J. Korean Math. Soc. **51** (2014), No. 3, pp. 545–565
http://dx.doi.org/10.4134/JKMS.2014.51.3.545

OPTIMAL L^2-ERROR ESTIMATES FOR EXPANDED MIXED FINITE ELEMENT METHODS OF SEMILINEAR SOBOLEV EQUATIONS

Mi Ray Ohm, Hyun Young Lee, and Jun Yong Shin

Abstract. In this paper we derive a priori $L^\infty(L^2)$ error estimates for expanded mixed finite element formulations of semilinear Sobolev equations. This formulation expands the standard mixed formulation in the sense that three variables, the scalar unknown, the gradient and the flux are explicitly treated. Based on this method we construct finite element semidiscrete approximations and fully discrete approximations of the semilinear Sobolev equations. We prove the existence of semidiscrete approximations of u, $-\nabla u$ and $-\nabla u - \nabla u_t$ and obtain the optimal order error estimates in the $L^\infty(L^2)$ norm. And also we construct the fully discrete approximations and analyze the optimal convergence of the approximations in $\ell^\infty(L^2)$ norm. Finally we also provide the computational results.

1. Introduction

Let Ω be an open bounded convex domain in R^d, $1 \le d \le 3$ with a boundary $\partial\Omega$ and let $0 < T < \infty$ be given. In this paper we consider the following semilinear Sobolev equation:

$$
\begin{aligned}
& u_t - \nabla \cdot (\nabla u + \nabla u_t) = f(x,t,u), && \text{in } \Omega \times (0,T], \\
& (\nabla u + \nabla u_t) \cdot \boldsymbol{n} = 0, && \text{on } \partial\Omega \times (0,T], \qquad (1.1)\\
& u(x,0) = u_0(x), && \text{on } \Omega,
\end{aligned}
$$

where $\boldsymbol{n}$ denotes the outward normal vector to $\partial\Omega$ and $u_0(x)$ and $f(x,t,u)$ are given functions assumed to be sufficiently smooth so that (1.1) has a unique sufficiently smooth solution. The problem (1.1) represents natural phenomena appearing in the research of the flow of fluids through fissured materials [4], thermodynamics [6] and other areas. For details about the physical significance and the existence and uniqueness of the solutions of the Sobolev equations, see

Received September 16, 2013; Revised January 13, 2014.
2010 *Mathematics Subject Classification.* 65M15, 65N30.
Key words and phrases. semilinear Sobolev equations, expanded mixed finite element method, semidiscrete approximations, fully discrete approximations, computational results.
This research was supported by Kyungsung University Research Grants in 2014.

[4, 5, 6, 12, 16, 28]. In the references [4, 5, 6, 12, 16, 28], we also find that the study of the properties of the equations (1.1) contributes to the development of the mathematical theories for the inverse problem of the heat equations.

In the past, several mathematicians [2, 3, 15, 17, 19, 20, 21, 26, 27] applied the classical Galerkin finite element method or discontinuous Galerkin method to construct the approximations of the scalar unknown $u(x,t)$ of the Sobolev equations combined with the various types of boundary conditions with $1 \leq d \leq 3$.

Compared with the classical Galerkin finite element method, the advantage of mixed finite element formulations is that one can simultaneously approximate both the displacement and the stress or the pressure and the flux. Another advantage of this procedure is that the flux or the stress can be approximated to the same order of convergence as the unknown scalar $u(x,t)$ itself. Recently due to these advantages, the authors [23, 25] have applied the mixed finite element method (MFEM) to some types of the Sobolev equations, construct the numerical solutions of u and the flux term and proved the optimal order of convergence. By implementing the standard mixed finite element method we may approximate simultaneously the unknown $u(x)$ and the flux term of the form $a(x)u(x)$[13, 14]. However in the case that $a(x)$ is small which may occur in many circumstances, $a(x)$ is not readily to be inverted to compute ∇u. Motivated by this, Wheeler et al [29] and Arbogast et al [1] proposed an expanded mixed finite element method.

EMFEM expands the classical MFEM in the sense that the scalar unknown, the gradient and the flux are separately treated, so that three variables can be approximated directly. Chen [7] also Independently developed expanded mixed method based on BDM method for elliptic problem. Chen [8, 9] analyzed the error analysis of the expanded mixed method for second-order elliptic problems. Adopting this method Woodward and Dawson [30] approximate the solution of Richards' equation. In the several literatures such as [10, 11, 18], the authors tried to apply an EMFEM to approximate the three variables corresponding to elliptic equations and semilinear reaction-diffusion equations.

Furthermore, in the case that the flux term contains the mixed derivative with respect to the spatial variable and temporal variable such as the problem (1.1), the classical MFEM is not useful to approximate the gradient from the flux. In this paper, we apply an expanded mixed finite element method (EMFEM) and construct semidiscrete approximations and fully discrete approximations of u, $-\nabla u$ and $-\nabla u - \nabla u_t$, respectively.

To approximate ∇u and $\nabla u + \nabla u_t$, instead of computing the derivatives of u_h, we construct the approximations of ∇u and $\nabla u + \nabla u_t$ directly, to obtain the optimal convergence results for ∇u and $\nabla u + \nabla u_t$. Compared to the standard mixed finite element method, our expanded mixed method does not require the LBB condition. The LBB condition is needed in the process of constructing finite element spaces, so that it confines the construction of finite element spaces. As far as we know, this paper will be the first trial to estimate both semidiscrete and fully discrete approximations using an expanded mixed finite element methods for the Sobolev equations and obtain the optimal L^2 error estimates. This paper is organized as follows. In Section 2, we introduce some notations and preliminaries. Next we construct finite element spaces and we construct the weak formulation of (1.1). Then in Section 3, we introduce our expanded mixed formulation, construct semidiscrete approximations and prove the existence of semidiscrete approximations. The results of the optimal error estimates of ∇u and $\nabla u + \nabla u_t$ as well as u in $L^\infty(L^2(\Omega))$ normed space are derived. In Section 4, we formulate the expanded fully discrete finite element approximations and analyze the optimal error estimates in $\ell^\infty(L^2)$ norm. Finally in Section 5, we provide the computational results to support our theoretical analysis suggested in Section 4. Throughout this paper, the vectors will be denoted by the bold face.

2. Finite element spaces

For an $s \geq 0$, $1 \leq p \leq \infty$ and Ω, we denote by $W^{s,p}(\Omega)$ the Sobolev space equipped with the usual Sobolev norm $\|u\|_{s,p}^p = \sum_{|\boldsymbol{k}|\leq s}\int_\Omega |D^{\boldsymbol{k}}u|^p dx$ where $\boldsymbol{k} = (k_1, k_2, \ldots, k_d)$, $|\boldsymbol{k}| = k_1 + k_2 + \cdots + k_d$, $D^{\boldsymbol{k}}u = \frac{\partial^{|\boldsymbol{k}|}u}{\partial x_1^{k_1}\partial x_2^{k_2}\cdots\partial x_d^{k_d}}$, and k_i is a nonnegative integer for each $1 \leq i \leq d$. For simplicity we denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$. Let $\boldsymbol{H^s}(\Omega) = \{(u_1, u_2, \ldots, u_d) | u_i \in H^s(\Omega), 1 \leq i \leq d\}$. If $\boldsymbol{u} = (u_1, u_2, \ldots, u_d) \in \boldsymbol{H}^s(\Omega)$, then $\|\boldsymbol{u}\|_s^2 = \sum_{i=1}^d \|u_i\|_s^2$. And also we skip 0 in the notation of the Sobolev norm $\|\cdot\|_0$, so we simply write $\|\cdot\|$. If for each $t \in [0, T]$, $u(x, t)$ belongs to a Sobolev space X equipped with a norm $\|\cdot\|_X$, then we define for $p \in [1, \infty)$, $\|u(x,t)\|_{L^p(0,t_0:X)}^p = \int_0^{t_0} \|u(x,t)\|_X^p dt$ and for $p = \infty$, $\|u(x,t)\|_{L^\infty(0,t_0:X)} = \operatorname{ess\,sup}_{0\leq t\leq t_0} \|u(x,t)\|_X$. If $t_0 = T$, then we simply write $L^p(X)$ and $L^\infty(X)$ instead of $L^p(0, T : X)$ and $L^\infty(0, T : X)$ respectively. And also (f, g) denotes the usual inner product given by $(f, g) = \int_\Omega fg dx$.

We denote $V = L^2(\Omega)$, $\boldsymbol{\Lambda} = (L^2(\Omega))^d$ and $\boldsymbol{W} = \{\boldsymbol{w} \in \boldsymbol{H}(\mathbf{div} : \Omega) \,|\, \boldsymbol{w}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$ where $\boldsymbol{H}(\mathbf{div} : \Omega) = \{\boldsymbol{w} \in (L^2(\Omega))^d \,|\, \nabla\cdot\boldsymbol{w} \in L^2(\Omega)\}$. Let $\mathcal{E}_h = \{E_1, E_2, \ldots, E_{N_h}\}$ be a regular quasi-uniform subdivision of Ω where E_i is a triangle or a quadrilateral if $d = 2$ and E_i is a 3-simplex or 3-rectangle if $d = 3$. Boundary triangles or rectangles (3-simplex or 3-rectangle) are allowed to have a curvilinear edge (a curved surface). Let $h_i = \operatorname{diam}(E_i)$ be the diameter of E_i and $h = \max\{h_i \mid 1 \leq i \leq N_h\}$. We assume that there exists a constant $\delta > 0$ such that each E_i contains a ball of radius δh_i. The quasi-uniformity requirement is that there is a constant $\tau > 0$ such that $h/h_i \leq \tau$, $i = 1, 2, \ldots, N_h$.

We denote by $\boldsymbol{\Lambda_h} \times V_h = \boldsymbol{\Lambda}(\Omega, \mathcal{E}_h, k) \times V(\Omega, \mathcal{E}_h, k)$ the Raviart-Thomas-Nedelec space associated with $\mathcal{E}_h$. Let $E \in \mathcal{E}_h$ and let $P_k(E)$ denote the restriction of the polynomials of total degree $\leq k$ to the set E. Similarly we let $Q_k(E)$ indicate the space of the restrictions of the polynomials of degree $\leq k$ with respect to each one of the d variables $x_1, x_2, \ldots, x_d$ to E. If $E \in \mathcal{E}_h$ is a triangle, we let

$$V_h(E) = P_k(E), \quad \boldsymbol{\Lambda_h}(E) = \boldsymbol{W_h}(E) = (P_k(E))^2 \oplus (x_1, x_2)^T P_k(E).$$

Similarly, if E is a rectangle, we let $V_h(E) = Q_k(E)$,
$\boldsymbol{\Lambda_h}(E) = \boldsymbol{W_h}(E) = \{\boldsymbol{\mu} \in \left(Q_{k+1}(E)\right)^2 : \frac{\partial \mu_i^{k+1}}{\partial x_j^{k+1}} = 0,\ j \neq i, 1 \leq i, j \leq 2\}$.
If E is a rectangular, then the above finite element space coincides with that of Raviart and Thomas [24]. If E is a triangle, then it is the modification due to Nedelec [22]. With the obvious modification, we define

$$\boldsymbol{\Lambda_h}(E) = \boldsymbol{W_h}(E) = \left(P_k(E)\right)^3 \oplus (x_1, x_2, x_3)^T P_k(E),$$

$$\boldsymbol{\Lambda_h}(E) = \boldsymbol{W_h}(E) = \left\{\boldsymbol{\mu} \in \left(Q_{k+1}(E)\right)^3 | \frac{\partial \mu_i^{k+1}}{\partial x_j^{k+1}} = 0,\ j \neq i, 1 \leq i, j \leq 3\right\},$$

for T being a 3-simplex or a parallelogram in $\mathbb{R}^3$.

Let $V_h \subset V$, $\boldsymbol{\Lambda_h} \subset \boldsymbol{\Lambda}$ and $\boldsymbol{W_h} \subset \boldsymbol{W}$ be the finite element spaces such that

$$\begin{aligned} V_h &= \{v \in V \mid v|_E \in V_h(E), \quad \forall E \in \mathcal{E}_h\}, \\ \boldsymbol{\Lambda_h} &= \{\boldsymbol{\mu} \in \boldsymbol{\Lambda} \mid \boldsymbol{\mu}|_E \in \boldsymbol{\Lambda_h}(E), \quad \forall E \in \mathcal{E}_h\}, \\ \boldsymbol{W_h} &= \{\boldsymbol{w} \in \boldsymbol{W} \mid \boldsymbol{w}|_E \in \boldsymbol{W_h}(E), \quad \forall E \in \mathcal{E}_h\}. \end{aligned}$$

From now on, we concentrate the case that E is a triangle or a 3-simplex, and analyze the approximation results for this case only. Following the similar process, we may obtain the corresponding results for the case that E is a rectangle or parallelogram.

To introduce an expanded mixed formulation, we let $\boldsymbol{\lambda} = -\nabla u$, $\boldsymbol{\sigma} = -(\nabla u + \nabla u_t) = \boldsymbol{\lambda} + \boldsymbol{\lambda}_t$. Thus the weak form of (1.1) that we shall treat is given by seeking a triple $(u, \boldsymbol{\lambda}, \boldsymbol{\sigma}) \in V \times \boldsymbol{\Lambda} \times \boldsymbol{W}$ such that

$$(\boldsymbol{\lambda}, \boldsymbol{w}) - (u, \nabla \cdot \boldsymbol{w}) = 0, \qquad \forall\, \boldsymbol{w} \in \boldsymbol{W}, \tag{2.1}$$

$$(\boldsymbol{\lambda}, \boldsymbol{\mu}) + (\boldsymbol{\lambda}_t, \boldsymbol{\mu}) - (\boldsymbol{\sigma}, \boldsymbol{\mu}) = 0, \qquad \forall\, \boldsymbol{\mu} \in \boldsymbol{\Lambda}, \tag{2.2}$$

$$(u_t, v) + (\nabla \cdot \boldsymbol{\sigma}, v) = (f(u), v), \qquad \forall\, v \in V. \tag{2.3}$$

3. Optimal L^2 error estimates of the expanded semidiscrete approximations of u, $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}$

In this section by applying an expanded mixed method we will construct the semidiscrete approximations of u, $-\nabla u$ and $-\nabla u - \nabla u_t$, prove their existence and analyze the L^2 error estimates of semidiscrete approximations of u, $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}$.

Raviart and Thomas [24] defined a projection $\boldsymbol{\Pi_h} \times P_h : \boldsymbol{W} \times V \to \boldsymbol{W_h} \times V_h$ satisfying the properties:

$$(\nabla \cdot \boldsymbol{w} - \nabla \cdot \boldsymbol{\Pi_h} \boldsymbol{w}, v) = 0, \qquad \forall\, v \in V_h, \tag{3.1}$$

$$(v - P_h v, \chi) = 0, \qquad \forall\, \chi \in V_h. \tag{3.2}$$

Then obviously we have $(\nabla\cdot \boldsymbol{w}, v - P_h v) = 0,\ \ \forall\, v \in V,\ \forall\, \boldsymbol{w} \in \boldsymbol{W_h}$, and we know that the following diagram commutes

$$\begin{array}{ccc} W & \xrightarrow{\nabla\cdot} & V \\ {\scriptstyle \boldsymbol{\Pi}_h}\downarrow & & \downarrow{\scriptstyle P_h} \\ W_h & \xrightarrow{\nabla\cdot} & V_h \end{array}$$

i.e., $\operatorname{div}\boldsymbol{\Pi}_h = P_h\operatorname{div}$ as functions from $\boldsymbol{W}$ onto V_h. And also the following approximation properties hold [24]:

(3.3) $\|\boldsymbol{w} - \boldsymbol{\Pi_h w}\| \le Ch^r\|\boldsymbol{w}\|_r,\ \forall\, \boldsymbol{w} \in (H^r(\Omega))^d,\ \ 1 \le r \le k+1,$

(3.4) $\|u - P_h u\|_{L_p} \le Ch^r\|u\|_{r,p},\ \forall\, u \in W^{r,p}(\Omega),\ 0 \le r \le k+1,\ 1 \le p \le \infty.$

And also we define $\boldsymbol{R_h} : \boldsymbol{\Lambda} \longrightarrow \boldsymbol{\Lambda_h}$ be the projection satisfying

(3.5) $\quad (\boldsymbol{\lambda} - \boldsymbol{R_h\lambda}, \boldsymbol{\mu}) = 0, \qquad \forall\, \boldsymbol{\mu} \in \boldsymbol{\Lambda_h},$

(3.6) $\quad \|\boldsymbol{\lambda} - \boldsymbol{R_h\lambda}\| \le Ch^r\|\boldsymbol{\lambda}\|_r, \quad \forall\, \boldsymbol{\lambda} \in (H^r(\Omega))^d,\ 0 \le r \le k+1.$

Now we formulate the expanded mixed finite element method as follows: find a triple $(u_h, \boldsymbol{\lambda_h}, \boldsymbol{\sigma_h}) \in V_h \times \boldsymbol{\Lambda_h} \times \boldsymbol{W_h}$ such that

(3.7) $\quad (\boldsymbol{\lambda_h}, \boldsymbol{w}) - (u_h, \nabla\cdot\boldsymbol{w}) = 0, \qquad \forall\, \boldsymbol{w} \in \boldsymbol{W_h},$

(3.8) $\quad (\boldsymbol{\lambda_h}, \boldsymbol{\mu}) + ((\boldsymbol{\lambda_h})_t, \boldsymbol{\mu}) - (\boldsymbol{\sigma_h}, \boldsymbol{\mu}) = 0, \qquad \forall\, \boldsymbol{\mu} \in \boldsymbol{\Lambda_h},$

(3.9) $\quad ((u_h)_t, v) + (\nabla\cdot\boldsymbol{\sigma_h}, v) = (f(u_h), v), \qquad \forall\, v \in V_h,$

where $u_h(0) = P_h(u_0(x))$, $\boldsymbol{\lambda_h}(0) = \boldsymbol{R_h\lambda}(0) = \boldsymbol{R_h}(-\nabla u_0(x))$.

Theorem 3.1. (i) *If f is a continuous function, then there exists a semidiscrete approximation $(u_h, \boldsymbol{\lambda}_h, \boldsymbol{\sigma}_h)$ satisfying* (3.7)–(3.9).

(ii) *If f satisfies Lipschitz continuous on a domain D containing $(x, 0, P_h(u_0(x)))$, then there exists a unique semidiscrete approximation $(u_h, \boldsymbol{\lambda}_h, \boldsymbol{\sigma}_h)$.*

Proof. Let $\{\phi_i : 1 \le i \le \ell\}$ be an orthogonal basis of V_h and $\{\boldsymbol{\psi}_i : 1 \le i \le m\}$, $\{\boldsymbol{\varphi}_i : 1 \le i \le n\}$, orthogonal bases of $\boldsymbol{\Lambda_h}$, $\boldsymbol{W_h}$, respectively. Since $\boldsymbol{W_h}$ is a subspace of $\boldsymbol{\Lambda_h}$, $\{\boldsymbol{\varphi_i} : 1 \le i \le n\}$ is a subset of $\{\boldsymbol{\psi_i} : 1 \le i \le m\}$. Let $u_h(x,t) = \sum_{i=1}^{\ell}\alpha_i(t)\phi_i(x)$, $\boldsymbol{\lambda_h}(x,t) = \sum_{i=1}^{m}\beta_i(t)\boldsymbol{\psi_i}(x)$, and $\boldsymbol{\sigma_h}(x,t) = \sum_{i=1}^{n} r_i(t)\boldsymbol{\varphi_i}(x)$.

From (3.7), (3.8) and (3.9), we obtain a system of ordinary differential equations:

(3.10)

$$\sum_{i=1}^{m}\beta_i(t)(\boldsymbol{\psi_i}, \boldsymbol{\varphi_j}) - \sum_{i=1}^{\ell}\alpha_i(t)(\phi_i, \nabla\cdot\boldsymbol{\varphi_j}) = 0, \quad j = 1, 2, \ldots, n,$$

(3.11)

$$\sum_{i=1}^{m}\beta_i(t)(\boldsymbol{\psi_i}, \boldsymbol{\psi_j}) + \sum_{i=1}^{m}\beta_i'(t)(\boldsymbol{\psi_i}, \boldsymbol{\psi_j}) - \sum_{i=1}^{n} r_i(t)(\boldsymbol{\varphi_i}, \boldsymbol{\psi_j}) = 0, \quad j = 1, 2, \ldots, m,$$

(3.12)

$$\sum_{i=1}^{\ell}\alpha_i'(t)(\phi_i,\phi_j)+\sum_{i=1}^{n}r_i(t)(\nabla\cdot\boldsymbol{\varphi_i},\phi_j)=\Big(f\Big(x,t,\sum_{i=1}^{\ell}\alpha_i\phi_i\Big),\phi_j\Big),\ j=1,2,\dots,\ell.$$

From (3.11), since $\boldsymbol{W_h}\subset\boldsymbol{\Lambda_h}$,

(3.13)

$$\sum_{i=1}^{m}\beta_i(t)(\boldsymbol{\psi_i},\boldsymbol{\varphi_j})+\sum_{i=1}^{m}\beta_i'(t)(\boldsymbol{\psi_i},\boldsymbol{\varphi_j})-\sum_{i=1}^{n}r_i(t)(\boldsymbol{\varphi_i},\boldsymbol{\varphi_j})=0,\quad j=1,2,\dots,n.$$

Let $\boldsymbol{r}(t)=(r_1(t),r_2(t),\dots,r_n(t))^T$, $\boldsymbol{\beta}(t)=(\beta_1(t),\beta_2(t),\dots,\beta_m(t))^T$, $\boldsymbol{A}=(a_{ij})_{1\le i,j\le n}$, $\boldsymbol{B}=(b_{ij})_{1\le i\le n,1\le j\le m}$, $a_{ij}=(\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_i)$ and $b_{ij}=(\boldsymbol{\psi}_j,\boldsymbol{\varphi}_i)$. (3.13) can be represented by

$$\boldsymbol{B\beta}(t)+\boldsymbol{B\beta}'(t)-\boldsymbol{Ar}(t)=0.$$

By the invertibility of $\boldsymbol{A}$, we get $\boldsymbol{r}(t)=\boldsymbol{A}^{-1}(\boldsymbol{B\beta}(t)+\boldsymbol{B\beta}'(t))$. Since $\nabla\cdot\boldsymbol{\varphi}_i\in V_h$, we can represent $\nabla\cdot\boldsymbol{\varphi}_i=\sum_{k=1}^{\ell}\gamma_{ki}\phi_k(x)$. From (3.12) we get

$$\sum_{i=1}^{\ell}\alpha_i'(t)(\phi_i,\phi_j)+\sum_{i=1}^{n}r_i(t)\Big(\sum_{k=1}^{\ell}\gamma_{ki}\phi_k,\phi_j\Big)=\Big(f\Big(x,t,\sum_{i=1}^{\ell}\alpha_i\phi_i\Big),\phi_j\Big),\ 1\le j\le\ell,$$

from which we have

(3.14)

$$\sum_{i=1}^{\ell}\alpha_i'(t)(\phi_i,\phi_j)+\sum_{i=1}^{n}\Big(\sum_{k=1}^{\ell}(\phi_k,\phi_j)\gamma_{ki}\Big)r_i(t)=\Big(f\Big(x,t,\sum_{i=1}^{\ell}\alpha_i\phi_i\Big),\phi_j\Big),\ 1\le j\le\ell.$$

If we let $\boldsymbol{\alpha}(t)=(\alpha_1(t),\alpha_2(t),\dots,\alpha_\ell(t))^T$, $\boldsymbol{\Phi}=(\gamma_{ki})_{1\le k\le\ell,1\le i\le n}$, $\boldsymbol{D}=(d_{ij})_{1\le i,j\le\ell}$ and $\boldsymbol{F}(\boldsymbol{\alpha})=(F_j(\boldsymbol{\alpha}))_{1\le j\le\ell}$ where $d_{ij}=(\phi_j,\phi_i)$ and $F_j(\boldsymbol{\alpha})=\Big(f\Big(x,t,\sum_{i=1}^{\ell}\alpha_i\phi_i\Big),\phi_j\Big)$, then (3.14) can be reduced to

$$\boldsymbol{D\alpha}'(t)+\boldsymbol{D\Phi r}(t)=\boldsymbol{F}(\boldsymbol{\alpha}).\tag{3.15}$$

And the equation (3.10) can be represented by

$$\sum_{i=1}^{m}\beta_i(t)(\boldsymbol{\psi_i},\boldsymbol{\varphi_j})-\sum_{i=1}^{\ell}\sum_{k=1}^{\ell}\gamma_{kj}(\phi_k,\phi_i)\alpha_i(t)=0,\quad j=1,2,\dots,n,$$

which implies

$$\boldsymbol{B\beta}(t)-\boldsymbol{\Phi}^T\boldsymbol{D\alpha}(t)=0.\tag{3.16}$$

By substituting $\boldsymbol{r}(t)=\boldsymbol{A}^{-1}(\boldsymbol{B\beta}(t)+\boldsymbol{B\beta}'(t))$ into (3.15), we obtain

$$\boldsymbol{D\alpha}'(t)+\boldsymbol{D\Phi A}^{-1}\boldsymbol{B\beta}(t)+\boldsymbol{D\Phi A}^{-1}\boldsymbol{B\beta}'(t)=\boldsymbol{F}(\boldsymbol{\alpha}).\tag{3.17}$$

Now we substitute (3.16) into (3.17) to get

$$\begin{cases}(\boldsymbol{D}+\boldsymbol{D\Phi A}^{-1}\boldsymbol{\Phi}^T\boldsymbol{D})\boldsymbol{\alpha}'(t)=-\boldsymbol{D\Phi A}^{-1}\boldsymbol{\Phi}^T\boldsymbol{D\alpha}(t)+\boldsymbol{F}(\boldsymbol{\alpha}),\\ \boldsymbol{\alpha}(0)=\boldsymbol{\alpha}_0,\end{cases}\tag{3.18}$$

where $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0$ can be determined uniquely from the initial condition that $u_h(0) = P_h(u_0(x))$. Since $\boldsymbol{D} + \boldsymbol{D}\boldsymbol{\Phi}\boldsymbol{A}^{-1}\boldsymbol{\Phi}^T\boldsymbol{D}$ is positive definite and f is continuous, (3.18) has a solution $\boldsymbol{\alpha}(t)$ by the theory of the system of ordinary differential equations. By combining (3.16) with the relation $\boldsymbol{r}(t) = \boldsymbol{A}^{-1}(\boldsymbol{B}\boldsymbol{\beta}(t) + \boldsymbol{B}\boldsymbol{\beta}'(t))$, we have

$$\boldsymbol{r}(t) = \boldsymbol{A}^{-1}\boldsymbol{\Phi}^T\boldsymbol{D}\boldsymbol{\alpha}(t) + \boldsymbol{A}^{-1}\boldsymbol{\Phi}^T\boldsymbol{D}\boldsymbol{\alpha}'(t),$$

which shows the existence of $\boldsymbol{r}(t)$. To prove the existence of $\boldsymbol{\beta}(t)$, if we let $\boldsymbol{E} = (e_{ij})_{1\le i,j\le m}$, $e_{ij} = (\boldsymbol{\psi}_j, \boldsymbol{\psi}_i)$, then $\boldsymbol{E}$ is symmetric and positive definite. (3.11) can be reduced to

$$\begin{cases} \boldsymbol{E}\boldsymbol{\beta}'(t) = \boldsymbol{E}\boldsymbol{\beta}(t) - \boldsymbol{B}^T\boldsymbol{r}(t), \\ \boldsymbol{\beta}(0) = \boldsymbol{\beta}_0, \end{cases}$$

where $\boldsymbol{\beta}(0) = \boldsymbol{\beta}_0$ can be determined uniquely from the initial condition that $\boldsymbol{\lambda}_h(0) = \boldsymbol{R_h}(-\nabla u_0(x))$. Therefore $\boldsymbol{\beta}(t)$ exists which completes the proof of (i). If f satisfies a Lipschitz condition on a domain D containing $\big(x, 0, P_h(u_0(x))\big)$ by the theory of the system of ordinary differential equations, (3.18) has a unique $\boldsymbol{\alpha}(t)$. Therefore in a consecutive order, we can prove that $\boldsymbol{r}(t)$ and $\boldsymbol{\beta}(t)$ exist uniquely, which completes the proof of (ii). □

The result of the previous theorem induces the continuity of $u^h(x,t) = \sum_{i=1}^{\ell} \alpha_i(t)\phi_i(x)$ with respect to t, so that $\|u^h(t)\|_{L^\infty}$ is continuous with respect to t. By (3.4), there exists a constant K^* such that for sufficiently small h,

$$\|u(x,0) - u^h(x,0)\|_{L^\infty} = \|u_0(x) - P_h(u_0(x))\|_{L^\infty} \le K^*, \tag{3.19}$$

holds. Throughout this paper C denotes a generic positive constant dependent on the domain $\Omega, K^*, u(x,t)$, the constants δ and τ which manage the regularity and quasi-uniformity of the subdivision of Ω, but independent of the discretization sizes of space variable and time variable. If the generic constant C depends on some specific constants besides the ones mentioned already, we will clearly state the dependency.

To continue the analysis of the convergence of semidiscrete approximations and fully discrete approximations, In the rest of this paper, we need to assume that f satisfies the following locally Lipschitz continuous at $u(x,t)$: if $|u(x,t) - u^*| \le 2K^*$, then $|f(x,t,u(x,t)) - f(x,t,u^*)| \le C(u,K^*)|u(x,t) - u^*|$ for all $(x,t) \in \Omega \times [0,T]$.

Theorem 3.2. *If f is locally Lipschitz continuous at $u(x,t)$ and u, $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}$ satisfy $u \in L^\infty(H^s)$, $\boldsymbol{\lambda} \in L^\infty(\boldsymbol{H^s})$ and $\boldsymbol{\sigma} \in L^\infty(\boldsymbol{H^s})$, respectively and $\nu > \frac{d}{2}$, then*

$$\begin{aligned} &\|u - u_h\|_{L^\infty(L^2)} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda_h}\|_{L^\infty(\boldsymbol{L^2})} \\ &\le Ch^\nu\big(\|u\|_{L^\infty(H^s)} + \|\boldsymbol{\lambda}\|_{L^\infty(\boldsymbol{H^s})} + \|\boldsymbol{\sigma}\|_{L^2(\boldsymbol{H^s})}\big), \end{aligned}$$

and

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma_h}\|_{L^\infty(\boldsymbol{L^2})} \le Ch^\nu\big(\|u\|_{L^\infty(H^s)}+\|\boldsymbol{\lambda}\|_{L^\infty(\boldsymbol{H^s})}+\|\boldsymbol{\sigma}\|_{L^\infty(\boldsymbol{H^s})}\big),$$

where $\nu=\min(k+1,s)$, *hold.*

Proof. By subtracting (3.7) from (2.1), (3.8) from (2.2), and (3.9) and (2.3) respectively, we have the followings:

(3.20)
$$(\boldsymbol{\lambda}-\boldsymbol{\lambda_h},\boldsymbol{w})-(u-u_h,\nabla\cdot\boldsymbol{w})=0, \qquad \forall\,\boldsymbol{w}\in\boldsymbol{W_h},$$

(3.21)
$$(\boldsymbol{\lambda}-\boldsymbol{\lambda_h},\boldsymbol{\mu})+(\boldsymbol{\lambda_t}-(\boldsymbol{\lambda_h})_t,\boldsymbol{\mu})-(\boldsymbol{\sigma}-\boldsymbol{\sigma_h},\boldsymbol{\mu})=0, \qquad \forall\,\boldsymbol{\mu}\in\boldsymbol{\Lambda_h},$$

(3.22)
$$(u_t-(u_h)_t,v)+(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma_h}),v)=(f(x,t,u)-f(x,t,u_h),v), \quad \forall\,v\in V_h.$$

For the time being, we assume that there exists a sufficiently small $\tilde{h}$ to be defined below so that

$$\|u(x,t)-u^h(x,t)\|_{L^\infty}<2K^*,\ \forall t,\ 0\le t\le T,\ \forall h\le\tilde{h}, \tag{3.23}$$

holds. We will prove the adequacy of this assumption (3.23) later. By choosing $\boldsymbol{w}=\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h}$ in (3.20), $\boldsymbol{\mu}=\boldsymbol{R_h\lambda}-\boldsymbol{\lambda_h}$ in (3.21) and $v=P_hu-u_h$ in (3.22) respectively, we have

$$(\boldsymbol{R_h\lambda}-\boldsymbol{\lambda_h},\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h})-(P_hu-u_h,\nabla\cdot(\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h}))=0, \tag{3.24}$$

$$\begin{aligned}&(\boldsymbol{R_h\lambda}-\boldsymbol{\lambda_h},\boldsymbol{R_h\lambda}-\boldsymbol{\lambda_h})+(\boldsymbol{R_h\lambda_t}-\boldsymbol{\lambda_{ht}},\boldsymbol{R_h\lambda}-\boldsymbol{\lambda_h})\\&-(\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h},\boldsymbol{R_h\lambda}-\boldsymbol{\lambda_h})=(\boldsymbol{\sigma}-\boldsymbol{\Pi_h\sigma},\boldsymbol{R_h\lambda}-\boldsymbol{\lambda_h}),\end{aligned} \tag{3.25}$$

$$\begin{aligned}&(P_hu_t-u_{ht},P_hu-u_h)+(\nabla\cdot(\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h}),P_hu-u_h)\\&=(f(x,t,u)-f(x,t,u_h),P_hu-u_h).\end{aligned} \tag{3.26}$$

Combining (3.24)–(3.26), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|P_hu-u_h\|^2+\|\boldsymbol{R_h\lambda}-\boldsymbol{\lambda_h}\|^2+\frac{1}{2}\frac{d}{dt}\|\boldsymbol{R_h\lambda}-\boldsymbol{\lambda_h}\|^2\\&=(\boldsymbol{\sigma}-\boldsymbol{\Pi_h\sigma},\boldsymbol{R_h\lambda}-\boldsymbol{\lambda_h})+(f(x,t,u)-f(x,t,u_h),P_hu-u_h)\\&\le\frac{1}{2}\|\boldsymbol{\sigma}-\boldsymbol{\Pi_h\sigma}\|^2+\frac{1}{2}\|\boldsymbol{R_h\lambda}-\boldsymbol{\lambda_h}\|^2+C\big(\|u-P_hu\|+\|P_hu-u_h\|\big)\|P_hu-u_h\|\\&\le\frac{1}{2}\|\boldsymbol{\sigma}-\boldsymbol{\Pi_h\sigma}\|^2+\frac{1}{2}\|\boldsymbol{R_h\lambda}-\boldsymbol{\lambda_h}\|^2+C\big(\|u-P_hu\|^2+\|P_hu-u_h\|^2\big),\end{aligned}$$

from which we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|P_hu-u_h\|^2+\frac{1}{2}\|\boldsymbol{R_h\lambda}-\boldsymbol{\lambda_h}\|^2+\frac{1}{2}\frac{d}{dt}\|\boldsymbol{R_h\lambda}-\boldsymbol{\lambda_h}\|^2\\&\le Ch^{2\nu}\big\{\|u\|_s^2+\|\boldsymbol{\sigma}\|_s^2\big\}+C\|P_hu-u_h\|^2.\end{aligned} \tag{3.27}$$

By taking the integration for both sides of (3.27) with respect to t from 0 to $\widetilde{t}$, we get

$$\|(P_h u - u_h)(\widetilde{t})\|^2 + \|(\boldsymbol{R_h\lambda} - \boldsymbol{\lambda_h})(\widetilde{t})\|^2 + \int_0^{\widetilde{t}} \|\boldsymbol{R_h\lambda} - \boldsymbol{\lambda_h}\| dt$$
$$\leq Ch^{2\nu}\big\{\|u\|^2_{L^2(H^s)} + \|\boldsymbol{\sigma}\|^2_{L^2(\boldsymbol{H^s})}\big\} + C\int_0^{\widetilde{t}} \|P_h u - u_h\|^2 dt.$$

By applying Gronwall's Lemma we have

$$\|(P_h u - u_h)(\widetilde{t})\|^2 + \|(\boldsymbol{R_h\lambda} - \boldsymbol{\lambda_h})(\widetilde{t})\|^2 \leq Ch^{2\nu}\Big\{\|u\|^2_{L^2(H^s)} + \|\boldsymbol{\sigma}\|^2_{L^2(\boldsymbol{H^s})}\Big\}, \tag{3.28}$$

which implies, by the approximation results (3.4) and (3.6).

$$\|u-u_h\|_{L^\infty(L^2)}+\|\boldsymbol{\lambda}-\boldsymbol{\lambda_h}\|_{L^\infty(\boldsymbol{L^2})} \leq Ch^\nu\Big(\|u\|_{L^\infty(H^s)}+\|\boldsymbol{\lambda}\|_{L^\infty(\boldsymbol{H^s})}+\|\boldsymbol{\sigma}\|_{L^2(\boldsymbol{H^s})}\Big). \tag{3.29}$$

Now by means of contradiction, we will show that the assumption (3.23) is appropriate. Suppose that there exists t^* such that $0 < t^* \leq T$, $\|u(t) - u_h(t)\|_{L^\infty} < 2K^*$ holds for $0 \leq t < t^*$ but $\|u_(t^*) - u_h(t^*)\|_{L^\infty} \geq 2K^*$. Now we take a sequence $\{t_n\}_{n=1}^\infty \subset [0, t^*)$ converging to t^*. Then we obviously have $\|(P_h u - u_h)(t_n)\| \leq Ch^\nu\{\|u\|_{L^2(H^s)} + \|\sigma\|_{L^2(H^s)}\}$. Choose $\tilde{h}$ sufficiently small so that $\|(u-u_h)(t_n)\|_{L^\infty} \leq \|(u-P_h u)(t_n)\|_{L^\infty} + \|(P_h u-u_h)(t_n)\|_{L^\infty} \leq c(h^r + h^{-\frac{d}{2}}h^\nu) \leq \frac{3}{2}K^*$ holds $\forall h \leq \tilde{h}$. Since $\|(u-u_h)(t)\|_{L^\infty}$ is continuous with respect to t, $\|(u-u_h)(t^*)\|_{L^\infty} \leq \frac{3}{2}K^*$ holds which contradicts to $\|(u-u_h)(t^*)\|_{L^\infty} \geq 2K^*$.

To estimate $\|\boldsymbol{\sigma}-\boldsymbol{\sigma_h}\|$, we differentiate the both sides of (3.20) with respect to t and choose $\boldsymbol{w} = \boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h}$ then we have $(\boldsymbol{\lambda_t} - \boldsymbol{\lambda_{ht}}, \boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h}) - (u_t - u_{ht}, \nabla\cdot \boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h}) = 0$. Combining this equation with (3.21) with $\boldsymbol{\mu} = \boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h}$ and (3.20) with $\boldsymbol{w} = \boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h}$ yields that

$$(u-u_h, \nabla\cdot(\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h})) + (u_t-u_{ht}, \nabla\cdot(\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h})) - (\boldsymbol{\sigma}-\boldsymbol{\sigma_h}, \boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h}) = 0,$$

from which by (3.2) we have the following error equation

$$\begin{aligned}&(P_h u - u_h, \nabla\cdot(\boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h})) + (P_h u_t - u_{ht}, \nabla\cdot(\boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h}))\\ &- (\boldsymbol{\sigma} - \boldsymbol{\sigma_h}, \boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h}) = 0.\end{aligned} \tag{3.30}$$

Now we take $v = \nabla\cdot(\boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h})$ in (3.22) to get

$$\begin{aligned}&(P_h u_t - u_{ht}, \nabla\cdot(\boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h})) + (\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma_h}), \nabla\cdot(\boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h}))\\ &= (f(x,t,u) - f(x,t,u_h), \nabla\cdot(\boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h})),\end{aligned}$$

which by (3.1) implies the following

$$\begin{aligned}&(P_h u_t - u_{ht}, \nabla\cdot(\boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h})) + (\nabla\cdot(\boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h}), \nabla\cdot(\boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h}))\\ &= (f(x,t,u) - f(x,t,u_h), \nabla\cdot(\boldsymbol{\Pi_h\sigma} - \boldsymbol{\sigma_h})).\end{aligned} \tag{3.31}$$

Now we subtract (3.30) from (3.31) to get

$$\|\nabla\cdot(\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h})\|^2-(P_hu-u_h,\nabla\cdot(\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h}))+(\boldsymbol{\sigma}-\boldsymbol{\sigma_h},\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h})$$
$$=(f(x,t,u)-f(x,t,u_h),\nabla\cdot(\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h})),$$

which yields that,

$$\begin{aligned}&\|\nabla\cdot(\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h})\|^2+(\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h},\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h})\\&=(P_hu-u_h,\nabla\cdot(\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h}))+(\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma},\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h})\\&\quad+(f(x,t,u)-f(x,t,u_h),\nabla\cdot(\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h})).\end{aligned}$$

Then by the Cauchy-Schwarz inequality we have

$$\begin{aligned}&\|\nabla\cdot(\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h})\|^2+\|\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma_h}\|^2\\&\le C\big(\|P_hu-u_h\|^2+\|\boldsymbol{\Pi_h\sigma}-\boldsymbol{\sigma}\|^2+\|u-u_h\|^2\big)\\&\le Ch^{2\nu}(\|u\|^2_{L^\infty(H^s)}+\|\boldsymbol{\lambda}\|^2_{L^\infty(\boldsymbol{H^s})}+\|\boldsymbol{\sigma}\|^2_{L^2(\boldsymbol{H^s})}+\|\boldsymbol{\sigma}\|^2_s).\end{aligned}$$

Therefore $\|\boldsymbol{\sigma}-\boldsymbol{\sigma_h}\|_{L^\infty(\boldsymbol{L^2})}\le Ch^\nu(\|u\|_{L^\infty(H^s)}+\|\boldsymbol{\lambda}\|_{L^\infty(\boldsymbol{H^s})}+\|\boldsymbol{\sigma}\|_{L^\infty(\boldsymbol{H^s})})$. □

4. Error estimates of the expanded fully discrete approximations of u, $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}$

In this section we construct the fully discrete approximations of u, $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}$ using an expanded mixed Galerkin method and we prove its optimal convergence in $\ell^\infty(L^2)$ normed space. For a positive integer N, we let $\Delta t=\frac{T}{N}$, $t^n=n(\Delta t)$ for $n=0,1,\ldots,N$, and $t^{n,\theta}=\alpha_1t^n+\alpha_2t^{n-1}$, with $\alpha_1=(1+\theta)/2$ and $\alpha_2=(1-\theta)/2$, where $0\le\theta\le1$.

Construct $(U^n,\boldsymbol{\Lambda}^n,\boldsymbol{\Sigma}^n_\theta)\in V_h\times\boldsymbol{\Lambda_h}\times\boldsymbol{W_h}$ such that $U^n\cong u(t^n)$, $\boldsymbol{\Lambda}^n\cong\boldsymbol{\lambda}(t^n)$, $\boldsymbol{\Sigma}^n_\theta\cong\boldsymbol{\sigma}(t^{n,\theta})$, $n=2,3,\ldots,N$ and satisfying

$$(4.1)\qquad(\boldsymbol{\Lambda}^n,\boldsymbol{w})-(U^n,\nabla\cdot\boldsymbol{w})=0,\qquad\forall\,\boldsymbol{w}\in\boldsymbol{W_h},$$

$$(4.2)\qquad(\boldsymbol{\Lambda}^{n,\theta}+\partial_t\boldsymbol{\Lambda}^n,\boldsymbol{\mu})-(\boldsymbol{\Sigma}^n_\theta,\boldsymbol{\mu})=0,\qquad\forall\,\boldsymbol{\mu}\in\boldsymbol{\Lambda_h},$$

$$(4.3)\qquad(\partial_tU^n,v)+(\nabla\cdot\boldsymbol{\Sigma}^n_\theta,v)=(f(x,t^{n,\theta},EU^n),v),\quad\forall\,v\in V_h,$$

where $\boldsymbol{\Lambda}^{n,\theta}=\alpha_1\boldsymbol{\Lambda}^n+\alpha_2\boldsymbol{\Lambda}^{n-1}$, $\partial_t\boldsymbol{\Lambda}^n=\frac{\boldsymbol{\Lambda}^n-\boldsymbol{\Lambda}^{n-1}}{\Delta t}$, $EU^n=\beta_1U^{n-1}+\beta_2U^{n-2}$, $\beta_1=(3+\theta)/2$ and $\beta_2=(-1-\theta)/2$.

Since the extrapolation method (4.1)–(4.3) requires the previously computed approximations, U^0, U^1, $\boldsymbol{\Lambda}^0$ and $\boldsymbol{\Lambda}^1$ a starting procedure is needed. So we define $U^0=P_h(u_0(x))$, $\boldsymbol{\Lambda}^0=\boldsymbol{R}_h(-\nabla u_0(x))$. Then obviously $(\boldsymbol{\Lambda}^0,\boldsymbol{w})-(U^0,\nabla\cdot\boldsymbol{w})=0$, $\forall\,\boldsymbol{w}\in\boldsymbol{W_h}$ holds.

Define $(U^1,\boldsymbol{\Lambda}^1,\boldsymbol{\Sigma}^1_\theta)\in V_h\times\boldsymbol{\Lambda_h}\times\boldsymbol{W_h}$ such that $U^1\cong u(t^1)$, $\boldsymbol{\Lambda}^1\cong\boldsymbol{\lambda}(t^1)$, $\boldsymbol{\Sigma}^1_\theta\cong\boldsymbol{\sigma}(t^{1,\theta})$ and $(U^1,\boldsymbol{\Lambda}^1,\boldsymbol{\Sigma}^1_\theta)$ satisfies

$$(4.4)\qquad(\boldsymbol{\Lambda}^1,\boldsymbol{w})-(U^1,\nabla\cdot\boldsymbol{w})=0,\qquad\forall\,\boldsymbol{w}\in\boldsymbol{W_h},$$

$$(4.5)\qquad(\boldsymbol{\Lambda}^{1,\theta}+\partial_t\boldsymbol{\Lambda}^1,\boldsymbol{\mu})-(\boldsymbol{\Sigma}^1_\theta,\boldsymbol{\mu})=0,\qquad\forall\,\boldsymbol{\mu}\in\boldsymbol{\Lambda_h},$$

$$(4.6)\qquad(\partial_tU^1,v)+(\nabla\cdot\boldsymbol{\Sigma}^1_\theta,v)=(f(x,t^{1,\theta},U^{1,\theta}),v),\quad\forall\,v\in V_h,$$

where $U^{1,\theta} = \boldsymbol{\alpha}_1 U^1 + \boldsymbol{\alpha}_2 U^0$. Concerning the starting procedures (4.4)–(4.6) it can be shown that if u, $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}$ satisfy the hypotheses of Theorem 4.1, then

$$\|u(t^1) - U^1\| \le C(h^\nu + (\Delta t)^{1+\delta_{\theta 0}}), \tag{4.7}$$

$$\|\boldsymbol{\lambda}(t^1) - \boldsymbol{\Lambda}^1\| \le C(h^\nu + (\Delta t)^{1+\delta_{\theta 0}}), \tag{4.8}$$

$$\|\boldsymbol{\sigma}(t^{1,\theta}) - \boldsymbol{\Sigma}_\theta^1\| \le C(h^\nu + (\Delta t)^{1+\delta_{\theta 0}}) \tag{4.9}$$

holds where $\nu = \min(k+1, s)$, $\delta_{\theta 0}$ denotes the Kronecker symbol and the constant C depending on the Sobolev norms of u, $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}$ appearing in Theorem 4.1. The proofs of these estimates are similar to the ones that will be given concerning the principal scheme (4.1)–(4.3), so we omit the proofs.

To analyze the order of convergence we introduce the following notations:
$\zeta_u(t) = u(t) - \widetilde{u}(t)$, $\boldsymbol{\zeta_\lambda}(t) = \boldsymbol{\lambda}(t) - \widetilde{\boldsymbol{\lambda}}(t)$, $\boldsymbol{\zeta_\sigma}(t) = \boldsymbol{\sigma}(t) - \widetilde{\boldsymbol{\sigma}}(t)$,
$\gamma_u^{n,\theta} = \widetilde{u}(t^{n,\theta}) - \widetilde{u}^{n,\theta}$, $\boldsymbol{\gamma}_{\boldsymbol{\lambda}}^{n,\theta} = \widetilde{\boldsymbol{\lambda}}(t^{n,\theta}) - \widetilde{\boldsymbol{\lambda}}^{n,\theta}$,
$\rho_u^{n,\theta} = \widetilde{u}_t(t^{n,\theta}) - \partial_t \widetilde{u}^n$, $\boldsymbol{\rho}_{\boldsymbol{\lambda}}^{n,\theta} = \widetilde{\boldsymbol{\lambda}}_t(t^{n,\theta}) - \partial_t \widetilde{\boldsymbol{\lambda}}^n$,
$e_u^n = \widetilde{u}(t^n) - U^n$, $\boldsymbol{e}_{\boldsymbol{\lambda}}^{\boldsymbol{n}} = \widetilde{\boldsymbol{\lambda}}(t^n) - \boldsymbol{\Lambda}^n$, $\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta} = \widetilde{\boldsymbol{\sigma}}(t^{n,\theta}) - \boldsymbol{\Sigma}_\theta^n$,
$\xi_u^{n,\theta} = \widetilde{u}^{n,\theta} - E\widetilde{u}^n$,
where $\widetilde{u}(t) = P_h u(t)$, $\widetilde{\boldsymbol{\lambda}}(t) = \boldsymbol{R_h}\boldsymbol{\lambda}(t)$ and $\widetilde{\boldsymbol{\sigma}}(t) = \boldsymbol{\Pi_h}\boldsymbol{\sigma}(t)$.

Lemma 4.1.

(i) *For $\theta = 0$, if $u_{ttt} \in L^\infty(t^{n-1}, t^n : L^2)$ and $u_{ttt} \in L^\infty(t^{n-1}, t^n : H^1)$, then*
$\|\rho_u^{n,\theta}\| \le C(\Delta t)^2 \|u_{ttt}\|_{L^\infty(t^{n-1},t^n:L^2)}$ *and*
$\|\nabla \rho_u^{n,\theta}\| \le C(\Delta t)^2 \|u_{ttt}\|_{L^\infty(t^{n-1},t^n:H^1)}$ *hold.*

(ii) *For $\theta \in (0,1]$, if $u_{tt} \in L^\infty(t^{n-1}, t^n : L^2)$ and $u_{tt} \in L^\infty(t^{n-1}, t^n : H^1)$, then*
$\|\rho_u^{n,\theta}\| \le C(\Delta t)\|u_{tt}\|_{L^\infty(t^{n-1},t^n:L^2)}$ *and*
$\|\nabla \rho_u^{n,\theta}\| \le C(\Delta t)\|u_{tt}\|_{L^\infty(t^{n-1},t^n:H^1)}$ *hold.*

Proof. By Taylor's expansion, we easily obtain for $\theta = 0$,

$$\|\rho_u^{n,\theta}\| \le C(\Delta t)^2 \|\widetilde{u}_{ttt}\|_{L^\infty(t^{n-1},t^n:L^2)}$$

holds and for $\theta \in (0,1]$,

$$\|\rho_u^{n,\theta}\| \le C(\Delta t)\|\widetilde{u}_{tt}\|_{L^\infty(t^{n-1},t^n:L^2)}$$

holds. Therefore if $\theta = 0$, then by (3.4) we have

$$\|\rho_u^{n,\theta}\| \le C(\Delta t)^2 (\|u_{ttt}\|_{L^\infty(t^{n-1},t^n:L^2)}).$$

And similarly for $0 < \theta \le 1$, $\|\rho_u^{n,\theta}\| \le C\Delta t(\|u_{tt}\|_{L^\infty(t^{n-1},t^n:L^2)})$ holds. By the similar method we can prove the approximation results of $\nabla \rho_u^{n,\theta}$. □

Now we state the following lemmas without proof. The proofs can be obtained by Taylor's expansion easily.

Lemma 4.2. *If $u_{tt} \in L^\infty(t^{n-2}, t^n : L^p)$ for $1 \le p \le \infty$, then*

$$\|\gamma_u^{n,\theta}\|_{L^p} \le C(\Delta t)^2 \|u_{tt}\|_{L^\infty(t^{n-1},t^n:L^p)} \text{ and}$$
$$\|\xi_u^{n,\theta}\|_{L^p} \le C(\Delta t)^2 \|u_{tt}\|_{L^\infty(t^{n-2},t^n:L^p)}$$

hold.

Lemma 4.3. *If $\boldsymbol{\lambda}_{tt} \in L^\infty(t^{n-1}, t^n : \boldsymbol{L^2})$, then the following holds:*

$$\|\boldsymbol{\gamma}_{\boldsymbol{\lambda}}^{n,\theta}\| \le C(\Delta t)^2 \|\boldsymbol{\lambda}_{tt}\|_{L^\infty(t^{n-1},t^n:\boldsymbol{L^2})}.$$

Theorem 4.1. *Suppose that f satisfies the locally Lipschitz continuity at $u(x, t)$. Let $\{U^n, \boldsymbol{\Lambda}^n, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{\boldsymbol{n}}\}_{n=2}^N \in V_h \times \boldsymbol{\Lambda_h} \times \boldsymbol{W_h}$ be the solutions of (4.1)–(4.3).*

(i) *For $\theta = 0$, if $u \in L^\infty(H^s)$, $u_t \in L^\infty(H^s)$, $u_{tt} \in L^\infty(L^2)$, $u_{ttt} \in L^\infty(L^2)$, $\boldsymbol{\lambda} \in L^\infty(\boldsymbol{H^s})$, $\boldsymbol{\lambda}_{tt} \in L^\infty(\boldsymbol{L^2})$, $\boldsymbol{\lambda}_{ttt} \in L^\infty(\boldsymbol{L^2})$, $\boldsymbol{\sigma} \in L^\infty(\boldsymbol{H^s})$ and $\Delta t = O(h)$, then*

$$\max_{2\le n\le N} \|u(t^n) - U^n\| \le C(h^\nu + (\Delta t)^2)\big\{\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(L^2)} + \|u_{ttt}\|_{L^\infty(L^2)} + \|\boldsymbol{\sigma}\|_{L^\infty(\boldsymbol{H^s})}\big\}, \tag{4.10}$$

$$\max_{2\le n\le N} \|\boldsymbol{\lambda}(t^n) - \boldsymbol{\Lambda}^n\| \le C(h^\nu + (\Delta t)^2)\big\{\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(L^2)} + \|u_{ttt}\|_{L^\infty(L^2)} + \|\boldsymbol{\lambda}\|_{L^\infty(\boldsymbol{H^s})} + \|\boldsymbol{\lambda}_{tt}\|_{L^\infty(\boldsymbol{L^2})} + \|\boldsymbol{\lambda}_{ttt}\|_{L^\infty(\boldsymbol{L^2})} + \|\boldsymbol{\sigma}\|_{L^\infty(\boldsymbol{H^s})}\big\}, \tag{4.11}$$

and

$$\max_{2\le n\le N} \|\boldsymbol{\sigma}(t^{n,\theta}) - \boldsymbol{\Sigma}_\theta^n\| \le C(h^\nu + (\Delta t)^2)\big\{\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(L^2)} + \|u_{ttt}\|_{L^\infty(L^2)} + \|\boldsymbol{\sigma}\|_{L^\infty(\boldsymbol{H^s})}\big\} \tag{4.12}$$

hold where $\nu = \min(s, k+1)$.

(ii) *For $0 < \theta \le 1$, if $u \in L^\infty(H^s)$, $u_t \in L^\infty(H^s)$, $u_{tt} \in L^\infty(L^2)$, $\boldsymbol{\lambda} \in L^\infty(\boldsymbol{H^s})$, $\boldsymbol{\lambda}_{tt} \in L^\infty(\boldsymbol{L^2})$, $\boldsymbol{\sigma} \in L^\infty(\boldsymbol{H^s})$ and $\Delta t = O(h^{\frac{d}{2}+\varepsilon})$ for some $\varepsilon > 0$, then*

$$\max_{2\le n\le N} \|u(t^n) - U^n\| \le C(h^\nu + \Delta t)\big\{\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(L^2)} + \|\boldsymbol{\sigma}\|_{L^\infty(\boldsymbol{H^s})}\big\}, \tag{4.13}$$

$$\max_{2\le n\le N} \|\boldsymbol{\lambda}(t^n) - \boldsymbol{\Lambda}^n\| \le C(h^\nu + \Delta t)\big\{\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(L^2)} + \|\boldsymbol{\lambda}\|_{L^\infty(\boldsymbol{H^s})} + \|\boldsymbol{\lambda}_{tt}\|_{L^\infty(\boldsymbol{L^2})} + \|\boldsymbol{\sigma}\|_{L^\infty(\boldsymbol{H^s})}\big\}, \tag{4.14}$$

and

$$\max_{2\le n\le N} \|\boldsymbol{\sigma}(t^{n,\theta}) - \boldsymbol{\Sigma}_\theta^n\| \le C(h^\nu + (\Delta t))\big\{\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(L^2)} + \|\boldsymbol{\sigma}\|_{L^\infty(\boldsymbol{H^s})}\big\}, \tag{4.15}$$

hold where $\nu = \min(s, k+1)$.

Proof. We will prove below the statement (i) only. Similarly for $0 < \theta \le 1$, we can get the error estimations (4.13)–(4.15) under the appropriate regularity conditions for u, $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}$. We first prove by mathematical induction that

$$\|e_u^n\|_{L^\infty} \le K^*/5,\ \forall m = 0, 1, \ldots, N. \tag{4.16}$$

For $n = 0$, (4.16) trivially holds since $U^0 = \tilde{u}(0)$. Now we assume that $\|e_u^n\|_{L^\infty} \le K^*/5, \forall n \le m-1$ for some m with $2 \le m \le N$. By (3.4) and (4.7), (4.16) holds for $n = 1$ as follows:

$$\|e_u^1\|_{L^\infty} = \|\tilde{u}^1 - U^1\|_{L^\infty} \le ch^{-\frac{d}{2}}(\|\tilde{u}^1 - U^1\|_{L^\infty}) \le ch^{-\frac{d}{2}}(h^\nu + \Delta t^2) \le K^*/5.$$

From (4.2),

$$(\boldsymbol{\Lambda}^{n,\theta} + \partial_t \boldsymbol{\Lambda}^n, \boldsymbol{w}) - (\boldsymbol{\Sigma}_\theta^n, \boldsymbol{w}) = 0, \qquad \forall\, \boldsymbol{w} \in \boldsymbol{W_h}. \tag{4.17}$$

Now substitute (4.1) into (4.17) to get

$$(U^{n,\theta}, \nabla\cdot\boldsymbol{w}) + (\partial_t U^n, \nabla\cdot\boldsymbol{w}) - (\boldsymbol{\Sigma}_\theta^n, \boldsymbol{w}) = 0. \tag{4.18}$$

Now we take $v = \nabla\cdot\boldsymbol{w}$ in (4.3) and substitute this result into (4.18) to obtain

$$(U^{n,\theta}, \nabla\cdot\boldsymbol{w}) - (\nabla\cdot\boldsymbol{\Sigma}_\theta^n, \nabla\cdot\boldsymbol{w}) + (f(x, t^{n,\theta}, EU^n), \nabla\cdot\boldsymbol{w}) - (\boldsymbol{\Sigma}_\theta^n, \boldsymbol{w}) = 0. \tag{4.19}$$

Substituting (2.1) into (2.2), we get

$$(u, \nabla\cdot\boldsymbol{w}) + (u_t, \nabla\cdot\boldsymbol{w}) - (\boldsymbol{\sigma}, \boldsymbol{w}) = 0, \tag{4.20}$$

and combining (4.20) and (2.3) with $v = \nabla\cdot\boldsymbol{w}$ we have

$$(u, \nabla\cdot\boldsymbol{w}) - (\nabla\cdot\boldsymbol{\sigma}, \nabla\cdot\boldsymbol{w}) + (f(x,t,u), \nabla\cdot\boldsymbol{w}) - (\boldsymbol{\sigma}, \boldsymbol{w}) = 0, \quad \forall\, t \in (0, T]. \tag{4.21}$$

Now we subtract (4.19) from (4.21) to obtain that

$$\begin{aligned} &(u(t^{n,\theta}) - U^{n,\theta}, \nabla\cdot\boldsymbol{w}) - (\nabla\cdot(\boldsymbol{\sigma}(t^{n,\theta}) - \boldsymbol{\Sigma}_\theta^n), \nabla\cdot\boldsymbol{w}) - (\boldsymbol{\sigma}(t^{n,\theta}) - \boldsymbol{\Sigma}_\theta^n, \boldsymbol{w}) \\ =\ & -(f(x, t^{n,\theta}, u(t^{n,\theta})) - f(x, t^{n,\theta}, EU^n), \nabla\cdot\boldsymbol{w}). \end{aligned} \tag{4.22}$$

Noting that $u(t^{n,\theta}) - U^{n,\theta} = \zeta_u(t^{n,\theta}) + \gamma_u^{n,\theta} + e_u^{n,\theta}$, $\boldsymbol{\sigma}(t^{n,\theta}) - \boldsymbol{\Sigma}_\theta^n = \boldsymbol{\zeta_\sigma}(t^{n,\theta}) + \boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}$, and $u(t^{n,\theta}) - EU^n = \zeta_u(t^{n,\theta}) + \gamma_u^{n,\theta} + \xi_u^{n,\theta} + Ee_u^n$ and applying (4.22) with $\boldsymbol{w} = \boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}$, we have

$$\begin{aligned} &(e_u^{n,\theta}, \nabla\cdot\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}) - (\nabla\cdot\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}, \nabla\cdot\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}) - (\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}, \boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}) \\ =\ & -(\zeta_u(t^{n,\theta}), \nabla\cdot\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}) - (\gamma_u^{n,\theta}, \nabla\cdot\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}) + (\nabla\cdot\boldsymbol{\zeta_\sigma}(t^{n,\theta}), \nabla\cdot\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}) \\ & + (\boldsymbol{\zeta_\sigma}(t^{n,\theta}), \boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}) - (f(x, t^{n,\theta}, u(t^{n,\theta})) - f(x, t^{n,\theta}, EU^n), \nabla\cdot\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}). \end{aligned} \tag{4.23}$$

Applying (3.4), Lemma 4.2 and (4.16), we have

$$\begin{aligned} \|u(t^{n,\theta}) - Ev^n\|_{L^\infty} &\le \|\zeta_u(t^{n,\theta})\|_{L^\infty} + \|\gamma_u^{n,\theta}\|_{L^\infty} + \|\xi_u^{n,\theta}\|_{L^\infty} + \|Ee_u^n\|_{L^\infty} \\ &\le 2K^*. \end{aligned} \tag{4.24}$$

Applying (3.1), we have $(\nabla\cdot\boldsymbol{\zeta_\sigma}(t^{n,\theta}), \nabla\cdot\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}) = 0$ and adopt this result, the Cauchy-Schwarz inequality and (4.24) to the equation (4.23) then we have

(4.25) $\|\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}\|^2 + \|\nabla\cdot\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}\|^2$

$$\le \|e_u^{n,\theta}\|^2 + C\big\{\|\zeta_u(t^{n,\theta})\|^2 + \|\gamma_u^{n,\theta}\|^2 + \|\boldsymbol{\zeta_\sigma}(t^{n,\theta})\|^2 + \|\xi_u^{n,\theta}\|^2 + \|Ee_u^n\|^2\big\}$$
$$+ \frac{1}{2}\|\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}\|^2 + \frac{1}{2}\|\nabla\cdot \boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}\|^2,$$

which by (3.4), (3.3) and Lemma 4.2, implies that

$$\|\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}\|^2 + \|\nabla\cdot \boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}\|^2 \tag{4.26}$$
$$\le 2\|e_u^{n,\theta}\|^2 + C\big\{h^{2\nu} + (\Delta t)^4\big\}\big(\|u\|^2_{L^\infty(t^{n-1},t^n:H^s)} + \|u_{tt}\|^2_{L^\infty(t^{n-2},t^n:L^2)}$$
$$+ \|\boldsymbol{\sigma}\|^2_{L^\infty(t^{n-1},t^n:\boldsymbol{H^s})}\big) + C\big(\|e_u^{n-1}\|^2 + \|e_u^{n-2}\|^2\big).$$

Now we subtract (4.3) from (2.3) to get the following

$$\big(u_t(t^{n,\theta}) - \partial_t U^n, v\big) + \big(\nabla\cdot(\boldsymbol{\sigma}(t^{n,\theta}) - \boldsymbol{\Sigma}_\theta^n), v\big)$$
$$= \big(f(x,t^{n,\theta},u(t^{n,\theta})) - f(x,t^{n,\theta},EU^n), v\big), \quad \forall\, v\in V_h,$$

which implies by (3.1) that

$$\big(\partial_t\widetilde{u}^n - \partial_t U^n, v\big) + (\nabla\cdot(\boldsymbol{\Pi_h}\boldsymbol{\sigma}(t^{n,\theta}) - \boldsymbol{\Sigma}_\theta^n), v)$$
$$= \big(\partial_t\widetilde{u}^n - u_t(t^{n,\theta}), v\big) + \big(f(x,t^{n,\theta},u(t^{n,\theta})) - f(x,t^{n,\theta},EU^n), v\big), \quad \forall\, v\in V_h.$$

Therefore we have

$$\big(\partial_t e_u^n, v\big) + (\nabla\cdot(\boldsymbol{\Pi_h}\boldsymbol{\sigma}(t^{n,\theta}) - \boldsymbol{\Sigma}_\theta^n), v)$$
$$= \big(\partial_t\widetilde{u}^n - u_t(t^{n,\theta}), v\big) + \big(f(x,t^{n,\theta},u(t^{n,\theta})) - f(x,t^{n,\theta},EU^n), v\big), \quad \forall\, v\in V_h.$$

from which we deduce the following

$$\big(e_u^n - e_u^{n-1}, e_u^n\big) + \Delta t(\nabla\cdot(\boldsymbol{\Pi_h}\boldsymbol{\sigma}(t^{n,\theta}) - \boldsymbol{\Sigma}_\theta^n), e_u^n)$$
$$= \Delta t\big(\partial_t\widetilde{u}^n - u_t(t^{n,\theta}), e_u^n\big) + \Delta t\big(f(x,t^{n,\theta},u(t^{n,\theta})) - f(x,t^{n,\theta},EU^n), e_u^n\big).$$

By noting that $(e_u^n - e_u^{n-1}, e_u^n) \ge (1/2)\|e_u^n\|^2 - (1/2)\|e_u^{n-1}\|^2$ and applying (3.4), Lemma 4.1 and Lemma 4.2, we get for $\theta = 0$,

$$\frac{1}{2}\|e_u^n\|^2 - \frac{1}{2}\|e_u^{n-1}\|^2$$
$$\le \frac{1}{2}(\Delta t)\|\nabla\cdot(\boldsymbol{\Pi_h}\boldsymbol{\sigma}(t^{n,\theta}) - \boldsymbol{\Sigma}_\theta^n)\|^2 + C(\Delta t)\|e_u^n\|^2$$
$$+ C(\Delta t)(h^{2\nu} + (\Delta t)^4)\big(\|u_t\|^2_{L^\infty(t^{n-1},t^n:H^s)} + \|u_{ttt}\|^2_{L^\infty(t^{n-1},t^n:L^2)}\big)$$
$$+ C(\Delta t)\big(\|\zeta_u(t^{n,\theta})\|^2 + \|\gamma_u^{n,\theta}\|^2 + \|\xi_u^{n,\theta}\|^2 + \|Ee_u^n\|^2\big)$$
$$\le \frac{1}{2}(\Delta t)\|\nabla\cdot(\boldsymbol{\Pi_h}\boldsymbol{\sigma}(t^{n,\theta}) - \boldsymbol{\Sigma}_\theta^n)\|^2 + C(\Delta t)\|e_u^n\|^2$$
$$+ C(\Delta t)(h^{2\nu} + (\Delta t)^4)\big(\|u_t\|^2_{L^\infty(t^{n-1},t^n:H^s)} + \|u_{ttt}\|^2_{L^\infty(t^{n-1},t^n:L^2)}$$
$$+ \|u\|^2_{L^\infty(t^{n-1},t^n:H^s)} + \|u_{tt}\|^2_{L^\infty(t^{n-2},t^n:L^2)}\big) + C(\Delta t)\big(\|e_u^{n-1}\|^2 + \|e_u^{n-2}\|^2\big).$$

Now we add the both sides of the above inequality from $n = 2$ to $n = m$ and applying (4.26) we get the following,

$$\frac{1}{2}\|e_u^m\|^2 - \frac{1}{2}\|e_u^1\|^2$$

$$\leq \frac{1}{2}\Delta t \sum_{n=2}^{m} \|\nabla\cdot(\mathbf{\Pi_h}\boldsymbol{\sigma}(t^{n,\theta}) - \mathbf{\Sigma}_\theta^n)\|^2 + C(\Delta t)\sum_{n=2}^{m}\|e_u^n\|^2$$
$$+ C(h^{2\nu} + (\Delta t)^4)\big(\|u\|^2_{L^\infty(H^s)} + \|u_t\|^2_{L^\infty(H^s)} + \|u_{tt}\|^2_{L^\infty(L^2)} + \|u_{ttt}\|^2_{L^\infty(L^2)}\big)$$
$$+ C(\Delta t)\sum_{n=2}^{m}\big(\|e_u^{n-1}\|^2 + \|e_u^{n-2}\|^2\big)$$
$$\leq \frac{1}{2}\Delta t\sum_{n=2}^{m}\Big\{2\|e_u^{n,\theta}\|^2 + C(h^{2\nu} + (\Delta t)^4)\big(\|u\|^2_{L^\infty(t^{n-1},t^n:H^s)}$$
$$+ \|u_{tt}\|^2_{L^\infty(t^{n-2},t^n:L^2)} + \|\boldsymbol{\sigma}\|^2_{L^\infty(t^{n-1},t^n:\boldsymbol{H^s})}\big) + C\big(\|e_u^{n-1}\|^2 + \|e_u^{n-2}\|^2\big)\Big\}$$
$$+ C(\Delta t)\sum_{n=2}^{m}\|e_u^n\|^2 + C(h^{2\nu} + (\Delta t)^4)\big(\|u\|^2_{L^\infty(H^s)} + \|u_t\|^2_{L^\infty(H^s)}$$
$$+ \|u_{tt}\|^2_{L^\infty(L^2)} + \|u_{ttt}\|^2_{L^\infty(L^2)}\big) + C(\Delta t)\sum_{n=2}^{m}\big(\|e_u^{n-1}\|^2 + \|e_u^{n-2}\|^2\big),$$

from which we have

$$\frac{1}{2}\|e_u^m\|^2 \leq C(h^{2\nu} + (\Delta t)^4)\big(\|u\|^2_{L^\infty(H^s)} + \|u_t\|^2_{L^\infty(H^s)} + \|u_{tt}\|^2_{L^\infty(L^2)}$$
$$+ \|u_{ttt}\|^2_{L^\infty(L^2)} + \|\boldsymbol{\sigma}\|^2_{L^\infty(\boldsymbol{H^s})}\big) + C(\Delta t)\sum_{n=0}^{m}\|e_u^n\|^2 + \frac{1}{2}\|e_u^1\|^2.$$

Since Δt is sufficiently small, we obtain

$$\|e_u^m\|^2 \leq C\Delta t\sum_{n=0}^{m-1}\|e_u^n\|^2 + C(h^{2\nu} + (\Delta t)^4)\big\{\|u\|^2_{L^\infty(H^s)} + \|u_t\|^2_{L^\infty(H^s)}$$
$$+ \|u_{tt}\|^2_{L^\infty(L^2)} + \|u_{ttt}\|^2_{L^\infty(L^2)} + \|\boldsymbol{\sigma}\|^2_{L^\infty(\boldsymbol{H^s})}\big\} + \|e_u^1\|^2.$$

By applying the discrete type of Gronwall inequality, we get

$$\|e_u^m\|^2 \leq C\|e_u^1\|^2 + C\big(h^{2\nu} + (\Delta t)^4\big)\big(\|u\|^2_{L^\infty(H^s)} + \|u_t\|^2_{L^\infty(H^s)}$$
$$+ \|u_{tt}\|^2_{L^\infty(L^2)} + \|u_{ttt}\|^2_{L^\infty(L^2)} + \|\boldsymbol{\sigma}\|^2_{L^\infty(\boldsymbol{H^s})}\big). \tag{4.27}$$

Therefore $\|e_u^m\|_{L^\infty} \leq Ch^{-\frac{d}{2}}\{h^\nu + (\Delta t)^2\} \leq K^*/5$ holds, so by (4.7) we proved the statement (4.10) as follows:

$$\max_{2\leq m\leq N}\|u(t^n) - U^n\| \leq C\big(h^\nu + (\Delta t)^2\big)\big(\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)}$$
$$+ \|u_{tt}\|_{L^\infty(L^2)} + \|u_{ttt}\|_{L^\infty(L^2)} + \|\boldsymbol{\sigma}\|_{L^\infty(\boldsymbol{H^s})}\big).$$

From (4.26) and (4.27) we get

$$\|\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}\|^2 \leq C\|u(t^1) - U^1\| + C\big(h^{2\nu} + (\Delta t)^4\big)\big(\|u\|^2_{L^\infty(H^s)} + \|u_t\|^2_{L^\infty(H^s)}$$
$$+ \|u_{tt}\|^2_{L^\infty(L^2)} + \|u_{ttt}\|^2_{L^\infty(L^2)} + \|\boldsymbol{\sigma}\|^2_{L^\infty(\boldsymbol{H^s})}\big).$$

Therefore by (4.7), we proved the statement (4.12) as follows:

$$\begin{aligned}\|\boldsymbol{\sigma}(t^{n,\theta})-\boldsymbol{\Sigma}_\theta^n\| &\le \|\boldsymbol{\zeta_\sigma}(t^{n,\theta})\|+\|\boldsymbol{e}_{\boldsymbol{\sigma}}^{n,\theta}\| \\ &\le C\big(h^\nu+(\Delta t)^2\big)\big\{\|u\|_{L^\infty(H^s)}+\|u_t\|_{L^\infty(H^s)}+\|u_{tt}\|_{L^\infty(L^2)} \\ &\qquad +\|u_{ttt}\|_{L^\infty(L^2)}+\|\boldsymbol{\sigma}\|_{L^\infty(\boldsymbol{H^s})}\big\}. \end{aligned} \tag{4.28}$$

To estimate $\|\boldsymbol{\lambda}(t^n)-\boldsymbol{\Lambda}^n\|$, we substract (4.2) from (2.2) to get

$$(\boldsymbol{\lambda}(t^{n,\theta})-\boldsymbol{\Lambda}^{n,\theta},\boldsymbol{\mu})+(\boldsymbol{\lambda}_t(t^{n,\theta})-\partial_t\boldsymbol{\Lambda}^n,\boldsymbol{\mu})-(\boldsymbol{\sigma}(t^{n,\theta})-\boldsymbol{\Sigma}_\theta^n,\boldsymbol{\mu})=0. \tag{4.29}$$

Since $\boldsymbol{\lambda}(t^{n,\theta})-\boldsymbol{\Lambda}^{n,\theta}=\boldsymbol{\zeta_\lambda}(t^{n,\theta})+\boldsymbol{\gamma}_{\boldsymbol{\lambda}}^{n,\theta}+\boldsymbol{e}_{\boldsymbol{\lambda}}^{n,\theta}$, we get

$$\begin{aligned}\boldsymbol{\lambda}_t(t^{n,\theta})-\partial_t\boldsymbol{\Lambda}^n &= \boldsymbol{\lambda}_t(t^{n,\theta})-\widetilde{\boldsymbol{\lambda}}_t(t^{n,\theta})+\widetilde{\boldsymbol{\lambda}}_t(t^{n,\theta})-\partial_t\widetilde{\boldsymbol{\lambda}}^n+\partial_t\widetilde{\boldsymbol{\lambda}}^n-\partial_t\boldsymbol{\Lambda}^n \\ &= \boldsymbol{\zeta}_{\boldsymbol{\lambda}t}(t^{n,\theta})-\boldsymbol{\rho}_{\boldsymbol{\lambda}}^{n,\theta}+\partial_t\boldsymbol{e}_{\boldsymbol{\lambda}}^n.\end{aligned}$$

Adopting this relation to (4.29) and applying (3.5), we have

$$\begin{aligned}(\boldsymbol{e}_{\boldsymbol{\lambda}}^{n,\theta},\boldsymbol{e}_{\boldsymbol{\lambda}}^n)+(\partial_t\boldsymbol{e}_{\boldsymbol{\lambda}}^n,\boldsymbol{e}_{\boldsymbol{\lambda}}^n) =& -(\boldsymbol{\zeta_\lambda}(t^{n,\theta})+\boldsymbol{\gamma}_{\boldsymbol{\lambda}}^{n,\theta},\boldsymbol{e}_{\boldsymbol{\lambda}}^n)-(\boldsymbol{\zeta}_{\boldsymbol{\lambda}t}(t^{n,\theta})-\boldsymbol{\rho}_{\boldsymbol{\lambda}}^{n,\theta},\boldsymbol{e}_{\boldsymbol{\lambda}}^n) \\ &+(\boldsymbol{\sigma}(t^{n,\theta})-\boldsymbol{\Sigma}_\theta^n,\boldsymbol{e}_{\boldsymbol{\lambda}}^n) \\ =& -(\boldsymbol{\gamma}_{\boldsymbol{\lambda}}^{n,\theta},\boldsymbol{e}_{\boldsymbol{\lambda}}^n)+(\boldsymbol{\rho}_{\boldsymbol{\lambda}}^{n,\theta},\boldsymbol{e}_{\boldsymbol{\lambda}}^n)+(\boldsymbol{\sigma}(t^{n,\theta})-\boldsymbol{\Sigma}_\theta^n,\boldsymbol{e}_{\boldsymbol{\lambda}}^n),\end{aligned}$$

which implies,

$$\begin{aligned}\frac{1}{2\Delta t}\big(\|\boldsymbol{e}_{\boldsymbol{\lambda}}^n\|^2-\|\boldsymbol{e}_{\boldsymbol{\lambda}}^{n-1}\|^2\big) &\le \Big(\Big(\alpha_1+\frac{\alpha_2}{2}\Big)\|\boldsymbol{e}_{\boldsymbol{\lambda}}^n\|^2+\frac{\alpha_2}{2}\|\boldsymbol{e}_{\boldsymbol{\lambda}}^{n-1}\|^2\Big)+\frac{1}{2}\|\boldsymbol{\gamma}_{\boldsymbol{\lambda}}^{n,\theta}\|^2 \\ &\quad +\frac{3}{2}\|\boldsymbol{e}_{\boldsymbol{\lambda}}^n\|^2+\frac{1}{2}\|\boldsymbol{\rho}_{\boldsymbol{\lambda}}^{n,\theta}\|^2+\frac{1}{2}\|\boldsymbol{\Sigma}_\theta^n-\boldsymbol{\sigma}(t^{n,\theta})\|^2 \\ &\le C\big\{\|\boldsymbol{e}_{\boldsymbol{\lambda}}^n\|^2+\|\boldsymbol{e}_{\boldsymbol{\lambda}}^{n-1}\|^2\big\}+\frac{1}{2}\|\boldsymbol{\gamma}_{\boldsymbol{\lambda}}^{n,\theta}\|^2+\frac{1}{2}\|\boldsymbol{\rho}_{\boldsymbol{\lambda}}^{n,\theta}\|^2 \\ &\quad +\frac{1}{2}\|\boldsymbol{\Sigma}_\theta^n-\boldsymbol{\sigma}(t^{n,\theta})\|^2.\end{aligned} \tag{4.30}$$

Now we add the both sides of (4.30) from $n=2$ to N to obtain

$$\|\boldsymbol{e}_{\boldsymbol{\lambda}}^N\|^2 \le C\Delta t\sum_{n=1}^N\|\boldsymbol{e}_{\boldsymbol{\lambda}}^n\|^2+\Delta t\sum_{n=2}^N\big(\|\boldsymbol{\gamma}_{\boldsymbol{\lambda}}^{n,\theta}\|^2+\|\boldsymbol{\rho}_{\boldsymbol{\lambda}}^{n,\theta}\|^2+\|\boldsymbol{\Sigma}_\theta^n-\boldsymbol{\sigma}(t^{n,\theta})\|^2\big)+\|\boldsymbol{e}_{\boldsymbol{\lambda}}^1\|^2.$$

Therefore for sufficiently small Δt, by applying Lemmas 4.3, 4.1 and (4.28) we conclude that

$$\begin{aligned}\|\boldsymbol{e}_{\boldsymbol{\lambda}}^N\|^2 &\le C\Delta t\sum_{n=1}^{N-1}\|\boldsymbol{e}_{\boldsymbol{\lambda}}^n\|^2+C(\Delta t)\big(h^{2\nu}+(\Delta t)^4\big)\sum_{n=2}^N\big(\|\boldsymbol{\lambda}_{tt}\|^2_{L^\infty(t^{n-1},t^n:\boldsymbol{L^2})} \\ &\quad +\|\boldsymbol{\lambda}_{ttt}\|^2_{L^\infty(t^{n-1},t^n:\boldsymbol{L^2})}+\|u\|^2_{L^\infty(H^s)}+\|u_t\|^2_{L^\infty(H^s)}+\|u_{tt}\|^2_{L^\infty(L^2)} \\ &\quad +\|u_{ttt}\|^2_{L^\infty(L^2)}+\|\boldsymbol{\sigma}\|^2_{L^\infty(\boldsymbol{H^s})}\big)+C\|\widetilde{\boldsymbol{\lambda}}(t^1)-\boldsymbol{\Lambda}^1\|^2.\end{aligned}$$

Now we apply the discrete-type Gronwall inequality and (4.8) to get

$$\|\boldsymbol{e}_{\boldsymbol{\lambda}}^N\|^2 \le C\big(h^{2\nu}+(\Delta t)^4\big)\big\{\|u\|^2_{L^\infty(H^s)}+\|u_t\|^2_{L^\infty(H^s)}+\|u_{tt}\|^2_{L^\infty(L^2)}$$

$$+ \|u_{ttt}\|^2_{L^\infty(L^2)} + \|\boldsymbol{\lambda}_{tt}\|^2_{L^\infty(\boldsymbol{L^2})} + \|\boldsymbol{\lambda}_{ttt}\|^2_{L^\infty(\boldsymbol{L^2})} + \|\boldsymbol{\sigma}\|^2_{L^\infty(\boldsymbol{H^s})}\big\},$$

from which we obtain the statement (4.11)

$$\|\boldsymbol{\lambda}(t^n) - \boldsymbol{\Lambda}^n\| \le C\big(h^\nu + (\Delta t)^2\big)\big\{\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(L^2)} + \|u_{ttt}\|_{L^\infty(L^2)} + \|\boldsymbol{\lambda}\|_{L^\infty(\boldsymbol{H^s})} + \|\boldsymbol{\lambda}_{tt}\|_{L^\infty(\boldsymbol{L^2})} + \|\boldsymbol{\lambda}_{ttt}\|_{L^\infty(\boldsymbol{L^2})} + \|\boldsymbol{\sigma}\|_{L^\infty(\boldsymbol{H^s})}\big\}. \qquad \square$$

5. Numerical results and conclusions

In this section, we will present some numerical results to verify the convergence order of the proposed EMFEM. For the sake of convenience we consider the Sobolev equation (1.1) with $\Omega = [0,1]$ and $T = 1.0$. The fully discrete scheme (4.1)-(4.6) is characterized by θ. For each θ, we provide a set of numerical results with $f(x,t)$ and $f(x,t,u)$ which is locally Lipschitz continuous in u. And also we choose $k = 0$ i.e. $V_h = \{v \in V \mid v|_E \in P_0(E),\ \forall E \in \mathcal{E}_h\}$,

$$\boldsymbol{\Lambda_h} = \{\boldsymbol{\mu} \in \boldsymbol{\Lambda} \mid \boldsymbol{\mu}|_E \in P_0(E) \oplus xP_0(E),\ \ \forall E \in \mathcal{E}_h\},$$
$$\boldsymbol{W_h} = \{\boldsymbol{w} \in \boldsymbol{W} \mid \boldsymbol{w}|_E \in P_0(E) \oplus xP_0(E),\ \ \forall E \in \mathcal{E}_h\}.$$

(I) In case of $\theta = 1$ (Bachward Euler method).

To prove the order of convergence we choose $\Delta t = h$.

(1) with $f(x,t) = (1 + 2\pi^2)e^t \cos(\pi x)$.

With $u_0(x) = \cos \pi x$, the solution of (1.1) is given by $u(x,t) = e^t \cos(\pi x)$.

Tables 1 and 2 show that the approximations of $u(x,t)$, $\lambda(x,t) = -u_x$ and $\sigma(x,t) = -(u_x + u_x)$ converge with convergence order = 1 for the space variable as well as the time variable as we expect from Theorem 4.1(ii).

TABLE 1

$h = \Delta t$	$\|u(t^N) - U^N\|$	convergence order
1/10	0.1765 e-0	
1/20	0.8897 e-1	0.99
1/40	0.4468 e-1	0.99
1/80	0.2239 e-1	1.00
1/160	0.1121 e-1	1.00
1/320	0.5607 e-2	1.00

(2) with $f(x,t,u) = u + u^2 + 2e^t(2x-1) - e^{2t}(\frac{1}{2}x^2 - \frac{1}{3}x^3)^2$.

With $u_0(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$, the solution of (1.1) is given by $u(x,t) = e^t(\frac{1}{2}x^2 - \frac{1}{3}x^3)$. Then $-u_x = e^t(x^2 - x)$ and $-u_x - u_{xt} = 2e^t(x^2 - x)$. Tables 3 and 4 show that the approximations of u, $-u_x$ and $-(u_x + u_{xt})$ converge with convergence order = 1.

(II) In case of $\theta = 0$ (Crank-Nicolson method).

(i) with $f(x,t) = (1 + 2\pi^2)e^t \cos(\pi x)$.

Table 2

$h=\Delta t$	$\|\boldsymbol{\lambda}(t^N)-\boldsymbol{\Lambda}^N\|$	convergence order	$\|\boldsymbol{\sigma}(t^N)-\boldsymbol{\Sigma}_1^N\|$	convergence order
1/80	0.1619 e-1		0.6988 e-3	
1/160	0.8258 e-2	0.97	0.4428 e-3	0.66
1/320	0.4170 e-2	0.99	0.2987 e-3	0.57
1/640	0.2095 e-2	0.99	0.1711 e-3	0.80
1/1280	0.1050 e-2	1.00	0.9116 e-4	0.91
1/2560	0.5258 e-3	1.00	0.4700 e-4	0.96

Table 3

$h=\Delta t$	$\|u(t^N)-U^N\|$	convergence order
1/10	0.1865 e-1	
1/20	0.9760 e-2	0.93
1/40	0.4992 e-2	0.97
1/80	0.2523 e-2	0.98
1/160	0.1268 e-2	0.99
1/320	0.6359 e-3	1.00

Table 4

$h=\Delta t$	$\|\boldsymbol{\lambda}(t^N)-\boldsymbol{\Lambda}^N\|$	convergence order	$\|\boldsymbol{\sigma}(t^N)-\boldsymbol{\Sigma}_1^N\|$	convergence order
1/40	0.2720 e-2		0.5365 e-3	
1/80	0.1421 e-2	0.94	0.3596 e-3	0.58
1/160	0.7258 e-3	0.97	0.2120 e-3	0.76
1/320	0.3668 e-3	0.99	0.1147 e-3	0.89
1/640	0.1843 e-3	0.99	0.5957 e-4	0.95
1/1280	0.9242 e-4	1.00	0.3035 e-4	0.97

Theorem 4.1 shows that for $\theta=0$, the scheme converges with convergence order 2 in temporal direction. As shown in Table 5, $U^N(x)$ converges to $u(t^N)$ with convergence order 1, since the spatial error $O(h)$ dominates the temporal error $O(\Delta t^2)$. Since with $d=1$ $\boldsymbol{\Lambda_h}(E)=\boldsymbol{W_h}(E)=P_0(E)\oplus xP_0(E)=P_1(E)$ holds, so that we have a chance to get the approximations Λ^N and Σ_θ^N of $\lambda(t^N)$ and $\sigma(t^N)$ which converge to $\lambda(t^N)$ and $\sigma(t^N)$ with convergence order 2 in spatial variable as shown in Table 6, though U^N converges to $u(t^N)$ with convergence order 1.

(2) with $f(x,t,u)=u+u^2+2e^t(2x-1)-e^{2t}(\frac{1}{2}x^2-\frac{1}{3}x^3)^2$.

As appeared in Tables 7 and 8, we have the computational convergence results which validate the theoretical proofs of Theorem 4.1 with a locally Lipschitz continuous $f(x,t,u)$ in u.

Conclusions. In this paper, applying the EMFEM to the problem (1.1) we approximate the scalar unknown, the gradient and the flux separately and prove

TABLE 5

$h = \Delta t$	$\|u(t^N) - U^N\|$	convergence order
1/10	0.1746 e-0	
1/30	0.5812 e-1	1.00
1/90	0.1937 e-1	1.00
1/270	0.6456 e-2	1.00
1/810	0.2152 e-2	1.00

TABLE 6

$h = \Delta t$	$\|\boldsymbol{\lambda}(t^N) - \boldsymbol{\Lambda}^N\|$	convergence order	$\|\boldsymbol{\sigma}(t^N) - \boldsymbol{\Sigma}_1^N\|$	convergence order
1/10	0.4958 e-1		0.9834 e-1	
1/30	0.5519 e-2	2.00	0.1094 e-1	2.00
1/90	0.6133 e-3	2.00	0.1216 e-2	2.00
1/270	0.6815 e-4	2.00	0.1351 e-3	2.00
1/810	0.7572 e-5	2.00	0.1501 e-4	2.00

TABLE 7

$h = \Delta t$	$\|u(t^N) - U^N\|$	convergence order
1/10	0.1444 e-1	
1/30	0.4781 e-2	1.01
1/90	0.1592 e-2	1.00
1/270	0.5306 e-3	1.00
1/810	0.1769 e-3	1.00

TABLE 8

$h = \Delta t$	$\|\boldsymbol{\lambda}(t^N) - \boldsymbol{\Lambda}^N\|$	convergence order	$\|\boldsymbol{\sigma}(t^N) - \boldsymbol{\Sigma}_1^N\|$	convergence order
1/10	0.4763 e-2		0.9815 e-2	
1/30	0.5311 e-3	2.00	0.1094 e-2	2.00
1/90	0.5906 e-4	2.00	0.1217 e-3	2.00
1/270	0.6565 e-5	2.00	0.1353 e-4	2.00
1/810	0.7295 e-6	2.00	0.1504 e-5	2.00

the convergence of optimal order. We prove the convergence of three unknowns theoretically as well as computationally. We present the numerical results with $d = 1$ which verify the theoretical analysis of the optimal order of convergence of three unknowns. We conclude that the EMFEM accomplishes our purpose in approximating the unknowns of the semilinear Sobolev equations.

References

[1] T. Arbogast, M. F. Wheeler, and I. Yotov, *Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences*, SIAM. J. Numer. Anal. **34** (1997), no. 2, 828–852.

[2] D. N. Arnold, *An interior penalty finite element method with discontinuous elements*, SIAM J. Numer. Anal. **19** (1982), no. 4, 724–760.

[3] D. N. Arnold, J. Jr. Douglas, and V. Thomée, *Superconvergence of a finite element approximation to the solution of a Sobolev equation in a single space variable*, Math. Comp. **36** (1981), no. 153, 53–63.

[4] G. I. Barenblatt, I. P. Zheltov, and I. N. Kochina, *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks*, J. Appl. Math. Mech. **24** (1960), no. 5, 1286–1303.

[5] R. W. Carroll and R. E. Showalter, *Singular and degenerate Cauchy problems*, Mathematics in Sciences and Engineering, Vol. 127, Academic Press, New York, 1976.

[6] P. J. Chen and M. E. Gurtin, *On a theory of heat conduction involving two temperatures*, Z. Angew Math. Phys. **19** (1968), no. 4, 614–627.

[7] Z. Chen, *BDM mixed methods for a nonlinear elliptic problem*, J. Comput. Appl. Math. **53** (1994), no. 2, 207–223.

[8] ———, *Expanded mixed finite element methods for linear second-order elliptic problems I*, RAIRO. Model. Math. Anal. Mumer. **32** (1998), no. 4, 479–499.

[9] ———, *Expanded mixed finite element methods for quasilinear second-order elliptic problems II*, RAIRO. Model. Math. Anal. Numer **32** (1998), no. 4, 501–520.

[10] Y. Chen, Y. Huang, and D. Yu, *A two-grid method for expanded mixed finite-element solution of semilinear reaction-diffusion equations*, Internat. J. Numer. Methods. Engrg **57** (2003), no. 2, 193–209.

[11] Y. Chena and L. Li, *L^p error estimates of two-grid schemes of expanded mixed finite element methods*, Appl. Math. Comp. **209** (2009), no. 2, 197–205.

[12] P. L. Davis, *A quasilinear parabolic and a related third order problem*, J. Math. Anal. Appl. **40** (1972), no. 2, 327–335.

[13] J. Douglas and J. E. Roberts, *Global estimates for mixed methods for second order elliptic equations*, Math. Comp. **44** (1985), no. 169, 39–52.

[14] R. Duran, *Error analysis in $L^p, 1 \le p \le \infty$ for mixed finite element mehtods for linear and quasi-linear elliptic problems*, RAIRO Mode. Math. Anal. Numer. **22** (1988), no. 3, 371–387.

[15] R. E. Ewing, *The approximation of certain parabolic equations backward in time by Sobolev equations*, SIAM J. Math. Anal. **6** (1975), no. 2, 283–294.

[16] ———, *Time-stepping Galerkin methods for nonlinear Sobolev partial differential equations*, SIAM J. Numer. Anal. **15** (1978), no. 6, 1125–1150.

[17] F. Gao, J. Qiu and Q. Zhang, *Local discontinuous Galerkin finite element method and error estimates for one class of Sobolev equation* J. Sci. Comput. **41** (2009), no. 3, 436–460.

[18] D. Kim and E.-J. Park, *A posteriori error estimator for expanded mixed hybrid methods*, Numer. Methods Partial Differential Equations **23** (2007), no. 2, 330–349.

[19] Y. Lin, *Galerkin methods for nonlinear Sobolev equations*, Aequationes Math. **40** (1990), no. 1, 54–66.

[20] Y. Lin and T. Zhang, *Finite element methods for nonlinear Sobolev equations with nonlinear boundary conditions*, J. Math. Anal. Appl. **165** (1992), no. 1, 180–191.

[21] M. T. Nakao, *Error estimates of a Galerkin method for some nonlinear Sobolev equations in one space dimension*, Numer. Math. **47** (1985), no. 1, 139–157.

[22] J. C. Nedelec, *Mixed finite elements in $\mathbb{R}^3$*, Numer. Math. **35** (1980), no. 3, 315–341.

[23] M. R. Ohm, H. Y. Lee, and J. Y. Shin, *Error analysis of a mixed finite element approximation of the semilinear Sobolev equations*, J. Appl. Math. Comput. **40** (2012), no. 1-2, 95–110.

[24] P. A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, in Proc. Conf. on Mathemaical Aspects of Finite Element Methods, pp. 292–315. Lecture Notes in Math., Vol. 606, Springer, Berlin, 1977.

[25] D. Shi and Y. Zhang, *High accuracy analysis of a new nonconforming mixed finite element scheme for Sobolev equations*, Appl. Math. Comput. **218** (2011), no. 7, 3176–3186.

[26] T. Sun and D. Yang, *A priori error estimates for interior penalty discontinuous Galerkin method applied to nonlinear Sobolev equations*, Appl. Math. Comput. **200** (2008), no. 1, 147–159.

[27] ———, *Error estimates for a discontinuous Galerkin method with interior penalties applied to nonlinear Sobolev equations*, Numer. Methods Partial Differential Equations **24** (2008), no. 3, 879–896.

[28] T. W. Ting, *A cooling process according to two-temperature theory of heat conduction*, J. Math. Anal. Appl. **45** (1974), 289–303.

[29] M. F. Wheeler, K. R. Roberson, and A. Chilakapati, *Three-dimensional bioremediation modeling in heterogeneous porous media*, Computational methods in water resources IX, Vol. 2, Mathematical modeling in water resources, T. F. Russell, R. E. Ewing, C. A. Brebbia, W. G. Gray and G. F. Pindar, editors, Computational Mechanics Publications, 299–315, Southampton, UK, 1992.

[30] C. S. Woodward and C. N. Dawson, *Analysis of expanded mixed finite element methods for a nonlinear parabolic equation modeling flow into variably saturated porous media*, SIAM. J. Numer. Anal. **37** (2000), no. 3, 701–724.

Mi Ray Ohm
Division of Information Systems Engineering
Dongseo University
Busan 617-716, Korea
E-mail address: mrohm@dongseo.ac.kr

Hyun Young Lee
Department of Mathematics
Kyungsung University
Busan 608-736, Korea
E-mail address: hylee@ks.ac.kr

Jun Yong Shin
Department of Applied Mathematics
Pukyong National University
Busan 608-737, Korea
E-mail address: jyshin@pknu.ac.kr