}^3}|n||\Delta c|dx\\ &:= (I)+(II).\end{aligned}$$

It follows from divergence free condition of u that

$$(I) = -\int_{\mathbb{R}^3}\sum_{i,j}\partial_i c\ \partial_j c\ \partial_i u_j = \int_{\mathbb{R}^3}\sum_{i,j} c\partial_j\partial_i c\ \partial_i u_j \le C_\epsilon\|\omega\|_2^2 + \epsilon\|\nabla^2 c\|_2^2. \tag{13}$$

Using the same method to (II) as in (12), we observe that

$$\frac{d}{dt}\|\nabla c\|_2^2 + \|\Delta c\|_2^2 \le C(\epsilon_4,\epsilon_5) + C(\epsilon_3)\|\omega\|_2^2 + (\epsilon_3+\epsilon_4)\|\nabla^2 c\|_2^2 + \epsilon_5\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2, \tag{14}$$

where ϵ_3, ϵ_4 and ϵ_5 are small numbers to be specified later. Multiplying ω with the vorticity equations for u and integrating it by parts gives

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \|\nabla\omega\|_2^2 \le \|\nabla\phi\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3}|n||\nabla\omega|dx. \tag{15}$$

Following similar procedures for deriving the estimate (12), we note that

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \|\nabla\omega\|_2^2 \le C(\epsilon_6,\epsilon_7) + \epsilon_6\|\nabla\omega\|_2^2 + \epsilon_7\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2, \tag{16}$$

where ϵ_6 and ϵ_7 are small numbers to be given later. Finally, in order to bound $\int n\log n$ in (12), multiply $(5)_1$ by the function $\langle x\rangle = (1+|x|^2)^{1/2}$ and integrate it. Then we have

$$\frac{d}{dt}\int_{\mathbb{R}^3}\langle x\rangle n dx = \int_{\mathbb{R}^3} nu\cdot\nabla\langle x\rangle dx + \int_{\mathbb{R}^3}\Delta(n+\varrho)^{1+\alpha}\langle x\rangle dx + \int_{\mathbb{R}^3}\nabla\langle x\rangle\cdot n\chi(c)\nabla c.$$

We estimate the second term, denote by J, in the righthand side. The integrating by parts gives

$$J := \int_{\mathbb{R}^3}\Delta(n+\varrho)^{1+\alpha}\langle x\rangle dx = -(\alpha+1)\int_{\mathbb{R}^3}(n+\varrho)^\alpha\nabla n\cdot\nabla\langle x\rangle dx. \tag{17}$$

In case that $0 < \alpha < 1$, using $(n+\varrho)^\ell \le C_\ell(n^\ell + \varrho^\ell)$ for any $\ell > 0$, we have

$$\begin{aligned}
J &= (\alpha+1)\alpha \int_{\mathbb{R}^3} (n+\varrho)^{\alpha-1} n \nabla n \cdot \nabla\langle x\rangle dx + (1+\alpha)\int_{\mathbb{R}^3} (n+\varrho)^\alpha n \Delta\langle x\rangle dx \\
&= (\alpha+1)\alpha \int_{\mathbb{R}^3} \left(\frac{n}{n+\varrho}\right)^{1-\alpha} n^\alpha \nabla n \cdot \nabla\langle x\rangle dx + (1+\alpha)\int_{\mathbb{R}^3} (n+\varrho)^\alpha n \Delta\langle x\rangle dx \\
&\le C(\alpha)\left(\int_{\mathbb{R}^3} \left| n^{\frac{1+\alpha}{2}} \nabla n^{\frac{1+\alpha}{2}} \right| |\nabla\langle x\rangle|\, dx + \int_{\mathbb{R}^3} (n^{1+\alpha} + n)\Delta\langle x\rangle dx\right),
\end{aligned}$$

where we used that $\Delta\langle x\rangle > 0$. On the other hands, in case that $\alpha \ge 1$, taking integration by part to (17) for $k-1$ times with $k = [1+\alpha]$, we have

$$\begin{aligned}
J = (-1)^k \binom{\alpha+1}{k} \int_{\mathbb{R}^3} (n+\varrho)^{\alpha-k+1} \nabla n^k \nabla\langle x\rangle dx \\
+ \sum_{j=1}^{k-1} (-1)^j \binom{\alpha+1}{j} \int_{\mathbb{R}^3} (n+\varrho)^{\alpha-j+1} n^j \Delta\langle x\rangle dx.
\end{aligned}$$

Hence, using Young's inequality and $\Delta\langle x\rangle > 0$, we obtain

$$\begin{aligned}
J &\le C(\alpha)\left(\int_{\mathbb{R}^3} \left|\nabla n^{1+\alpha} + \nabla n^k\right| |\nabla\langle x\rangle|\, dx + \int_{\mathbb{R}^3} (n^{1+\alpha}+n)\Delta\langle x\rangle dx\right) \\
&\le C(\alpha)\left(\int_{\mathbb{R}^3} \left(\left|n^{\frac{1+\alpha}{2}}\nabla n^{\frac{1+\alpha}{2}}\right| + \left|n^{k-\frac{1+\alpha}{2}}\nabla n^{\frac{1+\alpha}{2}}\right|\right)|\nabla\langle x\rangle|\right. \\
&\qquad \left. + \int_{\mathbb{R}^3} (n^{1+\alpha}+n)\Delta\langle x\rangle\right).
\end{aligned}$$

Since $\nabla\langle x\rangle, \Delta\langle x\rangle \in L^\infty$ and $k \le 1+\alpha$, via $\|n\|_{1+\alpha}^{1+\alpha} \le \|n_0\|_1^{\frac{2+2\alpha}{2+3\alpha}} \left\|\nabla n^{\frac{1+\alpha}{2}}\right\|_2^{\frac{6\alpha}{3\alpha+2}}$ and Young's inequality, we have

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^3} \langle x\rangle n dx &= \int_{\mathbb{R}^3} nu\cdot\nabla\langle x\rangle + \int_{\mathbb{R}^3} (n+\varrho)^{1+\alpha}\Delta\langle x\rangle + \int_{\mathbb{R}^3} \nabla\langle x\rangle \cdot n\chi(c)\nabla c \\
&\le C\left(1 + \|u\|_2^2 + \|\nabla c\|_2^2\right) + \left(C(\epsilon_8) + \epsilon_8 \|\nabla n^{\frac{1+\alpha}{2}}\|_2^2\right), \qquad (18)
\end{aligned}$$

where ϵ_8 is a small number to be specified later. We then obtain the following estimate (see also [9] and [1]):

$$\int_{\mathbb{R}^3} n|\log n| dx \le \int_{\mathbb{R}^3} n\log n dx + 2\int_{\mathbb{R}^3} \langle x\rangle n dx + C. \qquad (19)$$

By adding (12), (14), (16) and (18) with the choices of sufficiently small $\epsilon_1, \ldots, \epsilon_8 > 0$, the estimate (8) is achieved. Combining Gronwall's inequality for (8) and the estimate (19), we obtain the energy inequality (9).

Suppose that $\alpha = 1/3$. We have the following estimate:

$$\|n\|_2^2 \le C\|n\|_1^{\frac{2}{3}} \|n\|_4^{\frac{4}{3}} \le C\|n_0\|_1^{\frac{2}{3}} \|\nabla n^{\frac{1+\alpha}{2}}\|_2^2. \qquad (20)$$

Using (20) instead of (11) and applying the same procedure what is used for the case $\alpha > 1/3$, we obtain the following estimates for $\alpha = 1/3$:

$$\frac{d}{dt}\int_{\mathbb{R}^3} n\log n dx + \frac{4}{1+\alpha}\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 < \epsilon_1\|\nabla^2 c\|_2^2 + C(\epsilon_1)\|n_0\|_1^{\frac{2}{3}}\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2, \tag{21}$$

$$\frac{d}{dt}\|\nabla c\|_2^2 + \|\Delta c\|_2^2 \le C(\epsilon_2)\|\omega\|_2^2 + (\epsilon_2+\epsilon_3)\|\Delta c\|_2^2 + C(\epsilon_3)\|n_0\|_1^{\frac{2}{3}}\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2, \tag{22}$$

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \|\nabla\omega\|_2^2 \le \epsilon_4\|\nabla\omega\|_2^2 + C(\epsilon_4)\|n_0\|_1^{\frac{2}{3}}\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2, \tag{23}$$

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3}\langle x\rangle n dx &= \int_{\mathbb{R}^3} nu\cdot\nabla\langle x\rangle dx + \int_{\mathbb{R}^3}\Delta(n+\varrho)^{1+\alpha}\langle x\rangle dx + \int_{\mathbb{R}^3}\nabla\langle x\rangle\cdot n\chi(c)\nabla c\\ &\le C + C(\epsilon_5)\|u\|_2^2 + \epsilon_5\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla c\|_2^2,\end{aligned} \tag{24}$$

$$\int_{\mathbb{R}^3} n|\log n|dx \le \int_{\mathbb{R}^3} n\log n dx + 2\int_{\mathbb{R}^3}\langle x\rangle n dx + C. \tag{25}$$

Therefore, if $\|n_0\|_{L^1}$ is sufficiently small, then by adding (21), (22), (23) and (24), with sufficient choices of small $\epsilon_1,\ldots,\epsilon_5 > 0$, the estimate (8) is obtained and so is the energy inequality (9). This completes the proof of the case (i).

Next we give the proof of the case (ii). Testing n^α to $(5)_1$ and integrating it by parts, we note that

$$\frac{1}{1+\alpha}\frac{d}{dt}\|n\|_{1+\alpha}^{1+\alpha} + \int_{\mathbb{R}^3}\nabla n^\alpha\cdot\nabla(n+\varrho)^{1+\alpha}dx = \int_{\mathbb{R}^3}\nabla n^{1+\alpha}\cdot(\chi(c)\nabla c)\,dx.$$

Since $\nabla n^\alpha\cdot\nabla n = (4\alpha/(1+\alpha)^2)|\nabla n^{(1+\alpha)/2}|^2 \ge 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla n^\alpha\cdot\nabla(n+\varrho)^{1+\alpha}dx &= \int_{\mathbb{R}^3}\nabla n^\alpha\cdot(1+\alpha)(n+\varrho)^\alpha\nabla n\ dx\\ &\ge \int_{\mathbb{R}^3}\nabla n^\alpha\cdot(1+\alpha)n^\alpha\nabla n\ dx\\ &= \frac{4\alpha(1+\alpha)}{1+2\alpha}\left\|\nabla n^{\frac{1+2\alpha}{2}}\right\|_2^2.\end{aligned}$$

Thus via Young's inequality and the boundedness of c, we have

$$\begin{aligned}&\frac{1}{1+\alpha}\frac{d}{dt}\|n\|_{1+\alpha}^{1+\alpha} + \frac{4\alpha(1+\alpha)}{1+2\alpha}\|\nabla n^{\frac{1+2\alpha}{2}}\|_2^2\\ &\le \int_{\mathbb{R}^3}\nabla n^{1+\alpha}\cdot\chi(c)\nabla c dx\\ &\le \epsilon_1\|\nabla n^{\frac{1+2\alpha}{2}}\|_2^2 + C(\epsilon_1)\|n^{\frac{1}{2}}\nabla c\|_2^2,\end{aligned} \tag{26}$$

where ϵ_1 is a small number to be specified later. Let $M(\epsilon_1)$ be a positive constant satisfying that $M(\epsilon_1)\chi_0 > C(\epsilon_1)$. It follows by multiplying $(5)_1$ with $M(\epsilon_1)(1+\log n)$ and using the integration by parts that

$$\begin{aligned}
&\frac{d}{dt}M(\epsilon_1)\int_{\mathbb{R}^3} n\log n dx + \frac{4M(\epsilon_1)}{1+\alpha}\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 \\
&< M(\epsilon_1)\int_{\mathbb{R}^3}\nabla n\cdot(\chi(c)\nabla c)\,dx \\
&= -M(\epsilon_1)\int_{\mathbb{R}^3} n\chi'(c)\,|\nabla c|^2\,dx - M(\epsilon_1)\int_{\mathbb{R}^3} n\chi(c)\Delta c\;dx.
\end{aligned}$$

We then obtain via Young's inequality that

$$\begin{aligned}
&\frac{d}{dt}M(\epsilon_1)\int_{\mathbb{R}^3} n\log n\;dx + \frac{4M(\epsilon_1)}{1+\alpha}\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 + M(\epsilon_1)\chi_0\|n^{\frac{1}{2}}\nabla c\|_2^2 \\
&< M(\epsilon_1)\sup(|\chi(c)|)\left(\epsilon_2\|\Delta c\|_2^2 + C(\epsilon_2)\|n\|_2^2\right)
\end{aligned} \tag{27}$$

for small ϵ_2 to be specified later. On the other hand, since $\alpha > 1/6$, it follows that $3/(1+3\alpha) < 2$ and so we have for any small $\epsilon > 0$,

$$\begin{aligned}
\|n\|_2^2 &\le C\|n\|_1^{\frac{1+6\alpha}{2+6\alpha}}\|n\|_{3+6\alpha}^{\frac{3+6\alpha}{2+6\alpha}} \\
&\le C\|n_0\|_1^{\frac{1+6\alpha}{2+6\alpha}}\|\nabla n^{\frac{1+2\alpha}{2}}\|_2^{\frac{3}{1+3\alpha}} \\
&\le \|n_0\|_1^{\frac{1+6\alpha}{2+6\alpha}}\left(C(\epsilon_3) + \epsilon_3\|\nabla n^{\frac{1+2\alpha}{2}}\|_2^2\right)
\end{aligned} \tag{28}$$

for small ϵ_3, which will be specified with respect to ϵ_1 and ϵ_2 later. Combining (27) and (28), we note that

$$\begin{aligned}
&\frac{d}{dt}M(\epsilon_1)\int_{\mathbb{R}^3} n\log n dx + \frac{4M(\epsilon_1)}{1+\alpha}\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 + M_{\epsilon_1}\chi_0\|n^{\frac{1}{2}}\nabla c\|_2^2 \\
&< C + \epsilon_2 M(\epsilon_1)\|\Delta c\|_2^2 + C(\epsilon_1,\epsilon_2)\epsilon_3\|\nabla n^{\frac{1+2\alpha}{2}}\|_2^2.
\end{aligned} \tag{29}$$

Similarly as in (14), we can see that

$$\frac{d}{dt}\|\nabla c\|_2^2 + \|\Delta c\|_2^2 \le C(\epsilon_5,\epsilon_6) + C(\epsilon_4)\|\omega\|_2^2 + (\epsilon_4+\epsilon_5)\|\Delta c\|_2^2 + \epsilon_6\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 \tag{30}$$

for small ϵ_4, ϵ_5 and ϵ_6 to be specified later. As similarly in (16), (18) and (19), with the aid of (28) it follows that

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \|\nabla\omega\|_2^2 \le C(\epsilon_7,\epsilon_8) + \epsilon_7\|\nabla\omega\|_2^2 + \epsilon_8\|\nabla n^{\frac{1+2\alpha}{2}}\|_2^2, \tag{31}$$

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^3}\langle x\rangle n dx &= \int_{\mathbb{R}^3} nu\cdot\nabla\langle x\rangle dx + \int_{\mathbb{R}^3}\Delta(n+\varrho)^{1+\alpha}\langle x\rangle dx + \int_{\mathbb{R}^3}\nabla\langle x\rangle\cdot n\chi(c)\nabla c \\
&\le C\left(1+\|u\|_2^2+\|\nabla c\|_2^2\right) + \left(C(\epsilon_9) + \epsilon_9\|\nabla n^{\frac{1+2\alpha}{2}}\|_2^2\right),
\end{aligned} \tag{32}$$

$$\int_{\mathbb{R}^3} n|\log n|dx \le \int_{\mathbb{R}^3} n\log n dx + 2\int_{\mathbb{R}^3}\langle x\rangle n dx + C \tag{33}$$

for small ϵ_7, ϵ_8 and ϵ_9 to be specified later. Summing up all estimates with sufficient choices of $\epsilon_1, \dots, \epsilon_9 > 0$, we obtain the estimate (8) and the energy inequality (9). When $\|n_0\|_{L^1}$ is sufficiently small, the limiting case $\alpha = 1/6$ can be included. Since its verification is similar as in the case (i), the details are omitted.

Next we consider the case (iii). Reminding $\alpha < 1$ and testing $(1 + \log n)$ to $(5)_1$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3} n \log n dx + \frac{4}{1+\alpha}\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 \\ &\leq \int_{\mathbb{R}^3} \nabla n \cdot (\chi(c)\nabla c)\, dx \\ &\leq \sup|\chi(c)| \int_{\mathbb{R}^3} n^{\frac{1-\alpha}{2}} |\nabla n^{\frac{1+\alpha}{2}}||\nabla c| dx \\ &\leq C\int_{\mathbb{R}^3} \left(C(\epsilon_1) + \epsilon_1 n^{\frac{1}{2}}\right)|\nabla n^{\frac{1+\alpha}{2}}||\nabla c| dx \end{aligned}$$

for small ϵ_1 to be specified later. Therefore we note that

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3} n \log n dx + \frac{4}{1+\alpha}\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 \\ &\leq \epsilon_2 C(\epsilon_1)\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 + C(\epsilon_1,\epsilon_2)\|\nabla c\|_2^2 + C\epsilon_1\left(\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 + \|n^{\frac{1}{2}}\nabla c\|_2^2\right) \end{aligned} \tag{34}$$

for small number ϵ_2 depending on ϵ_1. Integrating it by parts after multiplying $(5)_1$ with n^α and applying the same procedures as in (26), we obtain that

$$\begin{aligned} &\frac{1}{1+\alpha}\frac{d}{dt}\|n\|_{1+\alpha}^{1+\alpha} + \frac{4\alpha(1+\alpha)}{1+2\alpha}\|\nabla n^{\frac{1+2\alpha}{2}}\|_2^2 \\ &\leq \epsilon_3\|\nabla n^{\frac{1+2\alpha}{2}}\|_2^2 + C(\epsilon_3)\|n^{\frac{1}{2}}\nabla c\|_2^2 \end{aligned} \tag{35}$$

for small ϵ_3 to be specified later. Multiplying $(5)_2$ with $-\Delta c$ and integrating it by parts gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla c\|_2^2 + \|\Delta c\|_2^2 = -\int_{\mathbb{R}^3} \nabla c\cdot\nabla(u\cdot\nabla c)dx - \int_{\mathbb{R}^3}\nabla c\cdot\nabla(n\kappa(c))dx = (I) + (II).$$

With the aid of $\operatorname{div} u = 0$, the first term (I) is estimated as follows:

$$(I) = -\int_{\mathbb{R}^3}\sum_{i,j}\partial_i c\, \partial_j c\, \partial_i u_j dx \leq C(\epsilon_4)\|\omega\|_2^2 + \epsilon_4\|\Delta c\|_2^2$$

for small ϵ_4 to be specified later. On the other hand, for the second term (II) we compute

$$(II) \leq -\int_{\mathbb{R}^3}\kappa'(c) n|\nabla c|^2 dx + \int_{\mathbb{R}^3}\kappa(c)|\nabla c||\nabla n| dx = -(II_1) + (II_2).$$

Keeping in mind that $\alpha < 1$ and continuing to compute (II_2), we obtain

$$(II_2) = \int_{\mathbb{R}^3}\kappa(c)\frac{2}{1+\alpha} n^{\frac{1-\alpha}{2}}|\nabla n^{\frac{1+\alpha}{2}}||\nabla c| dx$$

$$
\begin{aligned}
&\leq \sup|\kappa(c)| \int_{\mathbb{R}^3} \left(C(\epsilon_5) + \epsilon_5 n^{\frac{1}{2}}\right) |\nabla n^{\frac{1+\alpha}{2}}||\nabla c| dx \\
&\leq C(\epsilon_5)\epsilon_6 \|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 + C(\epsilon_5, \epsilon_6)\|\nabla c\|_2^2 + C\epsilon_5 \left(\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 + \|n^{\frac{1}{2}}\nabla c\|_2^2\right)
\end{aligned}
$$

for small ϵ_6 depending on ϵ_5. Summing up above estimates, we observe that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla c\|_2^2 + \|\Delta c\|_2^2 + \kappa_0 \|n^{\frac{1}{2}}\nabla c\|_2^2 \\
&\leq C(\epsilon_4)\|\omega\|_2^2 + \epsilon_4 \|\Delta c\|_2^2 + C(\epsilon_5)\|\nabla c\|_2^2 \\
&\quad + C(\epsilon_5)\cdot(\epsilon_5 + \epsilon_6)\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 + C\cdot \epsilon_5 \|n^{\frac{1}{2}}\nabla c\|_2^2.
\end{aligned} \tag{36}
$$

As in previous case, choosing sufficient small $\epsilon_1, \ldots, \epsilon_6$, we can derive the estimates (31), (32) and (33) and since computations are the same as before, we skip the details. Adding up all estimates leads to energy inequality and thus proof is completed. For the limiting case $\alpha = 1/6$ with sufficiently small $\|n_0\|_{L^1}$, we also skip its details, because its verification is rather straightforward.

The case (iv) is a direct consequence of the cases (i) and (iii) and therefore, we completes the proof. □

2.2. Uniform estimates for bounded weak solutions

In this subsection, we provide some uniform estimates of solutions to the system (5) under the hypothesis (A) and the Assumption 1.3. The estimate is used in Section 3 to construct the bounded weak solution of the equation (1). To be more precise, our result is read as follows:

Lemma 2.2. *Let $d = 3$, $\tau = 0$ and $T > 0$ be given. Suppose that (n, c, u) is a classical solution for the system $(5)_\varrho$, $\varrho \in (0, 1)$ with the smooth initial datum $(n_{0\varrho}, c_{0\varrho}, u_{0\varrho})$ satisfies the initial condition* (3) *and* (4)*. Assume further that κ satisfies the hypothesis* (A) *and the Assumption 1.3 holds. Then, for any $0 < t \leq T$*

$$
n \in L^\infty(0, T; L^p(\mathbb{R}^d)), \qquad 1 \leq p \leq \infty, \tag{37}
$$

$$
\nabla n^{\frac{p+\alpha}{2}} \in L^2(0, T; L^2(\mathbb{R}^d)), \qquad 2 \leq p < \infty, \tag{38}
$$

$$
c, u \in L^\infty(0, T; W^{1,q}(\mathbb{R}^d)) \cap L^q(0, T; W^{2,q}(\mathbb{R}^d)), \qquad 2 \leq q < \infty, \tag{39}
$$

$$
c_t, u_t \in L^q(0, T; L^q(\mathbb{R}^d)), \qquad 2 \leq q < \infty. \tag{40}
$$

Proof. We first show that for each $p > 1$, there exists constant $C(p, T)$ such that $\|n(t)\|_p \leq C(p, T)$ for all $t \in (0, T)$. We recall $\int_0^T \|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 < \infty$ from Lemma 2.1. Multiplying $(5)_1$ with n^{p-1} $(p > 1 + \alpha)$ and integrating it by parts implies that

$$
\frac{1}{p}\frac{d}{dt}\|n\|_p^p + \int_{\mathbb{R}^3} \nabla n^{p-1} \cdot \nabla (n + \varrho)^p dx = \frac{p-1}{p}\int_{\mathbb{R}^3} \nabla n^p \cdot (\chi(c)\nabla c)\, dx.
$$

Since $\nabla n^{p-1} \cdot \nabla n = (4(p-1)/p^2)|\nabla n^{p/2}|^2 \geq 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla n^{p-1} \cdot \nabla (n+\varrho)^{1+\alpha} dx &= \int_{\mathbb{R}^3} \nabla n^{p-1} \cdot (1+\alpha)(n+\varrho)^{\alpha} \nabla n \; dx \\ &\geq \int_{\mathbb{R}^3} \nabla n^{p-1} \cdot (1+\alpha) n^{\alpha} \nabla n \; dx \\ &= \frac{4(p-1)(1+\alpha)}{(p+\alpha)^2} \left\| \nabla n^{\frac{p+\alpha}{2}} \right\|_2^2 .\end{aligned}$$

Thus we have

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|n\|_p^p + \frac{4(p-1)(1+\alpha)}{(p+\alpha)^2}\|\nabla n^{\frac{p+\alpha}{2}}\|_2^2 \\ &\leq \int_{\mathbb{R}^3} \nabla n^p \cdot \chi(c) \nabla c dx \\ &= -\int_{\mathbb{R}^3} n^p \chi'(c) \left|\nabla c\right|^2 dx - \int_{\mathbb{R}^3} n^p \chi(c) \Delta c \; dx \leq \overline{\chi} \int_{\mathbb{R}^3} n^p |\Delta c| dx,\end{aligned} \tag{41}$$

where $\overline{\chi} := \sup|\chi(c)|$. Here we used that $-\int_{\mathbb{R}^3} n^p \chi'(c) \left|\nabla c\right|^2 dx$ in (41) is non-positive. We take $r_0 > 3$ and put $\ell = 2r_0^*/(3-r_0^*)$, where $r_0^* \in (1, \frac{3}{2})$ is the Hölder conjugate of r_0. Later, r_0 will be taken as $3(1+2\alpha)$ or $3(1+\alpha)$. We note that $1 < \ell < 2$. Via Sobolev imbedding inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^3} n^p |\Delta c| dx &\leq \|\Delta c\|_{r_0} \|n\|_{pr_0^*}^p \leq \|\Delta c\|_{r_0} \left[\; \|n\|_p^{\frac{1}{\ell}} \|n\|_{3p}^{\frac{\ell-1}{\ell}} \; \right]^p \\ &\leq \left[\; \|\Delta c\|_{r_0}^{\ell} \|n\|_p^p \; \right]^{\frac{1}{\ell}} \cdot \left[\; \|\nabla n^{\frac{p}{2}}\|_2^2 \; \right]^{\frac{\ell-1}{\ell}} .\end{aligned} \tag{42}$$

We note that for $0 < r < s < t < \infty$ and for any f with f^r, $f^s \in H^1$

$$\|\nabla f^s\|_2 \leq \frac{s}{r^{\theta} t^{1-\theta}} \|\nabla f^r\|_2^{\theta} \|\nabla f^t\|_2^{1-\theta},$$

where $\theta \in (0,1)$ satisfies $s = r\theta + t(1-\theta)$. Applying the above interpolation inequality, we obtain

$$\|\nabla n^{\frac{p}{2}}\|_2^2 \leq C_p \left(\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla n^{\frac{p+\alpha}{2}}\|_2^2 \right), \tag{43}$$

where

$$C_p = \frac{p}{4} \left(\frac{1+\alpha}{2} \right)^{-\frac{\alpha}{p-1}} \left(\frac{p+\alpha}{2} \right)^{-1+\frac{\alpha}{p-1}} . \tag{44}$$

It is easy to see that $\{C_p \;:\; p > 0\}$ is bounded. It follows from (41)∼(44) and Young's inequality that

$$\int_{\mathbb{R}^n} n^p |\Delta c| dx \leq M_p \|\Delta c\|_{r_0}^{\ell} \|n\|_p^p + \frac{1}{\overline{\chi}} \cdot \frac{p\alpha}{(p+\alpha)^2} \left[\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 + \|\nabla n^{\frac{p+\alpha}{2}}\|_2^2 \right],$$

where

$$M_p = \frac{1}{\ell} \left[\frac{p\alpha}{(p+\alpha)^2} \cdot \frac{\ell-1}{\ell} \cdot \frac{1}{C_p} \cdot \frac{1}{\overline{\chi}} \right]^{1-\ell} \leq \left[\overline{\chi} \cdot C_p \frac{(p+\alpha)^2}{p\alpha} \right]^{\ell-1} .$$

Thus we have

$$\frac{d}{dt}\|n\|_p^p \le \overline{\chi} p M_p \|\Delta c\|_{r_0}^{\ell} \|n\|_p^p + p\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2,$$

which implies

$$\|n\|_p^p \le \left[\|n_0\|_p^p + Cp\right] \exp\left(\overline{\chi} p M_p \int_0^t \|\Delta c(\tau)\|_{r_0}^{\ell} d\tau\right)$$

where $C = \int_0^T \left\|\nabla n^{\frac{1+\alpha}{2}}\right\|$. Hence we obtain

$$\|n\|_p \le C_1 \exp\left(C_2 M_p \int_0^t \|\Delta c(\tau)\|_{r_0}^{\ell} d\tau\right) + C_3$$

for some positive constants C_1, C_2 and $C_3 > 0$. Suppose the assumption (ii), (iii) or (iv) holds. For each case, it follows from Lemma 2.1 that

$$\int_0^T \|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 + \int_0^T \|\nabla n^{\frac{1+2\alpha}{2}}\|_2^2 < \infty. \tag{45}$$

We put $r_0 = 3(1+2\alpha)$ and we then note that $\ell = 2r_0^*/(3-r_0^*) = (2+4\alpha)/(1+4\alpha)$. It remains to show

$$\int_0^t \|\Delta c(\tau)\|_{r_0}^{\ell} d\tau < \infty, \quad t \in [0,T]. \tag{46}$$

We first consider the case $1/4 \le \alpha < 1/2$. Since

$$\frac{4}{1+4\alpha} \le 2, \quad r_0 < 6 \quad \text{and} \quad u \in L^\infty(0,T;L^6(\mathbb{R}^3)),$$

we obtain via mixed norm estimate of heat equation

$$\begin{aligned}
\int_0^T \|\nabla^2 c\|_{r_0}^{\ell} &\le M_1 \int_0^T \left(\|n\|_{r_0}^{\ell} + \|u\cdot\nabla c\|_{r_0}^{\ell}\right) \\
&\le M_1 \int_0^T \|n^{\frac{1+2\alpha}{2}}\|_6^{\frac{4}{1+4\alpha}} + M_2 \int_0^T \|u\|_6^{\ell} \|\nabla c\|_{\frac{6r_0}{6-r_0}}^{\ell} \\
&\le M_3 \int_0^T \|\nabla n^{\frac{1+2\alpha}{2}}\|_2^{\frac{4}{1+4\alpha}} + M_4 \int_0^T \|\Delta c\|_{\frac{6r_0}{6+r_0}}^{\ell} \\
&:= M_3(I) + M_4(II).
\end{aligned} \tag{47}$$

We set $\delta = 4/(1+4\alpha)$. It follows from Young's inequality that there exists $C_\delta \ge 0$ ($C_\delta = 0$ if and only if $\delta = 2$) such that

$$(I) \le \int_0^T C_\delta + \|\nabla n^{\frac{1+2\alpha}{2}}\|_2^2 dt < \infty.$$

On the other hand, again by the mixed norm estimate of heat equation, we have

$$(II) \le M_5 \int_0^T \|n\|_{\frac{6r_0}{6+r_0}}^{\ell} + M_6 \int_0^T \|\nabla c\|_{r_0}^{\ell}.$$

Due to (8) and (9), we note that

$$\nabla c \in L^{\ell}(0,T;L^2(\mathbb{R}^3)) \cap L^{\ell}(0,T;L^6(\mathbb{R}^3)),$$
$$n \in L^{\ell}(0,T;L^1(\mathbb{R})) \cap L^{\ell}(0,T;L^{r_0}(\mathbb{R}^3)).$$

Therefore, we have $(II) < \infty$. Therefore (46) holds, which completes the proof for the case $1/4 \le \alpha < 1/2$.

For the case $\alpha \ge 1/2$, we note that, from (45), we have

$$n \in L^{1+2\alpha}(0,T;L^{r_0}(\mathbb{R}^3)) \subset L^2(0,T;L^{r_0}(\mathbb{R}^3)). \tag{48}$$

Considering the vorticity equations $\omega_t - \Delta\omega = -\nabla \times (n\nabla\phi)$, we can obtain L^p estimate for ω. To be more precise,

$$\begin{aligned}\frac{1}{r_0}\frac{d}{dt}\|\omega\|_{r_0}^{r_0} + \left\| |\omega|^{\frac{r_0-2}{2}} \nabla\omega \right\|_2^2 &\le C_1 \int_{\mathbb{R}^3} n\,|\omega|^{r_0-2}\,|\nabla\omega| \\ &\le C_2 \|n\|_{r_0} \|\omega\|_{r_0}^{\frac{r_0-2}{2}} \left\| |\omega|^{\frac{r_0-2}{2}} \nabla\omega \right\|_2 \\ &\le C_\epsilon \|n\|_{r_0}^2 \|\omega\|_{r_0}^{r_0-2} + \epsilon \left\| |\omega|^{\frac{r_0-2}{2}} \nabla\omega \right\|_2^2 .\end{aligned}$$

Thus we have

$$\|\omega\|_{r_0}^{r_0-1} \frac{d}{dt}\|\omega\|_{r_0} \le C_\epsilon \|n\|_{r_0}^2 \|\omega\|_{r_0}^{r_0-2},$$

which implies

$$\frac{d}{dt}\|\omega\|_{r_0}^2 \le C_\epsilon \|n\|_{r_0}^2.$$

Thus it follows from (48) that

$$u \in L^{\infty}(0,T;L^{\infty}(\mathbb{R}^3)). \tag{49}$$

Hence, together with (49), we notice that

$$\begin{aligned}\int_0^T \|\Delta c\|_{r_0}^{\ell} &\le M_1 \int_0^T \left(\|n\|_{r_0}^{\ell} + \|u \cdot \nabla c\|_{r_0}^{\ell} \right) \\ &\le M_2 \int_0^T \|\nabla n^{\frac{1+2\alpha}{2}}\|_2^2 + M_3 \int_0^T \|u\|_{\infty}^{\ell} \|\nabla c\|_{r_0}^{\ell} < \infty.\end{aligned}$$

Thus (46) is verified, which completes the proof for the case $\alpha \ge 1/2$.

Now, it remains to estimate for the case (i). We recall again due to result of Lemma 2.1 that $\int_0^T \|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 < \infty$. We put $r_0 = 3(1+\alpha)$ and $\ell = (2 + 2\alpha)/(1+2\alpha)$. Following similar procedures for proof of the previous case, we need to show only

$$\int_0^t \|\Delta c(\tau)\|_{r_0}^{\ell} d\tau < \infty, \quad t \in [0,T]. \tag{50}$$

Since $\alpha \ge 1/2$ and $\int_0^T \|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 < \infty$, we have

$$n \in L^{1+\alpha}(0,T;L^{r_0}(\mathbb{R}^3)) \subset L^2(0,T;L^{r_0}(\mathbb{R}^3)). \tag{51}$$

As proved earlier, L^p estimate of vorticity ω leads that

$$u \in L^\infty(0,T;L^\infty(\mathbb{R}^3)). \tag{52}$$

Noting that $4/(1+2\alpha) \le 2$ and using (52), we obtain

$$\begin{aligned}\int_0^T \|\Delta c\|_{r_0}^\ell &\le M_1 \int_0^T \left(\|n\|_{r_0}^\ell + \|u\cdot\nabla c\|_{r_0}^\ell\right) \\ &\le M_2 \int_0^T \|\nabla n^{\frac{1+2\alpha}{2}}\|_2^{\frac{4}{1+2\alpha}} + M_3 \int_0^T \|u\|_\infty^\ell \|\nabla c\|_{r_0}^\ell < \infty.\end{aligned}$$

This deduce the boundness of L^p-norm of n, for each $p > 1$. Now adapting the well known Moser-Alikakos iteration procedure, (see, for example, [13], Lemma 4.1) we finally conclude the boundedness of L^∞-norm of n. Regularity of c and u, i.e., (39) and (40), is direct consequence of mixed norm estimate of heat equation and Stokes system. This completes the proof. □

2.3. Two dimensional case

To prove Theorem 1.8, we need to derive a uniform estimate of the solution of the equation (1) in the two dimensional case. In this case, as mentioned earlier, the Navier-Stokes equations are considered and we obtain the following estimate.

Lemma 2.3. *Let $d = 2$, $\tau = 1$ and $T > 0$ be given. Suppose that (n,c,u) is a classical solution for the system $(5)_\varrho$, $\varrho \in (0,1)$ and the smooth initial datum $(n_{0\varrho}, c_{0\varrho}, u_{0\varrho})$ satisfies the initial condition* (3) *and* (4). *Assume further that κ satisfies the hypothesis* (A) *and one of the following conditions holds*:

(i) *χ satisfies* (B1) *and $\alpha > 0$.*
(ii) *κ satisfies* (B3) *and $0 < \alpha < 1$.*

Then $E(t)$ and $D(t)$ defined in (6)-(7) *satisfy*

$$\frac{d}{dt}E(t) + D(t) \le CE(t). \tag{53}$$

Moreover, it satisfies the following energy inequality

$$\begin{aligned}&\int_{R^3} \left(n(|\log n| + 2\langle x\rangle) + \frac{|\nabla c|^2}{2} + \frac{|\omega|^2}{2}\right) dx \\ &+ \int_0^T \|\nabla\omega\|_{L^2}^2 + \|\nabla n^{\frac{1+\alpha}{2}}\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 dt \le C\end{aligned} \tag{54}$$

with $C = C(T,\ \|c_{0\varrho}\|_{L^\infty},\ \|\langle x\rangle n_{0\varrho}\|_{L^1},\ \|\nabla c_{0\varrho}\|_{L^2},\ \|n_{0\varrho}|\ln n_{0\varrho}|\|_{L^1},\ \|\nabla\phi\|_{L^\infty},$ $\|\phi\|_{L^\infty})$. Moreover, for any $0 < t \le T$

$$n \in L^\infty(0,T;L^p(\mathbb{R}^d)), \qquad 1 \le p \le \infty, \tag{55}$$

$$\nabla n^{\frac{p+\alpha}{2}} \in L^2(0,T;L^2(\mathbb{R}^d)), \qquad 2 \le p < \infty, \tag{56}$$

$$c, u \in L^\infty(0,T;W^{1,q}(\mathbb{R}^d)) \cap L^q(0,T;W^{2,q}(\mathbb{R}^d)), \qquad 2 \le q < \infty, \tag{57}$$

$$c_t, u_t \in L^q(0,T;L^q(\mathbb{R}^d)), \qquad 2 \le q < \infty. \tag{58}$$

Moreover, if the initial mass $\|n_{0\varrho}\|_{L^1(\mathbb{R}^d)}$ *is sufficiently small, then the limiting case* $\alpha = 0$ *can be included in the above hypothesis* (i) *and* (ii).

Proof. In the case of two dimensions, almost all the procedures are similar to those that were used to prove the cases (B1) or (B3) shown in the proof of Lemma 2.1. Therefore, we just present main stream of proof and specify some distinct points compared to three dimensional case, instead giving all the details. In addition, we consider the only the case (i), since the other case (ii) can be argued similarly.

Multiplying $(5)_1$ with $\log n$ and following the same procedure as before, we lead to the estimate (10). Here we recall so called Ladyzhenskaya inequality in $\mathbb{R}^2$ (see [8]):

$$\|f\|_4^2 \le C\|f\|_2\|\nabla f^m\|_2^{\frac{1}{m}}, \quad 1 \le m < \infty. \tag{59}$$

With the aid of (59), we derive the following inequality in place of (11): instead of using the interpolation inequality which have been used to derive (11), to obtain the following alternative inequality

$$\|n\|_2^2 = \|n^{\frac{1}{2}}\|_4^2 \le C\|n^{\frac{1}{2}}\|_2^2\|\nabla n^{\frac{1+\alpha}{2}}\|_2^{1-\delta}, \quad \delta = \frac{\alpha-1}{\alpha+1}.$$

Therefore, there exist sufficiently small $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that

$$\frac{d}{dt}\int_{\mathbb{R}^3} n\log n dx + \frac{4}{1+\alpha}\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 < C_{\epsilon_1,\epsilon_2} + \epsilon_1\|\Delta c\|_2^2 + \epsilon_2\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2. \tag{60}$$

The estimates (14), (15), (18) and (19) are derived in the same manners as three dimensional case, and thus we omit the details. Summing up the estimates, we obtain (53) and (54). Multiplying $(5)_1$ with n^{p-1} $(1+\alpha < p < \infty)$ and using integration by parts, we arrive at the estimate (41). We then estimate the righthand side as follows (compare to (42)):

$$\begin{aligned}\int_{\mathbb{R}^2} n^p|\Delta c|dx \le \|\Delta c\|_2\|n\|_{2p}^p \le C\|\Delta c\|_2\|n^{\frac{p}{2}}\|_2\|\nabla n^{\frac{p}{2}}\|_2 \\ \le C_\epsilon\left(\|\Delta c\|_2^2\|n\|_p^p\right) + \epsilon\|\nabla n^{\frac{p}{2}}\|_2^2,\end{aligned}$$

where the inequality (59) is used. Via the fact that $\int_0^T \|\nabla n^{\frac{1+\alpha}{2}}\|_2^2 < \infty$ and similar procedure as in the proof of Lemma 2.2 with $\ell = r_0 = 2$, we finally have

$$\frac{1}{p}\frac{d}{dt}\|n\|_p^p < C_1\|\Delta c\|_2^2\|n\|_p^p + C_2\|\nabla n^{\frac{1+\alpha}{2}}\|_2^2$$

for some $C_1, C_2 > 0$, independent of p. Since regularity of (n,c,u) can be shown as the case of three dimensions, we omit the details. This completes the proof. $\square$

3. Proofs of main theorems

In this section, we present proofs of our main results - Theorem 1.5, Theorem 1.7 and Theorem 1.8. Construction of weak and bounded weak solutions is based on the uniform estimates established in the previous section. Since the argument is rather standard, we just give the sketch of how our construction goes, instead presenting all the details.

Proofs of Theorem 1.5, Theorem 1.7 and Theorem 1.8. First, we recall the regularized system (5) with the initial data $(n_{0\varrho}, c_{0,\varrho}, u_{0\varrho})$ which are chosen as smooth approximations of (n_0, c_0, u_0):

$$n_{0\varrho} = \psi_\varrho * n_0, \quad c_{0\varrho} = \psi_\varrho * c_0 \quad \text{and} \quad u_{0\varrho} = \psi_\varrho * u_0$$

where ϕ_ϱ denotes the usual mollifier. The convergence of $(n_{0\varrho}, c_{0\varrho}, u_{0\varrho})$ entails that the estimates obtained in Lemma 2.1, Lemma 2.2 and Lemma 2.3 are uniform, independent of ϱ, precisely, the constant C in (9) and (54) can be chosen independent of ϱ. Likewise, there exists a constant C such that for $2 \le q < \infty$

$$\|n_\varrho\|_{L^\infty((0,T)\times\Omega)} + \left\|\nabla n_\varrho^{\frac{q+\alpha}{2}}\right\|_{L^2((0,T)\times\Omega)} < C, \tag{61}$$

$$\|c_\varrho\|_{L^\infty(0,T;W^{1,q}(\mathbb{R}^d))} + \|c_\varrho\|_{L^q(0,T;W^{2,q}(\mathbb{R}^d))} + \|\partial_t c_\varrho\|_{L^q(0,T;L^q(\mathbb{R}^d))} < C, \tag{62}$$

$$\|u_\varrho\|_{L^\infty(0,T;W^{1,q}(\mathbb{R}^d))} + \|u_\varrho\|_{L^q(0,T;W^{2,q}(\mathbb{R}^d))} + \|\partial_t u_\varrho\|_{L^q(0,T;L^q(\mathbb{R}^d))} < C. \tag{63}$$

According to the estimates we have derived, a bootstrap argument can extend the local solution to any given time interval $(0, T)$ (see e.g. [4] and [13], or alternatively, [12] for more detail). Let k be any number with $k \ge 2 + \alpha$. We then show that $\partial_t n_\varrho$ and $\partial_t n_\varrho^k$ are, independent of ϱ, in $L^1(0, T; W^{-2,2}(\mathbb{R}^d))$, where $W^{-2,2}(\mathbb{R}^d)$ is the dual space of $W^{2,2}(\mathbb{R}^d)$ (compare to [13]). Then via Aubin-Lions Lemma, by passing to the limit, we have some weak limit (n, c, u), which turns out to be a weak solution. Its verification is rather straightforward, and thus the details are skipped. It is also direct that (n, c, u) is a bounded weak solution and satisfies the estimates (61)-(63). This completes the proof. □

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