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Investigating Arithmetic Mean, Harmonic Mean, and Average Speed through Dynamic Visual Representations[1]

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Working with dynamic visual representations can help students-with-computer discover new mathematical ideas. Students translate among multiple representations as a strategy to investigate non-routine problems to explore possible solutions in mathematics classrooms. In this paper, we use the area models as new representations for our secondary students to investigate three problems related to the average speed of a particle. Students show their ideas in the process of investigating arithmetic mean, harmonic mean, and average speed through their created dynamic figures. These figures really utilize dynamic geometry software.

Keywords: Dynamic visual representations, arithmetic mean, harmonic mean, average speed
MESC Classification: D40, F80, G30
MSC2010Classification: 97D40, 97D80

INTRODUCTION

While written forms of representation are still important, it is necessary to consider how mathematical ideas can be represented through a visually dynamic medium. This strategy itself may help students to investigate and explore interesting mathematical ideas in a new way of mathematical representations.

In Vietnam, since the emphasis of the old curriculum was on procedural knowledge and memorization of algorithms, students often worked independently to complete exercises from textbooks and workbooks. The current reform curriculum tries to reduce the amount of basic skills and procedures in mathematics, while increasing hands-on activities that help students to grasp new ideas and develop mathematical thinking

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(MOET, 2009).

Mathematics educators and classroom teachers in Vietnam are seeking innovative strategies in teaching and learning. The teacher ought to think of teaching in terms of several principal hands-on activities with multiple representations, problematic real life situations, and open-ended problems. The innovation of mathematics teaching is to help students construct their own knowledge in an active way and to enhance their thinking through solving non-routine problems, while working cooperatively with classmates, so that their understanding and mathematical competencies are developed (Tran, 2006a, 2006b).

The use of multiple dynamic visual representations which promote students' exploration of mathematical ideas is relevant.Multiple modes of representation improve transitions from concrete manipulation to abstract thinking, and provide a foundation for continued learning. This study investigates the effectiveness of experimental environments for students-with-computers to explore mathematical ideas through dynamic multiple representations.We found out that students discover possible solutions in the process of solving problems with dynamic visual representations in mathematics classrooms. Students show their capability to construct their own dynamic models and conduct their experimentation.

INVESTIGATING AVERAGE SPEED, ARITHMETIC MEAN, AND HARMONIC MEAN

We construct dynamic visual representations for grade 10 students who have learnt the average speed of a particle, ratio and scale. The data emerged from classroom experiments in the 10^{th} grade (15-16 years old, in Hue City, Vietnam). Altogether 3 classes had been visited for 3 lessons in second semester, school year 2012-2013. The study aims at the use of dynamic representations to discover mathematical ideas in the process of investigating arithmetic mean, harmonic mean, average speed, and also the relationship among them. In the three following problems, we use the area models as new representations for our students to investigate the distance of a particle that travels in specific times and with given speeds.

For giving them the opportunities to come up with new mathematical ideas as possible solutions for a non-routine problem rooted from real life, the students were exposed to a problematic situation they could not solve with their former knowledge. They had to construct new ideas as plausible solutions. The general situation is: "*there are two arbitrary rectangles, construct a rectangle such that its area is the total of the two given rectangles*". To investigate this situation we use three following problems.

Problem 1(Arithmetic mean): A fellow travels from city **A** to city **B**. For the first half of the traveled time, he drove at the constant speed of *a* km per hour. Then he (instantaneously) increased his speed and, for the next half of the time, kept it at *b* km per hour. Find the average speed of the motion.

Dynamic Visual Representation1(Designed by the Geometer's Sketchpad, Tran et al. 2007).

There are two rectangles *AEGD* and *EBCH* with the bases $AE = EB$.
Assume that
$AD = a$, $AE = t = EB$, $BC = b$; $0 < a < b$.
Students work in group of four students to construct two rectangles with the same base as in the Fig. 1.
Construct a rectangle such that the base is $2t$ and its area is the total of the two given rectangles.
Students can drag points *D*, *C*, and *E* to change *a*, *b*, and *t* respectively.

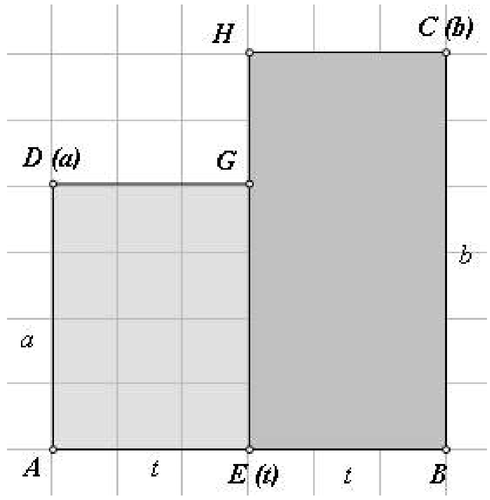

Fig. 1. Tworectangles with the same base

Students observe the area of rectangle *AEGD* representing the distance that the fellow travels in the first half of traveled time. They discover some interesting mathematical ideas.

Idea 1.
The distance $d_1 = a \times t$, $d_2 = b \times t$.
The total distance $d = (a + b)\, t$.

Average speed $v = \dfrac{d}{2t} = \dfrac{a+b}{2}$.

In the trapezoid *ABCD*, let *E, F*be the two midpoints of *AB* and *CD* respectively.

$EF - a = b - EF$; so $EF = \dfrac{a+b}{2}$.

The area of the rectangle *ABLM* is $\dfrac{a+b}{2} \times 2t = (a+b)t$.

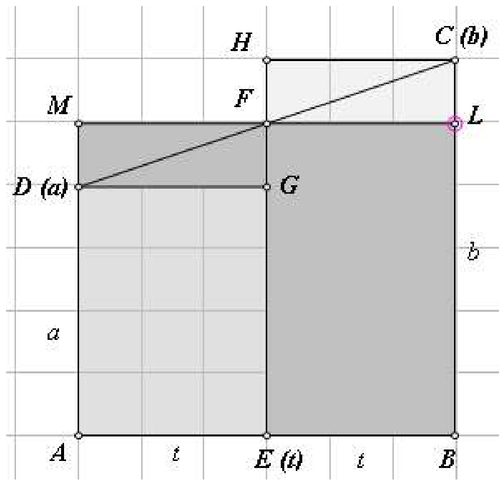

Fig. 2. "Arithmetic mean" in a trapezoid

Idea 2.

For the given two quantities, *a* and*b*, by dragging *E* to change *t*, the number $(a + b)/2$ is known as their *arithmetic mean* and represented by*EF*.Area of the rectangle *ABLM* represents the distance that a particle moves in the time of 2*t* with the average speed is $(a+b)/2$. The area of the trapezoid *ABCD* is equal to the area of the rectangle *ABLM*.

Idea 3.

Construct a similar area model such that $a + b$ is constant. Dragging *D* to change *a*.

If$a + b$ is constant, so $EF = \dfrac{a+b}{2}$ is also a constant. *CD* always passes through fixed point *F*.

Problem 2(Harmonic mean): A fellow travels from city **A** to city **B**. The first half of the way, he drove at the constant speed of *a* km per hour. Then he (instantaneously) increased his speed and traveled the remaining distance at *b* km per hour ($0 < a < b$). Find the average speed of the motion.

Dynamic Representation 2 (Designed by the Geometer's Sketchpad).

There are two rectangles *AMLD* and *MBCN* with the same area, assume that $AD = a$, $AM = t_1$, $MB = t_2$, $BC = b$; $0 < a < b$.

Students work in group of four students to construct two rectangles with the same area as in the Fig. 3.

Construct a rectangle such that the base is $t_1 + t_2$ and its area is the total of the two given rectangles.

Students can drag points *D*, *(b)*, and *M* to change *a*, *b*, and t_1 respectively.

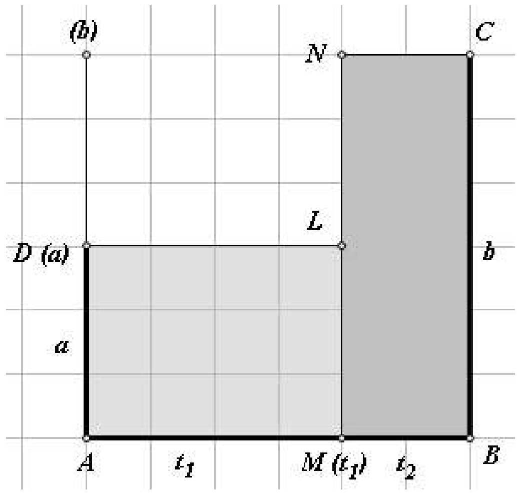

Fig. 3. Tworectangles with the same area

Idea 4.

a, b: speed, *t*: time, distance $d = a \times t_1 = b \times t_2$ is also the area of the two given rectangles.

The total distance is 2*d*. The average speed is:

$$S = \frac{2d}{t_1 + t_2} = \frac{2d}{\frac{d}{a} + \frac{d}{b}} = \frac{2ab}{a+b}$$

Construct a rectangle with the base is $t_1 + t_2$ and the height is $\frac{2ab}{a+b}$.

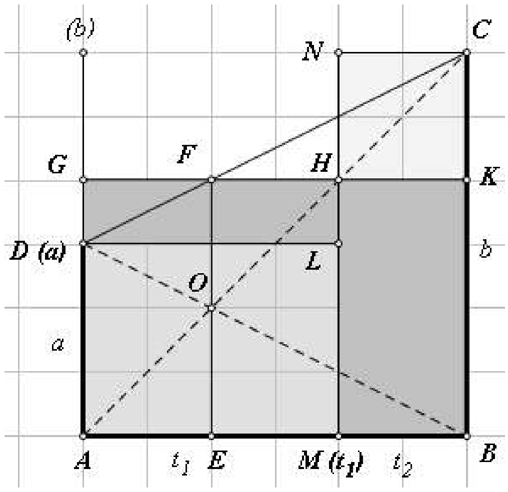

Fig. 4. "Harmonic mean" in a trapezoid

Idea 5.

In the trapezoid *ABCD*, let *EF* be the line parallel to the sides *AD* and *BC* through the intersection point *O* of the diagonals *AC* and *BD*.

Since $\frac{OE}{a} = \frac{BE}{BA}$ and $\frac{OF}{a} = \frac{CF}{CD} = \frac{BE}{BA}$, we have *OE* = *OF*.

From $\frac{OE}{a} = \frac{BE}{BA}$; $\frac{OF}{b} = \frac{AE}{AB}$, we derive $(\frac{1}{a} + \frac{1}{b})OE = 1 \Rightarrow OE = \frac{ab}{a+b}$;

so $EF = \frac{2ab}{a+b}$.

Area of the rectangle *ABHK* is $AB \times EF = (t_1 + t_2) \times \frac{2ab}{a+b} = 2d$.

Idea 6.

Area of the rectangle *DLHG* = $DL \times LH = \frac{d}{a} \times (\frac{2ab}{a+b} - a) = \frac{d(b-a)}{a+b}$

Area of the rectangle *HKCL* = $HK \times KC = \frac{d}{b} \times (b - \frac{2ab}{a+b}) = \frac{d(b-a)}{a+b}$

Area (*DLHG*) = Area (*HKCL*)

Idea 7.

If a vehicle travels a certain distance at a speed *a* (e.g. 40 kilometers per hour) and then the same distance again at a speed *b*(e.g. 60 kilometers per hour), then its average speed is the harmonic mean of *a* and *b* (48 kilometers per hour), and its total travel time is the same as if it had traveled the whole distance at that average speed.

Problem 3(Average speed)**:** A fellow travels from city A to city B. For the first period of time, he drove at the constant speed of *a* km per hour. Then he (instantaneously) increased his speed and, for the next period of time, kept it at *b* km per hour. Find the average speed of the motion.

Dynamic Representation 3 (Designed by the Geometer's Sketchpad)

There are two rectangles *AEGD* and *EBCH* with the arbitrary areas, assume that $AD = a$, $AE = t_1$, $EB = t_2$, $BC = b$; $0 < a < b$.

Students work in group of four students to construct two arbitrary rectangles whose bases are *AE* and *EB* as in the Fig. 5.

Construct a rectangle such that the base is $t_1 + t_2$ and its area is the total of the two given rectangles.

Students can drag points *D*, *C*, *E*, and *B* to change *a*, *b*, $t_1$and t_2 respectively.

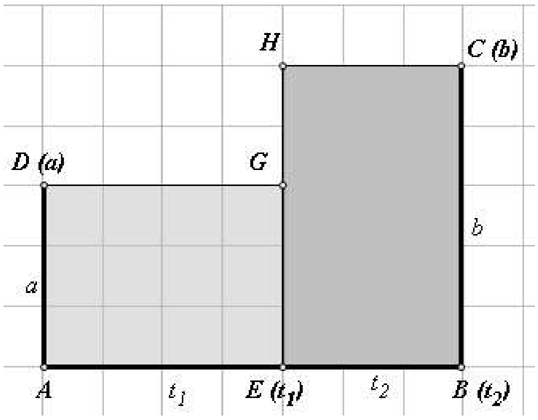

Fig. 5. Twoarbitraryrectangles

Idea 8. We try to define the point *F* by geometric approach by calculating some ratios.

$$EF = \frac{at_1 + bt_2}{t_1 + t_2} \qquad (1)$$

$$\frac{RB}{RE} = \frac{b}{a} = \frac{RE + t_2}{RE} = 1 + \frac{t_2}{RE}$$

$$RE = \frac{at_2}{b - a}; \qquad (2)$$

$$RB = \frac{bt_2}{b - a} \qquad (3)$$

$$\frac{QB}{QE} = \frac{b}{EF} = \frac{b(t_1 + t_2)}{at_1 + bt_2}$$

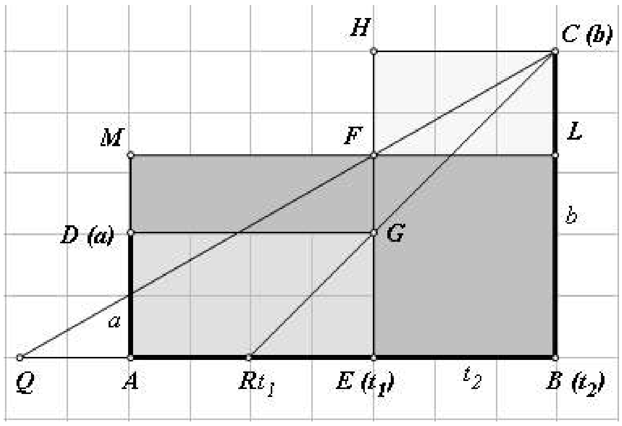

Fig. 6. "Average speed" in area models

$$\frac{QB}{QE} = \frac{QE + EB}{QE} = 1 + \frac{t_2}{QE} = \frac{b(t_1 + t_2)}{at_1 + bt_2}$$

$$QE = \frac{t_2(at_1 + bt_2)}{(b-a)t_1} \tag{4}$$

$$QB = \frac{b(t_1 + t_2)}{at_1 + bt_2} \times QE = \frac{b(t_1 + t_2)t_2}{(b-a)t_1} \tag{5}$$

$$\frac{BQ}{BR} = \frac{t_1 + t_2}{t_1} = \frac{AB}{AE} \tag{6}$$

From the ratio calculated by (6), we can define the point *F* as follows. We create point *Q* by dilating *R* by ratio (6) with center *B*. The segment *CQ* intersects *EH* at *F*. Then the length of *EF* will represent the average speed.

Idea 9.

If $t_1 = t_2$, then the average speed is the arithmetic mean $EF = \frac{a+b}{2}$.

If $at_1 = bt_2$, the average speed is the harmonic mean $ST = \frac{2ab}{a+b}$.

The arithmetic and harmonic means are special cases of "*average speed*".

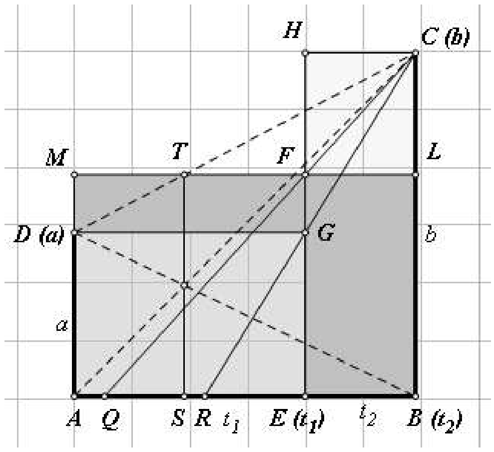

Fig. 7. Harmonic mean as a special case

USING MULTIPLE REPRESENTATIONS IN EXPLORING NEW IDEAS

In the process of solving three above problems with multiple representations such as geometric figures, algebraic equations, numerical values, students may make some mistakes, and give either correct or incorrect answers but, at the end of the problem-solving process, students will explore some new ideas in mathematics. The teacher encourages students to be interested in seeking alternative solutions, and promotes creativity when learning mathematics.

A discovery requires creative thinking:

- Which is executed on the basis of the knowledge of a rule, while the given facts have not been associated conceptually with that rule before; and/or

- Whichis consisted in the creation of a new rule.

Since the discovery of new knowledge alone does not guarantee certainty, the hypothetical knowledge has to be verified. To express a discovery only means an explanatory hypothesis is becoming plausible. Nevertheless, the correctness of the rule and the case, as well as the coherence between the rule and the observed fact, could remain vague (Meyer, 2007, 2010).

There is strong support in the mathematics education community that students can grasp the meaning of mathematical concepts by experiencing multiple mathematical representations in the process of solving challenging problems (Sierpinska, 1992).The term “representations” is interpreted as the tools used for representing mathematical ideas such as geometric figures, algebraic equations or numerical measurements (Confrey& Smith, 1991).

A student can demonstrate deep understanding of a concept by translating a representation of that concept to other modes of representation. For instance, asking a student to restate a familiar algebraic problem in geometric terms, to construct visualpictures to illustrate the problem, or to act out the problem are some ways of translating among representations. This translational skill among different modes of representation can support students’ relational thinking and algebraic reasoning (Suh& Moyer, 2007).

We try to make the best use of multiple representations for successful thinking and investigating school mathematics. Finding a new representation, that is, finding a new perception of the open-ended problem basically means a restructuring of the problem representation. For example, in finding the solution of a problem, the problem solver might suddenly become aware of new relations between elements of the given material by mentally changing, amplifying or restructuring the material (Montgomery, 1988). For a long time, mathematics educators have been designing and using multiple representations to make the formal, abstract mathematics accessible for students. The use of multiple mathematical representations has been shown to increase students’ capability in exploring mathematical ideas.

DYNAMIC VISUAL REPRESENTATIONS

This section emphasizes some of the positive effects of visualizing in mathematical concept formation and to show how dynamic visual representations can be used to achieve more than just a basic, procedural and mechanical understanding of mathematical concepts.

Arcavi (2003) proposed that:

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.

The computer is a rich source of visual and computational images that makes the exploration of mathematical conjectures possible. In this sense, the function of the software is important, providing the students with the opportunity to explore mathematical ideas, analyze examples and counter-examples, and then gain the necessary visual intuitions to attain powerful formal insights.

A *visual approach* in the mathematical thinking process would be characterized by:

- Use of graphical information to solve mathematical problems that could also be approached algebraically.
- Difficulty in establishing algebraic interpretations of graphical solutions.
- No need to first run through the algebra, when graphical solutions are requested.
- Facility in formulating conjectures and refutations or giving explanations using graphical information.

In this case, the computer is used to verify conjectures, to calculate, and to decide questions that have visual information as a starting point.

DISCUSSION AND CONCLUSION

At the beginning of the lessons, we realize that students got some difficulties in constructing two rectangles with the same area in the problem 2 with the Geometer's Sketchpad. Students were not familiar with the fact that every rectangle represents the distance of a particle traveling in a specific time with a given speed. When students really understand the situation, in the process of investigating three mentioned problems, students can demonstrate deep understanding of a concept by translating a representation of that concept to other modes of representation. In this study, asking students to construct the area models for arithmetic mean, harmonic mean and average speed, they constructed dynamic representations, dragged some sliding points to observe the invariant mathematical properties. Dynamic visual representations encourage students to incorporate many different types of representations into their sense-making; the students will become more capable of solving mathematical problems and exploring underlying mathematical ideas. Dynamic mathematical software generates environments that can be considered as laboratories where mathematical experiments are performed. Good dynamic geometric software gives students more opportunities to construct their own models and observe many mathematical facts. Students can use "*Measure Menu*" to measure length, perimeter, angle, area, ratio... to get more numerical data. From these data students make more conjectures based on their incomplete knowledge.

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