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SELF-ADJOINT INTERPOLATION ON $AX = Y$ IN A TRIDIAGONAL ALGEBRA ALG$\mathcal{L}$

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Abstract. Given operators X and Y acting on a separable Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator A such that $AX = Y$. In this article, we investigate self-adjoint interpolation problems for operators in a tridiagonal algebra : Let $\mathcal{L}$ be a subspace lattice acting on a separable complex Hilbert space $\mathcal{H}$ and let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators acting on $\mathcal{H}$. Then the following are equivalent:

(1) There exists a self-adjoint operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $AX = Y$.

(2) There is a bounded real sequence $\{\alpha_n\}$ such that $y_{ij} = \alpha_i x_{ij}$ for $i, j \in \mathbb{N}$.

1. Introduction

Let $\mathcal{C}$ be a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all operators acting on a Hilbert space $\mathcal{H}$ and let X and Y be operators acting on $\mathcal{H}$. An *interpolation question* for $\mathcal{C}$ asks for which X and Y is there a bounded operator $A \in \mathcal{C}$ such that $AX = Y$. A variation, the 'n-operator interpolation problems', asks for an operator A such that $AX_i = Y_i$ for fixed finite collections $\{X_1, X_2, \cdots, X_n\}$ and $\{Y_1, Y_2, \cdots, Y_n\}$. The n-operator interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison[4]. In case $\mathcal{U}$ is a nest algebra, the (one-operator) interpolation problem was solved by Lance[5]: his result was extended by Hopenwasser[2] to the case that $\mathcal{U}$ is a CSL-algebra. Munch[6] obtained conditions for interpolation in case A is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser[3] once again extended the

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interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contains a sufficient condition for interpolation n-operators, although necessity was not proved in that paper.

We establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, Alg$\mathcal{L}$ is mathcalled a CSL-algebra. The symbol Alg$\mathcal{L}$ is the algebra of all bounded operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let x and y be two vectors in a Hilbert space $\mathcal{H}$. Then $\langle x, y\rangle$ means the inner product of the vectors x and y. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ the orthogonal complement of $\overline{M}$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers.

2. Results

Let $\mathcal{H}$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \cdots\}$. Let $x_1, x_2, \cdots, x_n$ be vectors in $\mathcal{H}$. Then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors x_1,x_2,$\cdots$,x_n. Let $\mathcal{L}$ be the subspace lattice generated by the subspaces $[e_{2k-1}]$, $[e_{2k-1}, e_{2k}, e_{2k+1}]$ $(k = 1, 2, \cdots)$. Then the algebra Alg$\mathcal{L}$ is mathcalled a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson[1]. These algebras have been found to be useful counterexample to a number of plausible conjectures.

Let $\mathcal{A}$ be the algebra consisting of all bounded operators acting on $\mathcal{H}$ of the form

$$\begin{pmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \ddots \end{pmatrix}$$

with respect to the orthonormal basis $\{e_1, e_2, \cdots\}$, where all non-starred entries are zero. It is easy to see that Alg$\mathcal{L}$=$\mathcal{A}$.

Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded operators acting on $\mathcal{H}$.

Lemma 1. *Let $A=(a_{ij})$ be an operator in the tridiagonal algebra Alg$\mathcal{L}$. Then the following are equivalent:*

(1) A is self-adjoint.

(2) A is diagonal and a_{ii} is real for all $i \in \mathbb{N}$.

Proof. Suppose that A is self-adjoint. Since $A = A^*$, $a_{ij} = 0$ for all $i \neq j$ and a_{ii} is real. So A is a real diagonal matrix.

Conversely, it is clear. □

Theorem 2. *Let Alg$\mathcal{L}$ be the tridiagonal algebra and let $X=(x_{ij})$ and $Y=(y_{ij})$ be operators in $\mathcal{H}$. Then the following are equivalent:*

(1) There exists a self-adjoint operator $A=(a_{ij})$ in Alg$\mathcal{L}$ such that $AX=Y$.

(2) There is a bounded sequence $\{\alpha_n\}$ of real numbers such that $y_{ij} = \alpha_i x_{ij}$ for all $i,j \in \mathbb{N}$.

Proof. Suppose that A is a self-adjoint operator $A=(a_{ij})$ in Alg$\mathcal{L}$ such that $AX=Y$. By Lemma 1, A is diagonal and a_{ii} is real for all $i \in \mathbb{N}$. Let $\alpha_i = a_{ii}$ for $i = 1,2,\cdots$. Since $AX=Y$, $y_{ij} = a_{ii}x_{ij} = \alpha_i x_{ij}$ for $i,j = 1,2,\cdots$.

Conversely, assume that there is a bounded sequence $\{\alpha_n\}$ of real numbers such that $y_{ij} = \alpha_i x_{ij}$ for $i,j = 1,2,\cdots$. Let A be a diagonal matrix with the diagonal sequence $\{\alpha_n\}$. Since $\{\alpha_n\}$ is bounded, A is a bounded operator. Also A is self-adjoint and $AX=Y$. □

Theorem 3. *Let Alg$\mathcal{L}$ be the tridiagonal algebra and let $X_i = (x_{jk}^{(i)})$ and $Y_i = (y_{jk}^{(i)})$ be operators acting on $\mathcal{H}$ for $i = 1,2,\cdots,n$. Then the following are equivalent:*

(1) There exists a self-adjoint operator $A=(a_{ij})$ in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1,2,\cdots,n$.

(2) There is a bounded sequence $\{\alpha_n\}$ of real numbers such that $y_{jk}^{(i)} = \alpha_j x_{jk}^{(i)}$ for all $i = 1,2,\cdots,n$ and $j,k \in \mathbb{N}$.

Proof. Suppose that there exists a self-adjoint operator $A=(a_{ij})$ in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1,2,\cdots,n$. Then A is diagonal and a_{ii} is real for each $i \in \mathbb{N}$ by Lemma 1. Let $\alpha_i = a_{ii}$ for $i = 1,2,\cdots$. Then $\{\alpha_n\}$ is bounded. Since $AX_i = Y_i$, $y_{jk}^{(i)} = a_{jj}x_{jk}^{(i)} = \alpha_j x_{jk}^{(i)}$ for $i = 1,2,\cdots,n$ and $j,k = 1,2,\cdots$.

Conversely, assume that there is a bounded sequence $\{\alpha_n\}$ of real numbers such that $y_{jk}^{(i)} = \alpha_j x_{jk}^{(i)}$ for $i = 1,2,\cdots,n$ and $j,k = 1,2,\cdots$.

Let A be a diagonal matrix with the diagonal sequence $\{\alpha_n\}$. Since $\{\alpha_n\}$ is bounded, A is a bounded operator. Also A is self-adjoint and $AX_i = Y_i$ for $i = 1, 2, \cdots, n$. □

By the similar way with the above, we have the following.

Theorem 4. *Let* $Alg\mathcal{L}$ *be the tridiagonal algebra and let* $X_i = (x_{jk}^{(i)})$ *and* $Y_i = (y_{jk}^{(i)})$ *be operators acting on in* $\mathcal{H}$ *for* $i = 1, 2, \cdots$. *Then the following are equivalent:*

(1) There exists a self-adjoint operator $A = (a_{ij})$ *in* $Alg\mathcal{L}$ *such that* $AX_i = Y_i$ *for* $i = 1, 2, \cdots$.

(2) There is a bounded sequence $\{\alpha_n\}$ *of real numbers such that* $y_{jk}^{(i)} = \alpha_j x_{jk}^{(i)}$ *for all* $i, j, k \in \mathbb{N}$.

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