**Original Article**
International Journal of Fuzzy Logic and Intelligent Systems
Vol. 14, No. 1, March 2014, pp. 57-65
http://dx.doi.org/10.5391/IJFIS.2014.14.1.57

ISSN(Print) 1598-2645
ISSN(Online) 2093-744X

# The Properties of $L$-lower Approximation Operators

**Yong Chan Kim**
Department of Mathematics, Gangneung-Wonju National University, Gangneung, Korea

Ijfis

**Abstract**

In this paper, we investigate the properties of $L$-lower approximation operators as a generalization of fuzzy rough set in complete residuated lattices. We study relations lower (upper, join meet, meet join) approximation operators and Alexandrov $L$-topologies. Moreover, we give their examples as approximation operators induced by various $L$-fuzzy relations.

**Keywords:** Complete residuated lattices, $L$-upper approximation operators, Alexandrov $L$-topologies

## 1. Introduction

Pawlak [1, 2] introduced rough set theory as a formal tool to deal with imprecision and uncertainty in data analysis. Hájek [3] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Radzikowska and Kerre [4] developed fuzzy rough sets in complete residuated lattice. Bělohlávek [5] investigated information systems and decision rules in complete residuated lattices. Lai and Zhang [6, 7] introduced Alexandrov $L$-topologies induced by fuzzy rough sets. Kim [8, 9] investigate relations between lower approximation operators as a generalization of fuzzy rough set and Alexandrov $L$-topologies. Algebraic structures of fuzzy rough sets are developed in many directions [4, 8, 10]

In this paper, we investigate the properties of $L$-lower approximation operators as a generalization of fuzzy rough set in complete residuated lattices. We study relations lower (upper, join meet, meet join) approximation operators and Alexandrov $L$-topologies. Moreover, we give their examples as approximation operators induced by various $L$-fuzzy relations.

**Definition 1.1.** [3, 5] An algebra $(L, \wedge, \vee, \odot, \to, \perp, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$;
(C2) $(L, \odot, \top)$ is a commutative monoid;
(C3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$

**Remark 1.2.** [3, 5] (1) A completely distributive lattice $L = (L, \leq, \vee, \wedge = \odot, \to, 1, 0)$ is a complete residuated lattice defined by

$$x \to y = \bigvee \{z \mid x \wedge z \leq y\}.$$

Received: Dec. 10, 2013
Revised : Mar. 18, 2014
Accepted: Mar. 19, 2014

Correspondence to: Yong Chan Kim
(yck@gwnu.ac.kr)

(2) The unit interval with a left-continuous t-norm $\odot$,

$$([0,1], \vee, \wedge, \odot, \rightarrow, 0, 1),$$

is a complete residuated lattice defined by

$$x \rightarrow y = \bigvee\{z \mid x \odot z \leq y\}.$$

In this paper, we assume $(L, \wedge, \vee, \odot, \rightarrow, ^* \perp, \top)$ is a complete residuated lattice with the law of double negation;i.e. $x^{**} = x$. For $\alpha \in L, A, \top_x \in L^X$,

$$(\alpha \rightarrow A)(x) = \alpha \rightarrow A(x), \quad (\alpha \odot A)(x) = \alpha \odot A(x)$$

and

$$\top_x(x) = \top, \top_x(y) = \perp, \text{ otherwise.}$$

**Lemma 1.3.** [3, 5] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.
(2) $x \odot y \leq x \wedge y \leq x \vee y$.
(3) $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$.
(4) $x \rightarrow (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \rightarrow y_i)$
(5) $(\bigwedge_{i\in\Gamma} x_i) \rightarrow y \geq \bigvee_{i\in\Gamma}(x_i \rightarrow y)$.
(6) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.
(7) $x \odot (x \rightarrow y) \leq y$, $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$ and $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$.
(8) $y \leq x \rightarrow (x \odot y)$ and $x \leq (x \rightarrow y) \rightarrow y$.
(9) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$.
(10) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.
(11) $x \rightarrow y = \top$ iff $x \leq y$.
(12) $x \rightarrow y = y^* \rightarrow x^*$.
(13) $(x \odot y)^* = x \rightarrow y^* = y \rightarrow x^*$ and $x \rightarrow y = (x \odot y^*)^*$.
(14) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

**Definition 1.4.** [8, 9]

(1) A map $\mathcal{H} : L^X \rightarrow L^X$ is called an *L-upper approximation operator* iff it satisfies the following conditions

(H1) $A \leq \mathcal{H}(A)$,
(H2) $\mathcal{H}(\alpha \odot A) = \alpha \odot \mathcal{H}(A)$ where $\alpha(x) = \alpha$ for all $x \in X$,
(H3) $\mathcal{H}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{H}(A_i)$.

(2) A map $\mathcal{J} : L^X \rightarrow L^X$ is called an *L-lower approximation operator* iff it satisfies the following conditions

(J1) $\mathcal{J}(A) \leq A$,
(J2) $\mathcal{J}(\alpha \rightarrow A) = \alpha \rightarrow \mathcal{J}(A)$,
(J3) $\mathcal{J}(\bigwedge_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{J}(A_i)$.

(3) A map $\mathcal{K} : L^X \rightarrow L^X$ is called an *L-join meet approximation operator* iff it satisfies the following conditions

(K1) $\mathcal{K}(A) \leq A^*$,
(K2) $\mathcal{K}(\alpha \odot A) = \alpha \rightarrow \mathcal{K}(A)$,
(K3) $\mathcal{K}(\bigvee_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{K}(A_i)$.

(4) A map $\mathcal{M} : L^X \rightarrow L^X$ is called an *L-meet join approximation operator* iff it satisfies the following conditions

(M1) $A^* \leq \mathcal{M}(A)$,
(M2) $\mathcal{M}(\alpha \rightarrow A) = \alpha \odot \mathcal{M}(A)$,
(M3) $\mathcal{M}(\bigwedge_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{M}(A_i)$.

**Definition 1.5.** [6, 9] A subset $\tau \subset L^X$ is called an *Alexandrov L-topology* if it satisfies:

(T1) $\perp_X, \top_X \in \tau$ where $\top_X(x) = \top$ and $\perp_X(x) = \perp$ for $x \in X$.
(T2) If $A_i \in \tau$ for $i \in \Gamma$, $\bigvee_{i\in\Gamma} A_i, \bigwedge_{i\in\Gamma} A_i \in \tau$ .
(T3) $\alpha \odot A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.
(T4) $\alpha \rightarrow A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

**Theorem 1.6.** [8, 9]

(1) $\tau$ is an Alexandrov topology on $X$ iff $\tau_* = \{A^* \in L^X \mid A \in \tau\}$ is an Alexandrov topology on $X$.
(2) If $\mathcal{H}$ is an $L$-upper approximation operator, then $\tau_{\mathcal{H}} = \{A \in L^X \mid \mathcal{H}(A) = A\}$ is an Alexandrov topology on $X$.
(3) If $\mathcal{J}$ is an $L$-lower approximation operator, then $\tau_{\mathcal{J}} = \{A \in L^X \mid \mathcal{J}(A) = A\}$ is an Alexandrov topology on $X$.
(4) If $\mathcal{K}$ is an $L$-join meet approximation operator, then $\tau_{\mathcal{K}} = \{A \in L^X \mid \mathcal{K}(A) = A^*\}$ is an Alexandrov topology on $X$.
(5) If $\mathcal{M}$ is an $L$-meet join operator, then $\tau_{\mathcal{M}} = \{A \in L^X \mid \mathcal{M}(A) = A^*\}$ is an Alexandrov topology on $X$.

**Definition 1.7.** [8, 9] Let $X$ be a set. A function $R : X \times X \rightarrow L$ is called:

(R1) *reflexive* if $R(x, x) = \top$ for all $x \in X$,
(R2) *symmetric* if $R(x, x) = \top$ for all $x \in X$,
(R3) *transitive* if $R(x, y) \odot R(y, z) \leq R(x, z)$, for all $x, y, z \in X$.

(R4) *Euclidean* if $R(x,z) \odot R(y,z) \le R(x,y)$, for all $x,y,z \in X$.

If $R$ satisfies (R1) and (R3), $R$ is called a *$L$-fuzzy preorder*.

If $R$ satisfies (R1), (R2) and (R3), $R$ is called a *$L$-fuzzy equivalence relation*

## 2. The Properties of $L$-lower Approximation Operators

**Theorem 2.1.** Let $\mathcal{J} : L^X \to L^X$ be an $L$-lower approximation operator. Then the following properties hold.

(1) For $A \in L^X$, $\mathcal{J}(A)(y) = \bigwedge_{x\in X}(\mathcal{J}^*(\top_x^*)(y) \to A(x))$.

(2) Define $\mathcal{H}_J(B) = \bigwedge\{A \mid B \le \mathcal{J}(A)\}$. Then $\mathcal{H}_J : L^X \to L^X$ with

$$\mathcal{H}_J(B)(x) = \bigvee_{y\in X}(\mathcal{J}^*(\top_x^*)(y) \odot B(y))$$

is an $L$-upper approximation operator such that $(\mathcal{H}_J, \mathcal{J})$ is a residuated connection;i.e.,

$$\mathcal{H}_J(B) \le A \ \text{ iff } \ B \le \mathcal{J}(A).$$

Moreover, $\tau_{\mathcal{J}} = \tau_{\mathcal{H}_J}$.

(3) If $\mathcal{J}(\mathcal{J}(A)) = \mathcal{J}(A)$ for $A \in L^X$, then $\mathcal{H}_J(\mathcal{H}_J(A)) = \mathcal{H}_J(A)$ for $A \in L^X$ such that $\tau_{\mathcal{J}} = \tau_{\mathcal{H}_J}$ with

$$\tau_{\mathcal{J}} = \{\mathcal{J}(A) = \bigwedge_{x\in X}(\mathcal{J}^*(\top_x^*)(y) \to A(x)) \mid A \in L^X\},$$

$$\tau_{\mathcal{H}_J} = \{\mathcal{H}_J(A)(x) = \bigvee_{y\in X}(\mathcal{J}^*(\top_x^*)(y) \odot A(y)) \mid A \in L^X\}.$$

(4) If $\mathcal{J}(\mathcal{J}^*(A)) = \mathcal{J}^*(A)$ for $A \in L^X$, then $\mathcal{J}(\mathcal{J}(A)) = \mathcal{J}(A)$ such that

$$\{\mathcal{J}^*(A) = \bigvee_{x\in X}(A^*(x) \odot \mathcal{J}^*(\top_x^*)) \mid A \in L^X\} = \tau_{\mathcal{J}} = (\tau_{\mathcal{J}})_*.$$

(5) Define $\mathcal{H}_s(A) = \mathcal{J}(A^*)^*$. Then $\mathcal{H}_s : L^X \to L^X$ with

$$\mathcal{H}_s(B)(x) = \bigvee_{y\in X}(\mathcal{J}^*(\top_y^*)(x) \odot B(y))$$

is an $L$-upper approximation operator. Moreover, $\tau_{\mathcal{H}_s} = (\tau_{\mathcal{J}})_* = (\tau_{\mathcal{H}_J})_*$.

(6) If $\mathcal{J}(\mathcal{J}(A)) = \mathcal{J}(A)$ for $A \in L^X$, then

$$\mathcal{H}_s(\mathcal{H}_s(A)) = \mathcal{H}_s(A)$$

for $A \in L^X$ such that $\tau_{\mathcal{H}_s} = (\tau_{\mathcal{J}})_* = (\tau_{\mathcal{H}_J})_*$. with

$$\tau_{\mathcal{H}_s} = \{\mathcal{H}_s(A) = \bigvee_{y\in X}(\mathcal{J}^*(\top_y^*) \odot A(y)) \mid A \in L^X\}.$$

(7) If $\mathcal{J}(\mathcal{J}^*(A)) = \mathcal{J}^*(A)$ for $A \in L^X$, then

$$\mathcal{H}_s(\mathcal{H}_s^*(A)) = \mathcal{H}_s^*(A)$$

such that

$$\{\mathcal{H}_s^*(A) = \bigwedge_{y\in X}(A(y) \to \mathcal{J}(\top_y^*)) \mid A \in L^X\} = \tau_{\mathcal{H}_s} = (\tau_{\mathcal{H}_s})_*.$$

(8) Define $\mathcal{K}_J(A) = \mathcal{J}(A^*)$. Then $\mathcal{K}_J : L^X \to L^X$ with

$$\mathcal{K}_J(A) = \bigwedge_{y\in X}(A(y) \to \mathcal{J}(\top_y^*))$$

is an $L$-join meet approximation operator.

(9) If $\mathcal{J}(\mathcal{J}(A)) = \mathcal{J}(A)$ for $A \in L^X$, then

$$\mathcal{K}_J(\mathcal{K}_J^*(A)) = \mathcal{K}_J^*(A)$$

for $A \in L^X$ such that $\tau_{\mathcal{K}_J} = (\tau_{\mathcal{J}})_*$ with

$$\tau_{\mathcal{K}_J} = \{\mathcal{K}_J^*(A) = \bigvee_{y\in X}(\mathcal{J}^*(\top_y^*) \odot A(y)) \mid A \in L^X\}.$$

(10) If $\mathcal{J}(\mathcal{J}^*(A)) = \mathcal{J}^*(A)$ for $A \in L^X$, then

$$\mathcal{K}_J(\mathcal{K}_J(A)) = \mathcal{K}_J^*(A)$$

such that

$$\{\mathcal{K}_J(A) = \bigwedge_{y\in X}(A(y) \to \mathcal{J}(\top_y^*)) \mid A \in L^X\} = \tau_{\mathcal{K}_J} = (\tau_{\mathcal{K}_J})_*.$$

(11) Define $\mathcal{M}_J(A) = (\mathcal{J}(A))^*$. Then $\mathcal{M}_J : L^X \to L^X$ with

$$\mathcal{M}_J(A)(y) = \bigvee_{x\in X}(A^*(x) \odot \mathcal{J}^*(\top_x^*)(y))$$

is an $L$-meet join approximation operator. Moreover, $\tau_{\mathcal{M}_J} = \tau_{\mathcal{J}}$.

(12) If $\mathcal{J}(\mathcal{J}(A)) = \mathcal{J}(A)$ for $A \in L^X$, then $\mathcal{M}_J(\mathcal{M}_J^*(A)) =$

$\mathcal{M}_J(A)$ for $A \in L^X$ such that $\tau_{\mathcal{M}_J} = (\tau_{\mathcal{J}})_*$ with

$$\{\mathcal{M}^*_J(A)(y) = \bigwedge_{x\in X} (\mathcal{J}^*(\top^*_x)(y) \to A(x)) \mid A \in L^X\}$$
$$= \tau_{\mathcal{M}_J} = (\tau_{\mathcal{J}})_*.$$

(13) If $\mathcal{J}(\mathcal{J}^*(A)) = \mathcal{J}^*(A)$ for $A \in L^X$, then

$$\mathcal{M}_J(\mathcal{M}_J(A)) = \mathcal{M}^*_J(A)$$

such that

$$\tau_{\mathcal{M}_J} = (\tau_{\mathcal{M}_J})_* = \left\{\mathcal{M}_J(A)(y) = \bigvee_{x\in X} (A^*(x) \odot \mathcal{J}^*(\top^*_x)(y)) \mid A \in L^X\right\}.$$

(14) Define $\mathcal{K}_{H_J}(A) = (\mathcal{H}_J(A))^*$. Then $\mathcal{K}_{H_J} : L^X \to L^X$ with

$$\mathcal{K}_{H_J}(A)(y) = \bigwedge_{x\in X} (A(x) \to \mathcal{J}(\top^*_y)(x))$$

is an $L$-meet join approximation operator. Moreover, $\tau_{\mathcal{K}_{H_J}} = \tau_{\mathcal{J}}$.

(15) If $\mathcal{J}(\mathcal{J}(A)) = \mathcal{J}(A)$ for $A \in L^X$, then

$$\mathcal{K}_{H_J}(\mathcal{K}^*_{H_J}(A)) = \mathcal{K}_{H_J}$$

for $A \in L^X$ such that $\tau_{\mathcal{K}_{H_J}} = (\tau_{\mathcal{J}})_*$ with

$$\tau_{\mathcal{K}_{H_J}} = \{\mathcal{K}^*_{H_J}(y) = \bigvee_{x\in X} (\mathcal{J}^*(\top^*_y)(x) \odot A^*(x)) \mid A \in L^X\}.$$

(16) If $\mathcal{H}_J(\mathcal{H}^*_J(A)) = \mathcal{H}^*_J(A)$ for $A \in L^X$, then

$$\mathcal{K}_{H_J}(\mathcal{K}_{H_J}) = \mathcal{K}^*_{H_J}(A)$$

such that

$$\tau_{\mathcal{K}_{H_J}} = (\tau_{\mathcal{K}_{H_J}})_* = \{\mathcal{K}_{H_J}(A)(y) = \bigwedge_{x\in X} (A(x) \to \mathcal{J}(\top^*_y)(x)) \mid A \in L^X\}.$$

(17) Define $\mathcal{M}_{H_J}(A) = \mathcal{H}_J(A^*)$. Then $\mathcal{M}_{H_J} : L^X \to L^X$ with

$$\mathcal{M}_{H_J}(A)(y) = \bigvee_{x\in X} (A^*(x) \odot \mathcal{J}^*(\top^*_y)(x))$$

is an $L$-join meet approximation operator. Moreover, $\tau_{\mathcal{M}_{H_J}} = (\tau_{\mathcal{J}})_*$.

(18) If $\mathcal{J}(\mathcal{J}(A)) = \mathcal{J}(A)$ for $A \in L^X$, then

$$\mathcal{M}_{H_J}(\mathcal{M}^*_{H_J}(A)) = \mathcal{M}_{H_J}(A)$$

for $A \in L^X$ such that $\tau_{\mathcal{M}_{H_J}} = (\tau_{\mathcal{J}})_*$ with

$$\tau_{\mathcal{M}_{H_J}} = \{\mathcal{M}^*_{H_J}(A)(y) = \bigwedge_{x\in X} (\mathcal{J}^*(\top^*_y)(x) \to A(x)) \mid A \in L^X\}.$$

(19) If $\mathcal{H}_J(\mathcal{H}^*_J(A)) = \mathcal{H}^*_J(A)$ for $A \in L^X$, then

$$\mathcal{M}_{H_J}(\mathcal{M}_{H_J}(A)) = \mathcal{M}^*_{H_J}(A)$$

such that

$$\tau_{\mathcal{M}_{H_J}} = (\tau_{\mathcal{M}_{H_J}})_* == \bigvee_{x\in X} (A^*(x) \odot \mathcal{J}^*(\top^*_y)(x)) \mid A \in L^X\}.$$

(20) $(\mathcal{K}_{H_J}, \mathcal{K}_J)$ is a Galois connection;i.e,

$$A \le \mathcal{K}_{H_J}(B) \text{ iff } B \le \mathcal{K}_J(A).$$

Moreover, $\tau_{\mathcal{K}_J} = (\tau_{\mathcal{K}_{H_J}})_*$.

(21) $(\mathcal{M}_J, \mathcal{M}_{H_J})$ is a dual Galois connection;i.e,

$$\mathcal{M}_{H_J}(A) \le B \text{ iff } \mathcal{M}_J(B) \le A.$$

Moreover, $\tau_{\mathcal{M}_J} = (\tau_{\mathcal{M}_{H_J}})_*$.

**Proof.**

(1) Since $A = \bigwedge_{x\in X}(A^*(x) \to \top^*_x)$, by (J2) and (J3),

$$\mathcal{J}(A)(y) = \bigwedge_{x\in X} (A^*(x) \to \mathcal{J}(\top^*_x)(y)) = \bigwedge_{x\in X} (\mathcal{J}^*(\top^*_x)(y) \to A(x)).$$

(2) Since $B(y) \le \mathcal{J}(A)(y) = \bigwedge_{x\in X}(\mathcal{J}^*(\top^*_x)(y) \to A(x))$ iff $\bigvee_{y\in X}(\mathcal{J}^*(\top^*_x)(y) \odot B(y)) \le A(x)$, we have

$$\mathcal{H}_J(B)(x) = \bigvee_{y\in Y} (\mathcal{J}^*(\top^*_x)(y) \odot B(y)).$$

(H1) Since $\mathcal{H}_J(A) \leq \mathcal{H}_J(A)$ iff $A \leq \mathcal{J}(\mathcal{H}_J(A))$, we have $A \leq \mathcal{J}(\mathcal{H}_J(A)) \leq \mathcal{H}_J(A)$.

(H2)
$$\begin{aligned} & a \odot A \leq \mathcal{J}(\mathcal{H}_J(a \odot A)) \\ \text{iff} \;\; & A \leq a \to \mathcal{J}(\mathcal{H}_J(a \odot A)) \\ & \quad = \mathcal{J}(a \to \mathcal{J}(\mathcal{H}_J(a \odot A))) \\ \text{iff} \;\; & \mathcal{H}_J(A) \leq a \to \mathcal{H}_J(a \odot A) \\ \text{iff} \;\; & a \odot \mathcal{H}_J(A) \leq \mathcal{H}_J(a \odot A). \end{aligned}$$

$$\begin{aligned} & A \leq \mathcal{J}(\mathcal{H}_J(A)) \\ & \leq \mathcal{J}(a \to a \odot \mathcal{H}_J(A)) = a \to \mathcal{J}(a \odot \mathcal{H}_J(A)) \\ & \text{iff} \;\; a \odot A \leq \mathcal{J}(a \odot \mathcal{H}_J(A)) \\ & \text{iff} \;\; \mathcal{H}_J(a \odot A) \leq a \odot \mathcal{H}_J(A). \end{aligned}$$

(H3) By the definition of $\mathcal{H}_J$, since $\mathcal{H}_J(A) \leq \mathcal{H}_J(B)$ for $B \leq A$, we have

$$\bigvee_{i\in\Gamma} \mathcal{H}_J(A_i) \leq \mathcal{H}_J(\bigvee_{i\in\Gamma} A_i).$$

Since $\mathcal{J}(\bigvee_{i\in\Gamma} \mathcal{H}_J(A_i)) \geq \mathcal{J}(\mathcal{H}_J(A_i)) \geq A_i$, then $\mathcal{J}(\bigvee_{i\in\Gamma} \mathcal{H}_J(A_i)) \geq \bigvee_{i\in\Gamma} A_i$. Thus

$$\bigvee_{i\in\Gamma} \mathcal{H}_J(A_i) \geq \mathcal{H}_J(\bigvee_{i\in\Gamma} A_i).$$

Thus $\mathcal{H}_J : L^X \to L^X$ is an $L$-upper approximation operator. By the definition of $\mathcal{H}_J$, we have

$$\mathcal{H}_J(B) \leq A \;\text{ iff }\; B \leq \mathcal{J}(A).$$

Since $A \leq \mathcal{J}(A)$ iff $\mathcal{H}_J(A) \leq A$, we have $\tau_{\mathcal{H}_J} = \tau_{\mathcal{J}}$.

(3) Let $\mathcal{J}(\mathcal{J}(A)) = \mathcal{J}(A)$ for $A \in L^X$. Since $\mathcal{J}(B) \geq \mathcal{H}_J(A)$ iff $\mathcal{J}(\mathcal{J}(B)) = \mathcal{J}(B) \geq A$ from the definition of $\mathcal{H}_J$, we have

$$\begin{aligned} \mathcal{H}_J(\mathcal{H}_J(A)) & = \bigwedge\{B \mid \mathcal{J}(B) \geq \mathcal{H}_J(A)\} \\ & = \bigwedge\{B \mid \mathcal{J}(\mathcal{J}(B)) = \mathcal{J}(B) \geq A\} \\ & = \mathcal{H}_J(A). \end{aligned}$$

(4) Let $\mathcal{J}^*(A) \in \tau_{\mathcal{J}}$. Since $\mathcal{J}(\mathcal{J}^*(A)) = \mathcal{J}^*(A)$,

$$\mathcal{J}(\mathcal{J}(A)) = \mathcal{J}(\mathcal{J}^*(\mathcal{J}^*(A))) = (\mathcal{J}(\mathcal{J}^*(A)))^* = \mathcal{J}(A).$$

Hence $\mathcal{J}(A) \in \tau_{\mathcal{J}}$; i.e. $\mathcal{J}^*(A) \in (\tau_{\mathcal{J}})_*$. Thus, $\tau_{\mathcal{J}} \subset (\tau_{\mathcal{J}})_*$.

Let $A \in (\tau_{\mathcal{J}})_*$. Then $A^* = \mathcal{J}(A^*)$. Since $\mathcal{J}(A) = \mathcal{J}(\mathcal{J}^*(A^*)) = \mathcal{J}^*(A^*) = A$, then $A \in \tau_{\mathcal{J}}$. Thus, $(\tau_{\mathcal{J}})_* \subset \tau_{\mathcal{J}}$.

(5) (H1) Since $\mathcal{J}(A^*) \leq A^*$, $\mathcal{H}_s(A) = \mathcal{J}(A^*)^* \geq A$.

(H2)
$$\begin{aligned} \mathcal{H}_s(\alpha \odot A) & = (\mathcal{J}((\alpha \odot A)^*)^* \\ & = (\mathcal{J}(\alpha \to A^*))^* \\ & = (\alpha \to \mathcal{J}(A^*))^* \\ & = \alpha \odot \mathcal{J}(A^*)^* \\ & = \alpha \odot \mathcal{H}_s(A). \end{aligned}$$

(H3)
$$\begin{aligned} \mathcal{H}_s(\bigvee_{i\in\Gamma} A_i) & = (\mathcal{J}(\bigvee_{i\in\Gamma} A_i)^*)^* \\ & = (\mathcal{J}(\bigwedge_{i\in\Gamma} A_i^*))^* \\ & = (\bigwedge_{i\in\Gamma} \mathcal{J}(A_i^*))^* \\ & = \bigvee_{i\in\Gamma} (\mathcal{J}(A_i^*))^* \\ & = \bigvee_{i\in\Gamma} \mathcal{H}_s(A_i). \end{aligned}$$

Hence $\mathcal{H}_s$ is an $L$-upper approximation operator such that

$$\mathcal{H}_s(B)(x) = (\mathcal{J}(B^*)(x))^* = \bigvee_{y\in X} (\mathcal{J}^*(\top_y^*)(x) \odot B(y)).$$

Moreover, $\tau_{\mathcal{H}_s} = (\tau_{\mathcal{J}})_*$ from:

$$A = \mathcal{H}_s(A) \;\text{ iff }\; A = \mathcal{J}(A^*)^* \;\text{ iff }\; A^* = \mathcal{J}(A^*).$$

(6) Let $\mathcal{J}(\mathcal{J}(A)) = \mathcal{J}(A)$ for $A \in L^X$. Then

$$\begin{aligned} \mathcal{H}_s(\mathcal{H}_s(A)) & = \mathcal{J}^*(\mathcal{H}_s^*(A)) = (\mathcal{J}(\mathcal{J}(A^*)))^* \\ & = \mathcal{J}^*(A^*) = \mathcal{H}_s(A). \end{aligned}$$

Hence $\tau_{\mathcal{H}_s} = \{\mathcal{H}_s(A) = \bigvee_{y\in X}(\mathcal{J}^*(\top_y^*) \odot A(y)) \mid A \in L^X\}$.

(7) Let $\mathcal{J}(\mathcal{J}^*(A)) = \mathcal{J}^*(A)$ for $A \in L^X$. Then

$$\begin{aligned} \mathcal{H}_s(\mathcal{H}_s^*(A)) & = \mathcal{J}^*(\mathcal{H}_s(A)) = (\mathcal{J}(\mathcal{J}^*(A^*)))^* \\ & = (\mathcal{J}^*(A^*))^* = \mathcal{H}_s^*(A). \end{aligned}$$

Hence $\tau_{\mathcal{H}_s} = \{\mathcal{H}_s^*(A) = \bigwedge_{y\in X}(A(y) \to \mathcal{J}(\top_y^*)) \mid A \in L^X\}$.

$$\begin{aligned} \mathcal{H}_s(\mathcal{H}_s(A)) & = \mathcal{H}_s(\mathcal{H}_s^*(\mathcal{H}_s^*(A))) \\ & = \mathcal{H}_s^*(\mathcal{H}_s^*(A)) = \mathcal{H}_s(A). \end{aligned}$$

By a similar method in (4), $\tau_{\mathcal{H}_s} = (\tau_{\mathcal{H}_s})_*$.

(8) It is similarly proved as (5).

(9) If $\mathcal{J}(\mathcal{J}(A)) = \mathcal{J}(A)$ for $A \in L^X$, then $\mathcal{K}_J(\mathcal{K}_J^*(A)) =$

$\mathcal{K}_J(A)$

$$\begin{aligned}\mathcal{K}_J(\mathcal{K}^*_J(A)) &= \mathcal{K}_J(\mathcal{J}^*(A^*)) = \mathcal{J}(\mathcal{J}(A^*)) \\ &= \mathcal{J}(A^*) = \mathcal{K}_J(A).\end{aligned}$$

(10) If $\mathcal{J}(\mathcal{J}^*(A)) = \mathcal{J}^*(A)$ for $A \in L^X$, then $\mathcal{K}_J(\mathcal{K}_J(A)) = \mathcal{K}^*_J(A)$

$$\begin{aligned}\mathcal{K}_J(\mathcal{K}_J(A)) &= \mathcal{J}(\mathcal{K}^*_J(A)) = \mathcal{J}(\mathcal{J}^*(A^*)) \\ &= \mathcal{J}^*(A^*) = \mathcal{K}^*_J(A).\end{aligned}$$

Since $\mathcal{K}_J(\mathcal{K}_J(A)) = \mathcal{K}^*_J(A)$,

$$\begin{aligned}\mathcal{K}_J(\mathcal{K}^*_J(A)) &= \mathcal{K}_J(\mathcal{K}_J(\mathcal{K}_J(A))) \\ &= \mathcal{K}^*_J(\mathcal{K}_J(A)) = \mathcal{K}_J(A).\end{aligned}$$

Hence $\tau_{\mathcal{K}_J} = \{\mathcal{K}_J(A) \mid A \in L^X\} = (\tau_{\mathcal{K}_J})_*$.

(11) , (12), (13) and (14) are similarly proved as (5), (9), (10) and (5), respectively.

(15) If $\mathcal{J}(\mathcal{J}(A)) = \mathcal{J}(A)$ for $A \in L^X$, then $\mathcal{H}_J(\mathcal{H}_J(A)) = \mathcal{H}_J(A)$. Thus, $\mathcal{K}_{H_J}(\mathcal{K}^*_{H_J}(A)) = \mathcal{K}_{H_J}(A)$

$$\begin{aligned}&\mathcal{K}_{H_J}(\mathcal{K}^*_{H_J}(A)) = \mathcal{K}_{H_J}(\mathcal{H}_J(A)) \\ &= (\mathcal{H}_J(\mathcal{H}_J(A)))^* = (\mathcal{H}_J(A))^* = \mathcal{K}_{H_J}(A).\end{aligned}$$

Since $\mathcal{J}(A) = A$ iff $\mathcal{H}_J(A) = A$ iff $\mathcal{K}_{H_J}(A) = A^*$, $\tau_{\mathcal{K}_{H_J}} = (\tau_{\mathcal{J}})_*$ with

$$\begin{aligned}\tau_{\mathcal{K}_{H_J}} &= \{\mathcal{K}^*_{H_J}(A)(y) \\ &= \bigvee_{x\in X} (\mathcal{J}^*(\top^*_y)(x) \odot A(x)) \mid A \in L^X\}.\end{aligned}$$

(16) If $\mathcal{H}_J(\mathcal{H}^*_J(A)) = \mathcal{H}^*_J(A)$ for $A \in L^X$, then

$$\mathcal{K}_{H_J}(\mathcal{K}_{H_J}(A)) = \mathcal{K}^*_{H_J}(A)$$

$$\begin{aligned}\mathcal{K}_{H_J}(\mathcal{K}_{H_J}(A)) &= \mathcal{K}_{H_J}(\mathcal{K}^*_J(A)) = \mathcal{H}^*_J(\mathcal{H}^*_J(A)) \\ &= \mathcal{H}_J(A) = \mathcal{K}^*_{H_J}(A).\end{aligned}$$

(17) , (18) and (19) are similarly proved as (14), (15) and (16), respectively.

(20) $(\mathcal{K}_{H_J}, \mathcal{K}_J)$ is a Galois connection;i.e,

$$A \le \mathcal{K}_{H_J}(B) \text{ iff } A \le (\mathcal{H}_J(B))^*$$

$$\text{iff } \mathcal{H}_J(B) \le A^* \text{ iff } B \le \mathcal{J}(A^*) = \mathcal{K}_J(A)$$

Moreover, since $A^* \le \mathcal{K}_J(A)$ iff $A \le \mathcal{K}_{H_J}(A^*)$, $\tau_{\mathcal{K}_J} = (\tau_{\mathcal{K}_{H_J}})_*$.

(21) $(\mathcal{M}_J, \mathcal{M}_{H_J})$ is a dual Galois connection;i.e,

$$\mathcal{M}_{H_J}(A) \le B \text{ iff } \mathcal{H}_J(A^*) \le B$$

$$\text{iff } A^* \le \mathcal{J}(B) \text{ iff } \mathcal{M}_J(B) = (\mathcal{J}(B))^* \le A.$$

Since $\mathcal{M}_{H_J}(A^*) \le A$ iff $\mathcal{M}_J(A) \le A^*$, $\tau_{\mathcal{M}_J} = (\tau_{\mathcal{M}_{H_J}})_*$.

Let $R \in L^{X\times X}$ be an $L$-fuzzy relation. Define operators as follows

$$\begin{aligned}\mathcal{H}_R(A)(y) &= \textstyle\bigvee_{x\in X}(A(x) \odot R(x,y)), \\ \mathcal{J}_R(A)(y) &= \textstyle\bigwedge_{x\in X}(R(x,y) \to A(x)), \\ \mathcal{K}_R(A)(y) &= \textstyle\bigwedge_{x\in X}(A(x) \to R(x,y)) \\ \mathcal{M}_R(A)(y) &= \textstyle\bigvee_{x\in X}(A^*(x) \odot R(x,y)).\end{aligned}$$

**Example 2.2.** Let $R$ be a reflexive $L$-fuzzy relation. Define $\mathcal{J}_R : L^X \to L^X$ as follows:

$$\mathcal{J}_R(A)(y) = \bigwedge_{x\in X} (R(x,y) \to A(x)).$$

(1) (J1) $\mathcal{J}_R(A)(y) \le R(y,y) \to A(y) = A(y)$. $\mathcal{J}_R$ satisfies the conditions (J1) and (J2) from:

$$\begin{aligned}\mathcal{J}_R(a \to A)(y) &= \textstyle\bigwedge_{x\in X}(R(x,y) \to (a \to A)(x)) \\ &= a \to \textstyle\bigwedge_{x\in X}(R(x,y) \to A(x)), \\ \mathcal{J}_R(\textstyle\bigwedge_{i\in\Gamma} A_i)(y) &= \textstyle\bigwedge_{x\in X}(R(x,y) \to \bigwedge_{i\in\Gamma} A_i(x)) \\ &= \textstyle\bigwedge_{i\in\Gamma}\bigwedge_{x\in X}(R(x,y) \to A_i(x)).\end{aligned}$$

Hence $\mathcal{J}_R$ is an $L$-lower approximation operator.

(2) Define $\mathcal{H}_{J_R}(B) = \bigvee\{A \mid B \le \mathcal{J}_R(A)\}$. Since

$$\begin{aligned}B(y) \le \mathcal{J}_R(A)(y) &\text{ iff } B(y) \le \bigwedge_{x\in X} (R(x,y) \to A(x)) \\ &\text{ iff } \bigvee_{y\in X} (B(y) \odot R(x,y)) \le A(x),\end{aligned}$$

then

$$\mathcal{H}_{J_R}(B)(x) = \bigvee_{y\in X} (R(x,y) \odot B(y)) = \mathcal{H}_{R^{-1}}(B)(x).$$

By Theorem 2.1(2), $\mathcal{H}_{J_R} = \mathcal{H}_{R^{-1}}$ is an $L$-upper approximation operator such that $(\mathcal{H}_{J_R}, \mathcal{J}_R)$ is a residuated connection;i.e.,

$$\mathcal{H}_{J_R}(A) \le B \text{ iff } A \le \mathcal{J}_R(B).$$

Moreover, $\tau_{\mathcal{H}_{J_R}} = \tau_{\mathcal{J}_R}$.

(3) If $R$ is an $L$-fuzzy preorder, then $R^{-1}$ is an $L$-fuzzy preorder. Since

$$\begin{aligned}\mathcal{J}_R(\mathcal{J}_R(A))(z) &= \bigwedge_{y\in X}(R(y,z)\to \mathcal{J}_R(A)(y))\\ &= \bigwedge_{y\in X}(R(y,z)\to \bigwedge_{x\in X}(R(x,y)\to A(x)))\\ &= \bigwedge_{x\in X}\bigwedge_{y\in X}(R(y,z)\odot R(x,y)\to A(x)))\\ &= \bigwedge_{x\in X}(\bigvee_{y\in X}(R(y,z)\odot R(x,y))\to A(x))\\ &= \bigwedge_{x\in X}(R(x,z)\to A(x))\\ &= \mathcal{J}_R(A)(z),\end{aligned}$$

By Theorem 2.1(3), $\mathcal{H}_{J_R}(\mathcal{H}_{J_R}(A)) = \mathcal{H}_{J_R}(A)$. By Theorem 2.1(3), $\tau_{\mathcal{H}_{J_R}} = \tau_{\mathcal{J}_R}$ with

$$\begin{aligned}&\{\mathcal{H}_{R^{-1}}(A) = \bigvee_{x\in X}(R(-,x)\odot A(x)) \mid A\in L^X\}\\ &= \tau_{\mathcal{H}_{J_R}} = \tau_{\mathcal{H}_{R^{-1}}},\\ &\tau_{\mathcal{J}_R} = \{\mathcal{J}_R(A) = \bigwedge_{x\in X}(R(x,-)\to A(x)) \mid A\in L^X\}.\end{aligned}$$

(4) Let $R$ be a reflexive and Euclidean $L$-fuzzy relation. Since $R(x,z)\odot R(y,z)\odot A^*(x) \le R(x,y)\odot A^*(x)$ iff $R(x,z)\odot A^*(x) \le R(y,z)\to R(x,y) \le A^*(x)$,

$$\begin{aligned}&\mathcal{J}_R(\mathcal{J}^*_R(A))(z)\\ &= \bigwedge_{y\in X}(R(y,z)\to \mathcal{J}^*_R(A)(y))\\ &= \bigwedge_{y\in X}(R(y,z)\to \bigvee_{x\in X}R(x,y)\odot A^*(x)))\\ &\ge \bigvee_{x\in X}R(x,z)\odot A^*(x))).\end{aligned}$$

Thus, $\mathcal{J}_R(\mathcal{J}^*_R(A)) = \mathcal{J}^*_R(A)$.

By Theorem 2.1(4), $\mathcal{J}_R(\mathcal{J}_R(A)) = \mathcal{J}_R(A)$ for $A\in L^X$. Thus, $\tau_{\mathcal{J}_R} = (\tau_{\mathcal{J}_R})_*$ with

$$\tau_{\mathcal{J}_R} = \left\{\mathcal{J}^*_R(A) = \bigvee_{x\in X}(R(x,-)\odot A^*(x)) = \mathcal{M}_R(A) \mid A\in L^X\right\}.$$

(5) Define $\mathcal{H}_s(A) = \mathcal{J}_R(A^*)^*$. By Theorem 2.1(5), $\mathcal{H}_s = \mathcal{H}_R$ is an $L$-upper approximation operator such that

$$\begin{aligned}\mathcal{H}_s(A)(y) &= (\bigwedge_{x\in X}R(x,y)\to A^*(x))^*\\ &= \bigvee_{x\in X}(R(x,y)\odot A(x)).\end{aligned}$$

Moreover, $\tau_{\mathcal{H}_s} = \tau_{\mathcal{H}_R} = (\tau_{\mathcal{H}_{J_R}})_*$.

(6) If $R$ is an $L$-fuzzy preorder, then $\mathcal{J}_R(\mathcal{J}_R(A)) = \mathcal{J}_R(A)$ for $A\in L^X$. By Theorem 2.1(6), then $\mathcal{H}_s(\mathcal{H}_s(A)) = \mathcal{H}_s(A)$ for $A\in L^X$ such that $\tau_{\mathcal{H}_s} = (\tau_{\mathcal{J}_R})_* = (\tau_{\mathcal{H}_{J_R}})_*$ with

$$\tau_{\mathcal{H}_s} = \{\mathcal{H}_s(A) = \bigvee_{y\in X}(R(y,-)\odot A(y)) \mid A\in L^X\}.$$

(7) If $R$ is a reflexive and Euclidean $L$-fuzzy relation, then $\mathcal{J}_R(\mathcal{J}^*_R(A)) = \mathcal{J}^*_R(A)$ for $A\in L^X$. By Theorem 2.1(7), $\mathcal{H}_s(\mathcal{H}^*_s(A)) = \mathcal{H}^*_s(A)$ such that

$$\begin{aligned}\tau_{\mathcal{H}_s} &= (\tau_{\mathcal{H}_s})_*\\ &= \{\mathcal{H}^*_s(A)\\ &= \bigwedge_{y\in X}(A(y)\to R^*(y,-))\\ &= \mathcal{K}_{R^*}(A) \mid A\in L^X\}.\end{aligned}$$

(8) Define $\mathcal{K}_{J_R}(A) = \mathcal{J}_R(A^*)$. Then $\mathcal{K}_{J_R}: L^X\to L^X$ with

$$\mathcal{K}_{J_R}(A)(y) = \bigwedge_{x\in X}(R(x,y)\to A^*(x)) = \mathcal{K}_{R^*}(y)$$

is an $L$-join meet approximation operator. Moreover, $\tau_{\mathcal{K}_{J_R}} = (\tau_{\mathcal{J}_R})_*$.

(9) $R$ is an $L$-fuzzy preorder, then $\mathcal{J}_R(\mathcal{J}_R(A)) = \mathcal{J}_R(A)$ for $A\in L^X$. By Theorem 2.1(9), $\mathcal{K}_{J_R}(\mathcal{K}^*_{J_R}(A)) = \mathcal{K}_{J_R}(A)$ for $A\in L^X$ such that $\tau_{\mathcal{K}_{J_R}} = (\tau_{\mathcal{J}_R})_*$ with

$$\begin{aligned}\tau_{\mathcal{K}_{J_R}} &= \{\mathcal{K}^*_{J_R}(A)\\ &= \bigvee_{x\in X}(R(x,-)\odot A(x))\\ &= \mathcal{H}_R(A) \mid A\in L^X\}.\end{aligned}$$

(10) If $R$ is a reflexive and Euclidean $L$-fuzzy relation, then $\mathcal{J}_R(\mathcal{J}^*_R(A)) = \mathcal{J}^*_R(A)$ for $A\in L^X$. By Theorem 2.1(10), $\mathcal{K}_{J_R}(\mathcal{K}_{J_R}(A)) = \mathcal{K}^*_{J_R}(A)$ such that

$$\begin{aligned}&\{\mathcal{K}_{J_R}(A) = \textstyle\bigwedge_{x\in X}(A(x)\to R^*(x,-) \mid A\in L^X\}\\ &= \tau_{\mathcal{K}_{J_R}} = (\tau_{\mathcal{K}_{J_R}})_*.\end{aligned}$$

(11) Define $\mathcal{M}_{J_R}(A) = (\mathcal{J}_R(A))^*$. Then $\mathcal{M}_{J_R}: L^X\to L^X$

with

$$\mathcal{M}_{J_R}(A)(y) = \bigvee_{x\in X}(A^*(x)\odot R(x,y)) = \mathcal{M}_R(A)(y)$$

is an $L$-join meet approximation operator. Moreover, $\tau_{\mathcal{M}_{J_R}} = \tau_{\mathcal{J}_R}$.

(12) If $R$ is an $L$-fuzzy preorder, then $\mathcal{J}_R(\mathcal{J}_R(A)) = \mathcal{J}_R(A)$ for $A \in L^X$. By Theorem 2.1(12), $\mathcal{M}_{J_R}(\mathcal{M}^*_{J_R}(A)) = \mathcal{M}_{J_R}(A)$ for $A \in L^X$ such that $\tau_{\mathcal{M}_{J_R}} = \tau_{\mathcal{J}_R}$ with

$$\tau_{\mathcal{M}_{J_R}} = \{\mathcal{M}^*_{J_R}(A) = \bigwedge_{x\in X}(R(x,-)\to A(x)) = \mathcal{J}_R(A) \mid A\in L^X\}.$$

(13) If $R$ is a reflexive and Euclidean $L$-fuzzy relation, then $\mathcal{J}_R(\mathcal{J}^*_R(A)) = \mathcal{J}^*_R(A)$ for $A \in L^X$. By Theorem 2.1(13), $\mathcal{M}_{J_R}(\mathcal{M}_{J_R}(A)) = \mathcal{M}^*_{J_R}(A)$ such that

$$\tau_{\mathcal{M}_{J_R}} = \{\mathcal{M}_{J_R}(A) = \bigvee_{x\in X}(A(x)\odot R(x,-)) = \mathcal{H}_{J_R}(A) \mid A\in L^X\} = (\tau_{\mathcal{M}_{J_R}})_*.$$

(14) Define $\mathcal{K}_{H_{J_R}}(A) = (\mathcal{H}_{J_R}(A))^*$. Then

$$\mathcal{K}_{H_{J_R}} : L^X \to L^X$$

with

$$\mathcal{K}_{H_{J_R}}(A)(y) = \bigwedge_{x\in X}(A(x)\to R^*(y,x)) = \mathcal{K}_{R^{-1*}}(A)(y)$$

is an $L$-join meet approximation operator. Moreover, $\tau_{\mathcal{K}_{R^{-1}}} = \tau_{\mathcal{J}_R} = \tau_{\mathcal{H}_{R^{-1}}}$.

(15) If $R$ is an $L$-fuzzy preorder, then $\mathcal{J}_R(\mathcal{J}_R(A)) = \mathcal{J}_R(A)$ for $A \in L^X$. By Theorem 2.1(15), $\mathcal{K}_{R^{-1}}(\mathcal{K}^*_{R^{-1}}(A)) = \mathcal{K}_{R^{-1}}(A)$ for $A \in L^X$ such that $\tau_{\mathcal{K}_{R^{-1}}} = \tau_{\mathcal{J}_R} = \tau_{\mathcal{H}_{R^{-1}}}$ with

$$\tau_{\mathcal{K}_{R^{-1*}}} = \{\mathcal{K}^*_{R^{-1*}}(A)(y) = \bigvee_{x\in X}(R(y,x)\odot A(x)) = \mathcal{H}_{R^{-1}}(A)(y) \mid A\in L^X\}.$$

(16) Let $R^{-1}$ be a reflexive and Euclidean $L$-fuzzy relation. Since

$$\begin{aligned} R^{-1}(x,z)\odot R^{-1}(y,z) &\le R^{-1}(x,y)\\ &\text{iff } R^{-1}(y,z)\le R^{-1}(x,z)\to R^{-1}(x,y)\\ &\text{iff } R^{-1*}(y,z)\ge R^{-1}(x,z)\odot R^{-1*}(x,y),\end{aligned}$$

we have

$$\begin{aligned}&(A(x)\to R^{-1*}(x,y))\odot A(x)\odot R^{-1}(x,z)\\ &\le R^{-1}(x,y)\odot R^{-1*}(x,z)\le R^{-1*}(y,z).\end{aligned}$$

Thus,

$$\begin{aligned} A(x)\odot R^{-1}(x,z) &\le (A(x)\to R^{-1*}(x,y))\\ &\to R^{-1}(x,y)\odot R^{-1*}(x,z)\le R^{-1*}(y,z).\end{aligned}$$

Hence

$$\begin{aligned}&\mathcal{K}_{R^{-1*}}(\mathcal{K}_{R^{-1*}}(A))(z)\\ &= \bigwedge_{y\in X}(\mathcal{K}_{R^{-1*}}(A)(y)\to R^{-1*}(y,z))\\ &= \bigwedge_{y\in X}(\bigwedge_{x\in X}(A(x)\to R^{-1*}(x,y))\to R^{-1*}(y,z))\\ &\le \bigvee_{x\in X}(A(x)\odot R^{-1*}(x,z)) = \mathcal{K}_{R^{-1*}}(A)(z)\end{aligned}$$

By (K1), $\mathcal{K}_{R^{-1*}}(\mathcal{K}_{R^{-1*}}(A)) = \mathcal{K}^*_{R^{-1*}}(A)$ such that

$$\begin{aligned}&\{\mathcal{K}_{R^{-1*}}(A) = \bigwedge_{x\in X}(A(x)\to R^*(-,x)) \mid A\in L^X\}\\ &= \tau_{\mathcal{K}_{R^{-1}}} = (\tau_{\mathcal{K}_{R^{-1}}})_*.\end{aligned}$$

(17) Define $\mathcal{M}_{H_{J_R}}(A) = \mathcal{H}_{J_R}(A^*)$. Then

$$\mathcal{M}_{H_{J_R}} : L^X \to L^X$$

is an $L$-meet join approximation operator as follows:

$$\mathcal{M}_{H_{J_R}}(A)(y) = \bigvee_{x\in X}(R(y,x)\odot A^*(x)) = \mathcal{M}_{R^{-1}}(A)(y).$$

Moreover, $\tau_{\mathcal{M}_{H_{J_R}}} = (\tau_{\mathcal{J}_R})_*$.

(18) If $R$ is an $L$-fuzzy preorder, then $\mathcal{J}_R(\mathcal{J}_R(A)) = \mathcal{J}_R(A)$ for $A \in L^X$. By Theorem 2.1(18),

$$\mathcal{M}_{H_{J_R}}(\mathcal{M}^*_{H_{J_R}}(A)) = \mathcal{M}_{H_{J_R}}(A)$$

for $A \in L^X$ such that $\tau_{\mathcal{M}_{H_{J_R}}} = (\tau_{\mathcal{J}})_*$ with

$$\tau_{\mathcal{M}_{H_{J_R}}} = \left\{\mathcal{M}^*_{H_{J_R}}(A)(y) = \bigwedge_{x\in X}(R(y,x)\to A(x)) = \mathcal{J}_{R^{-1}}(A)(y) \mid A\in L^X\right\}.$$

(19) Let $R^{-1}$ be a reflexive and Euclidean $L$-fuzzy relation. Since

$$\begin{aligned}&(R(y,x)\to A(x))\odot R(z,y)\odot R(z,x)\\ &\le R(y,x)\to A(x))\odot R(y,x)\le A(x),\end{aligned}$$

then $(R(y,x)\to A(x))\odot R(z,y)\le R(z,x)\to A(x)$.

Thus,

$$\begin{aligned}&\mathcal{M}_{R^{-1}}(\mathcal{M}_{R^{-1}}(A))(z)\\&=\bigvee_{y\in X}(\mathcal{M}_{R^{-1}}(A)(y)\odot R(z,y))\\&=\bigvee_{y\in X}(\bigwedge_{x\in X}(R(y,x)\to A(x))\odot R(z,y))\\&\le\bigwedge_{x\in X}(R(z,x)\to A(x))=\mathcal{M}_{R^{-1}}(A)(z).\end{aligned}$$

By (M1), $\mathcal{M}_{R^{-1}}(\mathcal{M}_{R^{-1}}(A))=\mathcal{M}^*_{R^{-1}}(A)$ such that

$$\{\mathcal{M}_{R^{-1}}(A)=\bigvee_{x\in X}(A^*(x)\odot R(-,x))\mid A\in L^X\}=\tau_{\mathcal{M}_{R^{-1}}}=(\tau_{\mathcal{M}_{R^{-1}}})_*.$$

(20) $(\mathcal{K}_{H_{J_R}}=\mathcal{K}_{R^{-1*}},\mathcal{K}_{J_R}=\mathcal{K}_{R^*})$ is a Galois connection;i.e, $A\le\mathcal{K}_{H_{J_R}}(B)$ iff $B\le\mathcal{K}_{J_R}(A)$. Moreover, $\tau_{\mathcal{K}_{J_R}}=(\tau_{\mathcal{K}_{H_{J_R}}})_*$.

(21) $(\mathcal{M}_{J_R}=\mathcal{M}_R,\mathcal{M}_{H_{J_R}}=\mathcal{M}_{R^{-1}})$ is a dual Galois connection;i.e, $\mathcal{M}_{H_{J_R}}(A)\le B$ iff $\mathcal{M}_{J_R}(B)\le A$. Moreover, $\tau_{\mathcal{M}_{J_R}}=(\tau_{\mathcal{M}_{H_{J_R}}})_*$.

## 3. Conclusions

In this paper, $L$-lower approximation operators induce $L$-upper approximation operators by residuated connection. We study relations lower (upper, join meet, meet join) approximation operators, Galois (dual Galois, residuated, dual residuated) connections and Alexandrov $L$-topologies. Moreover, we give their examples as approximation operators induced by various $L$-fuzzy relations.

## Conflict of Interest

No potential conflict of interest relevant to this article was reported.

## Acknowledgements

This work was supported by the Research Institute of Natural Science of Gangneung-Wonju National University.

## References

[1] Z. Pawlak, "Rough sets," *International Journal of Computer & Information Sciences*, vol. 11, no. 5, pp. 341-356, Oct. 1982. http://dx.doi.org/10.1007/BF01001956

[2] Z. Pawlak, "Rough probability," *Bulletin of Polish Academy of Sciences: Mathematics*, vol. 32, no. 9-10, pp. 607-615, 1984.

[3] P. Hájek,*Metamathematics of Fuzzy Logic*, Dordrecht, The Netherlands: Kluwer, 1998.

[4] A. M. Radzikowska and E. E. Kerre, "A comparative study of fuzzy rough sets," *Fuzzy Sets and Systems*, vol. 126, no. 2, pp. 137-155, Mar. 2002. http://dx.doi.org/10.1016/S0165-0114(01)00032-X

[5] R. Bělohlávek, *Fuzzy Relational Systems: Foundations and Principles*, New York, NY: Kluwer Academic/Plenum Publishers, 2002.

[6] H. Lai and D. Zhang, "Fuzzy preorder and fuzzy topology," *Fuzzy Sets and Systems*, vol. 157, no. 14, pp. 1865-1885, Jul. 2006. http://dx.doi.org/10.1016/j.fss.2006.02.013

[7] H. Lai and D. Zhang, "Concept lattices of fuzzy contexts: formal concept analysis vs. rough set theory," *International Journal of Approximate Reasoning*, vol. 50, no. 5, pp. 695-707, May. 2009. http://dx.doi.org/10.1016/j.ijar.2008.12.002

[8] Y. C. Kim, "Alexandrov $L$-topologiesand $L$-join meet approximatin operators," *International Journal of Pure and Applied Mathematics*, vol. 91, no. 1, pp. 113-129, 2014. http://dx.doi.org/10.12732/ijpam.v91i1.12

[9] Y. C. Kim, "Alexandrov $L$-topologies," *International Journal of Pure and Applied Mathematics*, 2014 [in press].

[10] Y. H. She and G. J. Wang, "An axiomatic approach of fuzzy rough sets based on residuated lattices," *Computers & Mathematics with Applications*, vol. 58, no. 1, pp. 189-201, Jul. 2009. http://dx.doi.org/10.1016/j.camwa.2009.03.100

**Yong Chan Kim** received the B.S., M.S. and Ph.D. degrees in Mathematics from Yonsei University, Seoul, Korea, in 1982, 1984 and 1991, respectively. He is currently Professor of Gangneung-Wonju University, his research interests is a fuzzy topology and fuzzy logic.
E-mail: yck@gwnu.ac.kr