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ANNIHILATORS IN ONE-SIDED IDEALS GENERATED BY COEFFICIENTS OF ZERO-DIVIDING POLYNOMIALS

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Abstract. Nielsen and Rege-Chhawchharia called a ring R *right McCoy* if given nonzero polynomials $f(x), g(x)$ over R with $f(x)g(x) = 0$, there exists a nonzero element $r \in R$ with $f(x)r = 0$. Hong et al. called a ring R *strongly right McCoy* if given nonzero polynomials $f(x), g(x)$ over R with $f(x)g(x) = 0$, $f(x)r = 0$ for some nonzero r in the right ideal of R generated by the coefficients of $g(x)$. Subsequently, Kim et al. observed similar conditions on linear polynomials by finding nonzero r's in various kinds of one-sided ideals generated by coefficients. But almost all results obtained by Kim et al. are concerned with the case of products of linear polynomials. In this paper we examine the nonzero annihilators in the products of general polynomials.

1. Introduction

A ring is usually called *reduced* if it has no nonzero nilpotent elements. Cohn [5] called a ring R *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Due to Narbonne [21], a ring R is called *semicommutative* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Reduced rings are reversible and reversible rings are semicommutative, but not conversely in general. Rege and Chhawchharia called R an *Armendariz* ring [24, Definition 1.1] if whenever any polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $ab = 0$ for each coefficient a of $f(x)$ and b of $g(x)$. Any reduced ring is Armendariz by [3, Lemma 1], but the class of semicommutative rings and the class of Armendariz rings don't imply each other by [24, Example 3.2] and [8, Example 14]. McCoy [20] showed that if two polynomials annihilate each other over a commutative ring then each polynomial has a nonzero annihilator in the base ring. In [10], Weiner showed this fact fails in noncommutative rings. Based on this result, Nielsen [22] and Rege and Chhawchharia [24] called a noncommutative ring R *right McCoy* (resp., left McCoy) if whenever any nonzero polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $f(x)c = 0$ (resp., $cg(x) = 0$) for some nonzero $c \in R$,

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and a ring R is called *McCoy* if it is both left and right McCoy. Armendariz rings are clearly McCoy but the converse does not hold by [24, Remark 4.3]. A ring is called *Abelian* if every idempotent is central. It is well-known that semicommutative rings and Armendariz rings are Abelian. Nielsen developed and extended the concept of a McCoy ring. In particular, he showed that any reversible ring is McCoy [22, Theorem 2] and gave an example that is a semicommutative ring but not McCoy [22, Section 3]. Nielsen also showed that the McCoy condition is not left-right symmetric [22, Section 3 and Section 4].

Hong et al. [7] called a ring R (possibly without identity) *strongly right* (resp., *left*) *McCoy* if $f(x)g(x) = 0$ implies $f(x)r = 0$ (resp., $rg(x) = 0$) for some nonzero r in the right (resp., left) ideal of R generated by the coefficients of $g(x)$ (resp., $f(x)$), where $f(x)$ and $g(x)$ are nonzero polynomials in $R[x]$. Strongly McCoy property for rings is not left-right symmetric by [11, Remark 2.6]. Reversible rings are strongly both left and right McCoy by [7, Theorem 1.6] or the proof of [22, Theorem 2]. Strongly right McCoy rings are clearly right McCoy, but not conversely by [7, Example 1.9].

Recently, the strongly McCoy condition for a ring is generalized by Kim et al. [11]. A ring R (possibly without identity) is called *right linearly right-ideal-McCoy* (resp., *right linearly left-ideal-McCoy*) [11, Definition 2.1] if $f(x)g(x) = 0$ implies $f(x)r = 0$ for some nonzero r in the right (resp., left) ideal of R generated by the set of all coefficients of $g(x)$, where $f(x)$ and $0 \neq g(x)$ are linear polynomials in $R[x]$. The *left linearly left-ideal-McCoy* and *left linearly right-ideal-McCoy* can be defined symmetrically. Strongly right McCoy rings are clearly right linearly right-ideal-McCoy but not conversely by [11, Example 2.5(1)].

In this paper, we continue study of the McCoy condition for one-sided ideals generated by the coefficients of zero-dividing polynomials which extends the concept of linearly left-ideal-McCoy property.

Throughout this note every ring is associative with identity unless otherwise stated. We use $R[x]$ to denote the polynomial ring with an indeterminate x over a ring R. Let $C_{f(x)}$ denote the set of all coefficients of $f(x) \in R[x]$ and $RC_{f(x)}$ (resp., $C_{f(x)}R$) denote the left (resp., right) ideal of R generated by $C_{f(x)}$. Denote the n by n full (resp., upper triangular) matrix ring over a ring R by $Mat_n(R)$ (resp., $U_n(R)$). Use E_{ij} for the matrix with (i, j)-entry 1 and elsewhere 0. $\mathbb{Z}_n$ denotes the ring of integers modulo n.

2. Property of right left-ideal-McCoy rings

Due to Lambek [18], an ideal I of a ring R is called *symmetric* if $rst \in I$ implies $rts \in I$ for all $r, s, t \in R$. If the zero ideal of a ring R is symmetric then R is also called *symmetric*; while Anderson and Camillo [2] used the term ZC_3 for this concept. Commutative rings and reduced rings are clearly symmetric. Symmetric rings are clearly reversible but not conversely in general.

Thinking about [7, Corollary 1.2] in the context of a noncommutative ring, we will say that a ring R has the condition (†) if whenever $f(x)g(x) = 0$ where $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ are nonzero in $R[x]$, there is a product $a_{t_1} a_{t_2} \cdots a_{t_h}$, with $a_{t_i} \in \{a_0, \ldots, a_m\}$ for each i (if any), and some $b_t \in \{b_0, \ldots, b_n\}$, such that $(a_{t_1} \cdots a_{t_h}) b_t \neq 0$ but $f(x)(a_{t_1} \cdots a_{t_h}) b_t = 0$.

Symmetric rings satisfy the condition (†) by [7, Proposition 1.7], and reversible rings also satisfy the condition (†) for the product of linear polynomials by the proof of [4, Proposition 5.3].

We start with the following definition which not only generalizes the condition (†) but also strengthens the concept of right linearly left-ideal-McCoy rings.

Using the definitions in [11], a ring R (possibly without identity) is called *right left-ideal-McCoy* (resp., *right right-ideal-McCoy*) if $f(x)g(x) = 0$ implies $f(x)r = 0$ for some nonzero r in the left (resp., right) ideal of R generated by $C_{g(x)}$ for two polynomials $f(x)$ and $0 \neq g(x)$ in $R[x]$. The *left right-ideal-McCoy* and *left left-ideal-McCoy* can be defined symmetrically.

Remark 2.1. (1) Note that our definition of a right right-ideal McCoy (resp., left left-ideal McCoy) ring is precisely what Hong et al. defined as a strongly right (resp., left) McCoy ring in [7]. The class of right left-ideal-McCoy rings and the class of strongly right McCoy rings do not imply each other by [11, Example 2.5].

(2) The class of right left-ideal-McCoy rings and the class of left right-ideal-McCoy rings are independent of each other by [11, Example 2.4].

(3) Armendariz rings are obviously both right left(right)-ideal-McCoy and left right(left)-ideal-McCoy.

(4) Right left-ideal-McCoy rings are clearly right McCoy, but not conversely by [11, Example 2.2(2)]. Right left-ideal-McCoy rings are also right linearly left-ideal-McCoy. The ring R in [22, Section 3] is a semicommutative ring but not right McCoy and hence it is not right left-ideal-McCoy. However, R is right linearly left-ideal-McCoy by [11, Proposition 2.3]. On the other hand, the ring in [11, Example 2.5(2)] (see also [8, Example 14]) is right left-ideal-McCoy but not semicommutative.

(5) $\mathrm{Mat}_n(A)$ and $U_n(A)$ over any ring A for $n \geq 2$ are not right McCoy by [9, Example 1.3 and Example 1.6] and so they are not right left-ideal-McCoy.

Theorem 2.2. (1) *A reversible ring is both left left-ideal-McCoy and right right-ideal-McCoy.*

(2) *If R is a reversible ring, then R is either left right-ideal McCoy or right left-ideal McCoy.*

Proof. (1) The case of right right-ideal-McCoy is proved by the proof of [7, Theorem 1.6(1)], and the case of left left-ideal-McCoy is proved by the symmetric version of the proof of [7, Theorem 1.6(1)].

(2) Let R be a reversible ring and suppose that $f(x)g(x) = 0$ for $0 \neq f(x) = \sum_{i=0}^{m} a_i x^m, g(x) = \sum_{j=0}^{n} b_j x^n \in R[x]$. By the proof of [7, Theorem 1.6(1)],

there exists $r = a_0^{l_0}a_1^{l_1}\cdots a_t^{l_t} \in R$ ($t \leq m$ and $l_k \geq 0$ for $k \in \{0, \ldots, t\}$) with $g(x)r \neq 0$ and $a_ib_jr = 0$ for all i and j. Since R is reversible, we also have

$$0 = (a_0^{l_0}a_1^{l_1}\cdots a_t^{l_t})a_ib_j \text{ for all } i \text{ and } j.$$

`Case 1.` If $(a_0^{l_0}a_1^{l_1}\cdots a_t^{l_t})f(x) = 0$, then $f(x)(a_0^{l_0}a_1^{l_1}\cdots a_t^{l_t}) = 0$ since R is reversible; hence we get

$$a_i(a_0^{l_0}a_1^{l_1}\cdots a_t^{l_t})b_j = 0 \text{ for all } i \text{ and } j.$$

But $(a_0^{l_0}a_1^{l_1}\cdots a_t^{l_t})g(x) \neq 0$, so $(a_0^{l_0}a_1^{l_1}\cdots a_t^{l_t})b_{j_0} \neq 0$ for some j_0. This yields

$$f(x)(a_0^{l_0}a_1^{l_1}\cdots a_t^{l_t})b_{j_0} = 0 \text{ with } (a_0^{l_0}a_1^{l_1}\cdots a_t^{l_t})b_{j_0} \in RC_{g(x)}.$$

`Case 2.` If $(a_0^{l_0}a_1^{l_1}\cdots a_t^{l_t})f(x) \neq 0$, then $(a_0^{l_0}a_1^{l_1}\cdots a_t^{l_t})a_{i_0} \neq 0$ for some i_0. This yields

$$(a_0^{l_0}a_1^{l_1}\cdots a_t^{l_t})a_{i_0}g(x) = 0 \text{ with } (a_0^{l_0}a_1^{l_1}\cdots a_t^{l_t})a_{i_0} \in C_{f(x)}R. \qquad \square$$

In fact, we do not know whether reversible rings are both left right-ideal-McCoy and right left-ideal-McCoy.

Question. Are reversible rings both left right-ideal-McCoy and right left-ideal-McCoy?

Observe that the right left-ideal-McCoy condition and semicommutativity of rings do not imply each other by Remark 2.1(4). However, we have the following.

Theorem 2.3. *If $R[x]$ is a semicommutative ring, then R satisfies the condition* (†) (*hence is right left-ideal-McCoy*).

Proof. Let $R[x]$ be a semicommutative ring. We apply the method in the proof of [7, Theorem 1.1]. Put $f(x)g(x) = 0$ with $0 \neq f(x) = \sum_{i=0}^{m} a_ix^i$ and $0 \neq g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x]$. We can assume that $a_m \neq 0$, $b_n \neq 0$. If $f(x)b_n = 0$, then we are done. Assume $f(x)b_n \neq 0$. Then $a_ig(x) \neq 0$ for some $i \in \{0, 1, \ldots, m\}$. Let k be the largest integer such that $a_kg(x) \neq 0$. Since $f(x)g(x) = 0$, we get $a_kb_n = 0$. Note that $a_kg(x)$ is a nonzero polynomial and its degree is less than n. Since $R[x]$ is semicommutative, we have $f(x)(a_kg(x)) = 0$.

If we replace $g(x)$ with $a_kg(x)$ in the argument above, then we can also find a_l or $0 \neq a_kb_h$ for some $0 \leq h \leq n-1$ such that $a_l(a_kg(x)) \neq 0$ and $f(x)a_kb_h = 0$. Note $f(x)(a_la_k)g(x) = 0$. Continuing this process, we finally obtain $a_{t_1}, \ldots, a_{t_h}$ and b_t with $t_1 = k$, $h \leq n$, $\{a_{t_1}, \ldots, a_{t_h}\} \subseteq \{a_0, \ldots, a_m\}$, and $t \in \{0, 1, \ldots, n\}$ such that $0 \neq (a_{t_h}\cdots a_{t_1})g(x) = (a_{t_h}\cdots a_{t_1})b_t \in R$ and

$$f(x)((a_{t_h}\cdots a_{t_1})b_t) = 0.$$

Thus R satisfies the condition (†), entailing that R is right left-ideal-McCoy. $\square$

This theorem provides the following interesting new result.

Corollary 2.4. *If $R[x]$ is a semicommutative ring, then R is right McCoy.*

The converse of Theorem 2.3 does not hold: Indeed, the ring R in [11, Example 2.5(2)] is right left-ideal-McCoy (and so right McCoy) but not semicommutative. Thus $R[x]$ is not semicommutative.

Example 2.5. (1) The class of right left-ideal-McCoy rings is not closed under homomorphic images. For the ring of quaternions R with integer coefficients is a domain, and so right left-ideal-McCoy. For any odd prime integer q, the ring R/qR is isomorphic to $\mathrm{Mat}_2(\mathbb{Z}_q)$ by [6, Exercise 2A], and thus R/qR is not right left-ideal-McCoy by Remark 2.1(4).

(2) The class of right left-ideal-McCoy rings is not closed under subrings. We refer to [11, Example 2.13(2)]. Let K be a field and $K\langle e, a, b, c\rangle$ be the free algebra with noncommuting indeterminates e, a, b, c over K. Set R be the factor ring of $K\langle e, a, b, c\rangle$ with the relations

$$e^2 = e,\ ae = a,\ ea = 0, eb = be = 0, ec = ce = c$$
$$a^2 = b^2 = c^2 = ab = ac = ba = bc = ca = cb = 0.$$

By the same argument as in [11, Example 2.13(2)], we can see that R is right left-ideal-McCoy. Consider the subring $S = \{\alpha + \beta e + \gamma a \mid \alpha, \beta, \gamma \in K\}$ of R. Let $f(x) = a + (e - 1)x, g(x) = a + ex$ in $S[x]$. Then $f(x)g(x) = 0$ and $SC_{g(x)} = \{\beta e + \gamma a \mid \beta, \gamma \in K\}$. If $f(x)d = 0$ for $0 \neq d = \beta e + \gamma a \in SC_{g(x)}$ then $\beta a = 0$ and $\beta e + \gamma a = 0$ and so $\beta = 0 = \gamma$, a contradiction. Thus S is not right left-ideal-McCoy.

Rings satisfying the condition (†) are clearly right left-ideal-McCoy but not conversely by Example 2.5(2). In fact, consider the right left-ideal-McCoy ring R in Example 2.5(2) and the polynomials $f(x) = a + (e - 1)x, g(x) = a + ex$ in $R[x]$ with $f(x)g(x) = 0$. Then $f(x)ae = -ax \neq 0$ and $f(x)(e - 1)a = ax \neq 0$. Thus R does not satisfy the condition (†).

Note that Abelian rings and right left-ideal-McCoy rings are independent each other by [11, Example 2.13(3)] and Example 2.5(2): Indeed, the right left-ideal-McCoy ring R in Example 2.5(2) is not Abelian since e is an idempotent but it does not commute with a.

A ring R is called (*von Neumann*) *regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$.

Proposition 2.6. *For a regular ring R, the following conditions are equivalent*:

(1) *R is reduced*;
(2) *R is semicommutative*;
(3) *R is Abelian*;
(4) *R is right left-ideal-McCoy*; *and*
(5) *R is right McCoy.*

Proof. (1)⇔(2)⇔(3) are well-known. (3)⇔(5) follows from [15, Theorem 20]. (3)⇒(4) Suppose that R is Abelian. Then R is reduced and so R is right left-ideal-McCoy. (4)⇒(5) is clear. □

A ring R is usually called *π-regular* if for each $a \in R$ there exist a positive integer n, depending on a, and $b \in R$ such that $a^n = a^n b a^n$. Regular rings are obviously π-regular. The condition "R is regular" in Proposition 2.6 cannot be weakened by the condition "R is π-regular". That is, there exists a π-regular and right left-ideal-McCoy ring R which is not reduced: For example,

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in A \right\}$$

for a division ring A is Armendariz by [12, Proposition 2] and hence right left-ideal-McCoy.

There exists a ring R which is not right left-ideal-McCoy such that R/I and I are right left-ideal-McCoy for any proper ideal I of R.

Example 2.7. For a division ring F, we consider a ring $R = U_2(F)$. Then R is not right left-ideal-McCoy by Remark 2.1(5). The only nonzero proper ideals of R are

$$I_1 = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, I_2 = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix} \text{ and } I_3 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}.$$

Then R/I_1 and R/I_2 are isomorphic to F and

$$R/I_3 = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + I_3 \mid a, c \in F \right\}$$

is a reduced ring, and hence each R/I_i (for $i = 1, 2, 3$) is right left-ideal-McCoy.

Note that I_3 is clearly right left-ideal-McCoy. By [16, Example 2.14], I_1 and I_2 are Armendariz, and hence they are right left-ideal-McCoy.

Recall that a ring R is *left* (resp., *right*) *weakly regular* [23] if $I^2 = I$ for every left (resp., right) ideal I of R.

By the similar argument to the proofs of [11, Proposition 2.10(2) and Proposition 2.12], we have the following proposition.

Proposition 2.8. (1) *Let R be a right left-ideal-McCoy ring and I a proper ideal of R. If R is a left weakly regular ring, then I is right left-ideal-McCoy as a ring.*

(2) *The class of right left-ideal-McCoy rings is closed under direct limits.*

(3) *Let Γ be a chain, R_γ ($\gamma \in \Gamma$) be rings with $R_\alpha \subseteq R_\beta$ for $\alpha < \beta$, and R_λ ($\lambda \in \Lambda \subseteq \Gamma$) be right left-ideal-McCoy rings. If Λ is dense in Γ, then $R = \cup_{\gamma \in \Gamma} R_\gamma$ is right left-ideal-McCoy.*

(4) *Let $R = \prod_{\lambda \in \Lambda} R_\lambda$ be the direct product of rings R_λ. Then R is right left-ideal-McCoy if and only if R_λ is for every $\lambda \in \Lambda$.*

(5) *Let $R = \sum_{\lambda \in \Lambda} R_\lambda$ be a direct sum of rings R_λ. Then R is right left-ideal-McCoy if and only if R_λ is for every $\lambda \in \Lambda$.*

(6) *For a central idempotent e of a ring R, R is right left-ideal-McCoy if and only if eR and $(1-e)R$ are.*

3. Extensions of right left-ideal-McCoy rings

Given a ring R and $n \geq 2$, consider the subrings

$$D_n(R) = \{(m_{ij}) \in U_n(R) \mid m_{11} = \cdots = m_{nn}\} \text{ and}$$

$$V_n(R) = \{m = (m_{ij}) \in D_n(R) \mid m_{st} = m_{(s+1)(t+1)} \text{ for } s = 1, \ldots, n-2 \text{ and } t = 2, \ldots, n-1\}$$

of $U_n(R)$.

Recall that for a reduced ring R and $3 \geq n$, $D_n(R)[x]$ is semicommutative by [16, Remark 2.2(3)] and hence $D_n(R)$ is right left-ideal-McCoy by Theorem 2.3.

Proposition 3.1. *For a ring R and $n \geq 2$, if $D_n(R)$ is right left-ideal-McCoy, then so is R.*

Proof. Suppose that $D_n(R)$ is right left-ideal-McCoy. Let $f(x)$ and $0 \neq g(x)$ be polynomials in $R[x]$ with $f(x)g(x) = 0$ where $f(x) = \sum_{i=0}^{m} a_i x^i, 0 \neq g(x) = \sum_{j=0}^{l} b_j x^j$. We take $F(x) = \sum_{i=0}^{m} A_i x^i$ and $G(x) = \sum_{j=0}^{l} B_j x^j$, where

$$A_i = \begin{pmatrix} a_i & 0 & \cdots & 0 \\ 0 & a_i & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_i \end{pmatrix}, \ B_j = \begin{pmatrix} 0 & 0 & \cdots & b_j \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

for each i and j. Then

$$F(x) = \begin{pmatrix} f(x) & 0 & \cdots & 0 \\ 0 & f(x) & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & f(x) \end{pmatrix}, \ G(x) = \begin{pmatrix} 0 & 0 & \cdots & g(x) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and $F(x)G(x) = 0$. Since $D_n(R)$ is right left-ideal-McCoy, there exists a nonzero $C = (c_{st}) \in D_n(R)C_{G(x)}$ such that $F(x)C = 0$. Say $C = \sum_{j=0}^{l} D_j B_j$ where $D_j \in D_n(R)$. For $j = 0, 1, \ldots, l$, let $D_j = (d_{st}^{(j)})$ with $d_{st}^{(j)} = d^{(j)}$ for $s = t$. Then $c_{st} = \sum_{j=0}^{l} d^{(j)} b_j \neq 0$ for $s = 1, t = n$ and otherwise 0. Thus $f(x)c_{1n} = 0$ and $0 \neq c_{1n} \in RC_{g(x)}$, showing that R is right left-ideal-McCoy. $\square$

Considering the structure of $D_n(R)$ related to ideals, one may think of the possibility of the existence of annihilators in the ideals generated by the coefficients of given zero-dividing polynomials in Proposition 3.1. We will study this in near future works.

Question. If R is a right left-ideal-McCoy ring, is $D_n(R)$ also right left-ideal-McCoy ($n \geq 3$)?

However, we have the following.

Theorem 3.2. *For a ring R and $n \geq 2$, the following conditions are equivalent*:
(1) *R is right left-ideal-McCoy*; *and*
(2) *$V_n(R)$ is right left-ideal-McCoy.*

Proof. We apply the proof of [11, Theorem 2.8]. (1)$\Rightarrow$(2) Note that $V_n(R)[x] \cong V_n(R[x])$ for any $n \geq 2$. Suppose that R is right left-ideal-McCoy. Let $f(x)$ and $0 \neq g(x)$ be polynomials in $V_n(R)[x]$ with $f(x)g(x) = 0$ where $f(x) = \sum_{i=0}^{m} A_i x^i, 0 \neq g(x) = \sum_{j=0}^{l} B_j x^j$ with $A_i = (a_{st}^{(i)})$ and $B_j = (b_{uv}^{(j)})$ for $s, t, u, v \in \{1, 2, \ldots, n\}$. We can write

$$f(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) & \cdots & f_{1n}(x) \\ 0 & f_{11}(x) & \cdots & f_{2n}(x) \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & f_{11}(x) \end{pmatrix}, \ g(x) = \begin{pmatrix} g_{11}(x) & g_{12}(x) & \cdots & g_{1n}(x) \\ 0 & g_{11}(x) & \cdots & g_{2n}(x) \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & g_{11}(x) \end{pmatrix}$$

with $f_{st}(x) = \sum_{i=0}^{m} a_{st}^{(i)} x^i, g_{uv}(x) = \sum_{j=0}^{l} b_{uv}^{(j)} x^j$. Note that

$f_{st}(x) = f_{(s+1)(t+1)}(x)$ and

$g_{st}(x) = g_{(s+1)(t+1)}(x)$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1$.

Since $f(x)g(x) = 0$, we have $f_{11}(x)g_{11}(x) = 0$.

`Case 1.` $g_{11}(x) \neq 0$.

Since R is right left-ideal-McCoy and $f_{11}(x)g_{11}(x) = 0$, there exists $0 \neq \alpha = \sum_{j=0}^{l} h_j b_{11}^{(j)} \in RC_{g_{11}(x)}$ such that $f_{11}(x)\alpha = 0$, where $h_j \in R$ for any j. If we let

$$d = \sum_{j=0}^{l} (E_{1n} h_j) B_j \in V_n(R) C_{g(x)}$$

then $d = E_{1n}\alpha \neq 0$ and $f(x)d = 0$.

`Case 2.` $g_{11}(x) = 0$.

In this case, we can find the largest k with respect to the property of $g_{kt}(x) \neq 0$ for some t. Then $k < t$ and $g_{kj}(x) = 0$ for all $j = 1, \ldots, t-1$. This yields $f_{11}(x)g_{kt}(x) = 0$. Since R is right left-ideal-McCoy, there exists $0 \neq \beta = \sum_{j=0}^{l} h_j b_{kt}^{(j)} \in RC_{g_{kt}(x)}$ such that $f_{11}(x)\beta = 0$, where $h_j \in R$ for any j. If we let

$$d' = \sum_{j=0}^{l} ((E_{1k} + E_{2(1+k)} + \cdots + E_{(t-k+1)t}) h_j) B_j \in V_n(R) C_{g(x)},$$

then $d' = E_{1n}\beta \neq 0$ and $f(x)d' = 0$.

By Cases 1 and 2, $V_n(R)$ is right left-ideal-McCoy.

(2)$\Rightarrow$(1) is the same as the proof of Proposition 3.1. $\square$

Recall that for a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$. This is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Corollary 3.3. *For a ring R, the following conditions are equivalent*:
(1) *R is right left-ideal-McCoy*;
(2) *The trivial extension $T(R, R)$ of R is right left-ideal-McCoy*; *and*
(3) *$R[x]/(x^n)$ is a right left-ideal-McCoy ring for any positive integer n, where (x^n) is an ideal of $R[x]$ generated by x^n.*

Proof. (1)$\Leftrightarrow$(2) follows directly from Theorem 3.2, since $T(R, R) \cong V_2$.
(1)$\Leftrightarrow$(3) $V_n(R) \cong R[x]/(x^n)$ by [19]. $\square$

Let R be an algebra over a commutative ring S. Recall that the *Dorroh extension* of R by S is the Abelian group $R \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in R$ and $s_i \in S$.

Recall that an element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, *left regular* elements can be defined. An element is *regular* if it is both left and right regular (and hence not a zero divisor).

Applying the proofs of [11, Theorem 3.1, Proposition 3.2 and Proposition 3.3], we have the following.

Proposition 3.4. (1) *Let R be an algebra over a commutative domain S, and D be the Dorroh extension of R by S. Then R is right left-ideal-McCoy if and only if D is.*
(2) *Let Δ be a multiplicatively closed subset of a ring R consisting of central regular elements. Then R is right left-ideal-McCoy if and only if $\Delta^{-1}R$ is.*
(3) *If $R[x]$ is a right left-ideal-McCoy ring, then so is R.*

We do not know whether the converse of Proposition 3.4(3) holds.

The ring of *Laurent polynomials* in x, coefficients in a ring R, consists of all formal sum $\sum_{i=k}^{n} r_i x^i$ with obvious addition and multiplication, where $r_i \in R$ and k, n are (possibly negative) integers. We denote this ring by $R[x; x^{-1}]$.

Corollary 3.5. *For a ring R, the following conditions are equivalent*:
(1) *$R[x]$ is right left-ideal-McCoy*; *and*
(2) *$R[x; x^{-1}]$ is right left-ideal-McCoy.*

Proof. (1)$\Leftrightarrow$(2) It directly follows from Proposition 3.4(2). For, let $\Delta = \{1, x, x^2, \ldots\}$, then Δ is clearly a multiplicatively closed subset of $R[x]$ and $R[x; x^{-1}] = \Delta^{-1}R[x]$. $\square$

A multiplicatively closed (m.c. for short) subset S of a ring R is said to satisfy the *right Ore condition* if for each $a \in R$ and $b \in S$, there exist $a_1 \in R$ and $b_1 \in S$ such that $ab_1 = ba_1$. It is well-known that S satisfies the right (resp., left) Ore condition and S consists of regular elements if and only if the right quotient ring RS^{-1} of R with respect to S exists.

Theorem 3.6. *Let S be an m.c. subset of a ring R, and suppose that S satisfies the Ore condition and S consists of regular elements. If R is right left-ideal-McCoy, then so is RS^{-1}.*

Proof. Let $F(x)G(x) = 0$ where $F(x) = a_0u^{-1} + a_1u^{-1}x + \cdots + a_mu^{-1}x^m$ and $0 \neq G(x) = b_0v^{-1} + b_1v^{-1}x + \cdots + b_nv^{-1}x^n \in RS^{-1}[x]$ for $a_i, b_j \in R$ with u, v regular. By hypothesis, there exists a regular u_1 for all j's such that $u^{-1}b_j = b'_ju_1^{-1}$ for some $b'_j \in R$. Now let $f(x) = \sum_{i=0}^m a_ix^i$, $g(x) = \sum_{j=0}^n b_jx^j$, $g_1(x) = \sum_{j=0}^n b'_jx^j$. Then

$$F(x)G(x) = f(x)u^{-1}g(x)v^{-1} = f(x)g_1(x)u_1^{-1}v^{-1}$$

and hence $f(x)g_1(x) = 0$, noting that $g(x) \neq 0$ and $g_1(x) \neq 0$. Since R is right left-ideal-McCoy, there exists $0 \neq r \in RC_{g_1(x)}$ such that $f(x)r = 0$. So we get $ur \neq 0$ and

$$0 = f(x)r = f(x)u^{-1}ur = F(x)ur.$$

Let $r = r_0b'_0 + r_1b'_1 + \cdots + r_nb'_n$ for some $r_0, \ldots, r_n \in R$. Since $b'_j = u^{-1}b_ju_1$ for all j, we have

$$\begin{aligned} 0 \neq uru_1^{-1} &= u(r_0b'_0 + r_1b'_1 + \cdots + r_nb'_n)u_1^{-1} \\ &= u(r_0u^{-1}b_0u_1 + r_1u^{-1}b_1u_1 + \cdots + r_nu^{-1}b_nu_1)u_1^{-1} \\ &= ur_0u^{-1}b_0 + ur_1u^{-1}b_1 + \cdots + ur_nu^{-1}b_n \in \sum_{j=0}^n RS^{-1}b_j. \end{aligned}$$

Thus $0 \neq uru_1^{-1}v^{-1} \in RS^{-1}C_{G(x)}$ and $F(x)uru_1^{-1}v^{-1} = 0$, showing that RS^{-1} is right ideal-McCoy. □

We do not know the answer to the following.

Question. Let S, R and RS^{-1} as before. If RS^{-1} is right left-ideal-McCoy then is R right left-ideal-McCoy?

Finally, we investigate minimal noncommutative right left-ideal-McCoy rings. The construction of the following rings is due to Xu and Xue [25, Example 7].

Let $A = \mathbb{Z}_2\langle x, y\rangle$, the free algebra with noncommuting indeterminates x, y over the field $\mathbb{Z}_2$. Let I be the ideal of A generated by

$$x^3, y^2, yx, x^2 - xy$$

and $R_1 = A/I$. Then R_1 has 16 elements. Since $R_1 \cong D_3(\mathbb{Z}_2)$, R_1 is Armendariz by [16, Remark 2.2(3)] and so right left-ideal-McCoy.

Let $B = \mathbb{Z}_4\langle x, y\rangle$ be the free algebra with noncommuting indeterminates x, y over the field $\mathbb{Z}_4$. Let I be the ideal of B generated by

$$x^3, y^2, yx, x^2 - xy, x^2 - \bar{2}, \bar{2}x, \bar{2}y$$

and $R_2 = B/I$. Then R_2 is Armendariz with 16 elements by [16, Example 2.10] and [25, Example 7]. Thus R_2 is obviously right left-ideal-McCoy.

Let $C = \mathbb{Z}_2\langle x, y\rangle$ be the free algebra with noncommuting indeterminates x, y over the field $\mathbb{Z}_2$. Let I be the ideal of C generated by

$$x^3, y^3, yx, x^2 - xy, y^2 - xy$$

and $R_3 = C/I$. Then R_3 is an Armendariz ring by [16, Example 2.10] with 16 elements, and so R_3 is right left-ideal-McCoy.

Let $D = \mathbb{Z}_4\langle x, y\rangle$ be the free algebra with noncommuting indeterminates x, y over $\mathbb{Z}_4$. Let $R_4 = D/I$ where I is the ideal of D generated by

$$x^3, y^3, yx, x^2 - xy, x^2 - \bar{2}, y^2 - \bar{2}, \bar{2}x, \bar{2}y.$$

Then R_4 is an Armendariz ring by [16, Example 2.10] with 16 elements, and so R_4 is right left-ideal-McCoy.

Let

$$R_5 = \left\{ \begin{pmatrix} a & b \\ 0 & a^2 \end{pmatrix} \mid a, b \in GF(2^2) \right\},$$

where $GF(2^2)$ is the Galois field of order 2^2. Then R_5 is an Armendariz ring by [16, Example 2.10] with 16 elements by [26, Example 2], and so R_5 is right left-ideal-McCoy.

Let R be a finite noncommutative ring and R_i for $i \in \{1, 2, 3, 4, 5\}$ be rings above. If R is a minimal Armendariz and semicommutative ring, then R is of order 16 and is isomorphic to R_i for some i by [16, Theorem 2.11], or if R is a minimal Abelian ring, then R is of order 16 and is isomorphic to R_i for some i by [14, Proposition 2.3]. Notice that every R_i is Abelian right left-ideal-McCoy. Therefore, if R is a minimal noncommutative Abelian right left-ideal-McCoy ring, then R is of order 16 and is isomorphic to R_i for some $i \in \{1, 2, 3, 4, 5\}$, where R_i's are the rings in the above.

As a corollary of this fact and [16, Corollary 2.12], R is a minimal noncommutative Abelian right left-ideal-McCoy ring ring if and only if R is a minimal noncommutative semicommutative ring if and only if R is a minimal noncommutative Armendariz ring.

Now, we consider the structure of minimal right left-ideal-McCoy rings without identity. Recall that a semicommutative ring with identity is Abelian, but this is no longer valid for the case of rings without identity as follows.

Let D be a domain and

$$R_6 = \begin{pmatrix} D & D \\ 0 & 0 \end{pmatrix}, R_7 = \begin{pmatrix} 0 & D \\ 0 & D \end{pmatrix}$$

be subrings of $U_2(D)$. Then both R_6 and R_7 are non-Abelian, semicommutative and Armendariz by [16, Example 2.13 and Example 2.14]. Thus both R_6 and R_7 are right left-ideal-McCoy.

Theorem 3.7. *Let R be a ring without identity. If R is a minimal non-Abelian right left-ideal-McCoy ring, then R is isomorphic to*

$$\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix} \quad or \quad \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}.$$

Proof. Let R be a minimal right left-ideal-McCoy ring. Then $|R| = 4$ by the existence of the right left-ideal-McCoy ring as above, R_6 or R_7. If R is nilpotent then R is commutative by [16, Lemma 2.7], a contradiction. If $|J(R)| = 0$ where $J(R)$ is the Jacobson radical of R, then R is also commutative by the proof of [14, Theorem 1.15], a contradiction. Thus we have the result of $|J(R)| = 2$, whence we also follow the proof [14, Theorem 1.15] to conclude that R is isomorphic to

$$\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}.$$

□

Hence, by Theorem 3.7 and [16, Corollary 2.16], if R is a ring without identity, then R is a minimal non-Abelian right left-ideal-McCoy ring if and only if R is a minimal non-Abelian semicommutative ring if and only if R is a minimal non-Abelian Armendariz ring.

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References

[1] D. D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra **26** (1998), no. 7, 2265–2272.

[2] ———, *Semigroups and rings whose zero products commute*, Comm. Algebra **27** (1999), no. 6, 2847–2852.

[3] E. P. Armendariz, *A note on extensions of Baer and P.P.-rings*, J. Aust. Math. Soc. **18** (1974), 470–473.

[4] V. Camillo and P. P. Nielsen, *McCoy rings and zero-divisors*, J. Pure Appl. Algebra **212** (2008), no. 3, 599–615.

[5] P. M. Cohn, *Reversible rings*, Bull. London Math. Soc. **31** (1999), no. 6, 641–648.

[6] K. R. Goodearl and R. B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings*, Cambridge University Press, 1989.

[7] C. Y. Hong, Y. C. Jeon, N. K. Kim, and Y. Lee, *The McCoy condition on noncommutative rings*, Comm. Algebra **39** (2011), no. 5, 1809–1825.

[8] C. Huh, Y. Lee, and A. Smoktunowicz, *Armendariz rings and semicommutative rings*, Comm. Algebra **30** (2002), no. 2, 751–761.

[9] Y. C. Jeon, H. K. Kim, N. K. Kim, T. K. Kwak, Y. Lee, and D. E. Yeo, *On a generalization of the McCoy condition*, J. Korean Math. Soc. **47** (2010), no. 6, 1269–1282.

[10] L. G. Jones and L. Weiner, *Advanced Problems and Solutions, Solutions 4419*, Amer. Math. Monthly **59** (1952), no. 5, 336–337.

[11] B. O. Kim, T. K. Kwak, and Y. Lee, *On constant zero-divisors of linear polynomials*, Comm. Algebra (to appear).

[12] N. K. Kim and Y. Lee, *Armendariz rings and reduced rings*, J. Algebra **223** (2000), no. 2, 477–488.

[13] ______, *Extensions of reversible rings*, J. Pure Appl. Algebra **185** (2003), no. 1-3, 207–223.

[14] N. K. Kim, Y. Lee, and Y. Seo, *Structure of idempotents in rings*, (submitted).

[15] T. K. Kwak and Y. Lee, *Rings over which coefficients of nilpotent polynomials are nilpotent*, Internat. J. Algebra Comput. **21** (2011), no. 5, 745–762.

[16] T. K. Kwak, Y. Lee, and S. J. Yun, *The Armendariz property on ideals*, J. Algebra **354** (2012), 121–135.

[17] J. Lambek, *Lectures on Rings and Modules*, Blaisdell Publishing Company, Waltham, 1966.

[18] ______, *On the representation of modules by sheaves of factor modules*, Canad. Math. Bull. **14** (1971), 359–368.

[19] T. K. Lee and Y. Q. Zhou, *Armendariz and reduced rings*, Comm. Algebra **32** (2004), no. 6, 2287–2299.

[20] N. H. McCoy, *Remarks on divisors of zero*, Amer. Math. Monthly **49** (1942), 286–295.

[21] L. Motais de Narbonne, *Anneaux semi-commutatifs et unis riels anneaux dont les id aux principaux sont idempotents*, In: Proceedings of the 106th National Congress of Learned Societies (Perpignan, 1981), 71–73, Bib. Nat., Paris, 1982.

[22] P. P. Nielsen, *Semi-commutativity and the McCoy condition*, J. Algebra **298** (2006), no. 1, 134–141.

[23] V. S. Ramamurthi, *Weakly regular rings*, Canad. Math. Bull. **16** (1973), 317–321.

[24] M. B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), no. 1, 14–17.

[25] L. Xu and W. Xue, *Structure of minimal non-commutative zero-insertive rings*, Math. J. Okayama Univ. **40** (1998), 69–76.

[26] W. Xue, *Structure of minimal noncommutative duo rings and minimal strongly bounded non-duo rings*, Comm. Algebra **20** (1992), no. 9, 2777–2788.

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