Journal for History of Mathematics
Vol. 27 No. 1 (Feb. 2014), 1–12
http://dx.doi.org/10.14477/jhm.2014.27.1.001

Mathematical Structures of Joseon mathematician Hong JeongHa

朝鮮 算學者 洪正夏의 數學的 構造

HONG Sung Sa 홍성사 HONG Young Hee 홍영희 LEE Seung On * 이승온

Dedicated to our teacher Professor Han Taidong (心齋 韓泰東) with hearty gratitude on his ninetieth birthday

From the mid 17th century, Joseon mathematics had a new beginning and developed along two directions, namely the traditional mathematics and one influenced by western mathematics. A great Joseon mathematician if not the greatest, Hong JeongHa was able to complete the Song–Yuan mathematics in his book GuIlJib based on his studies of merely Suanxue Qimeng, YangHui Suanfa and Suanfa Tongzong. Although Hong JeongHa did not deal with the systems of equations of higher degrees and general systems of linear congruences, he had the more advanced theories of right triangles and equations together with the number theory. The purpose of this paper is to show that Hong was able to realize the completion through his perfect understanding of mathematical structures.

Keywords: Hong JeongHa (洪正夏, 1684–?), GuIlJib (九一集), mathematical structures.

MSC: 01A13, 01A50, 11A25, 12–03, 12D05

0 Introduction

Joseon mathematics was completely devastated by successive foreign invasions, but Kim SiJin (金始振, 1618–1667) republished Suanxue Qimeng (算學啓蒙, 1299) in 1660 which formed main sources for Joseon mathematicians with YangHui Suanfa (楊輝算法, 1274–1275) and XiangMing Suanfa (詳明算法, 1373). Further, Suanfa Tongzong (算法統宗, 1592) was brought into Joseon in the 17th century. Using these, the

*Corresponding Author.
HONG Sung Sa: Dept. of Math., Sogang Univ. E-mail: sshong@sogang.ac.kr
HONG Young Hee: Dept. of Math., Sookmyung Women's Univ. E-mail: yhhong@sookmyung.ac.kr
LEE Seung On: Dept. of Math., Chungbuk National Univ. E-mail: solee@chungbuk.ac.kr
Receiveded on Jan. 8. 2014, revised on Feb. 2. 2014, accepted on Feb. 8. 2014.

revival of the traditional mathematics in Joseon was successfully achieved.

In the mean time, the Qing dynasty adopted a new system of calendar, Shixianli (時憲曆, 1645) based on the western mathematics and astronomy. In the same year, Shixianli was introduced to Joseon so that Joseon astronomers and mathematicians had to learn mathematics related to the new system. Thus Xiyang Xinfa Lishu (西洋新法曆書) and Tianxue Chuhan (天學初函, 1623) were brought into Joseon in the late 17th century. In the mid 18th century, Shuli Jingyun (1723), a compendium of Qing mathematics including the traditional and western mathematics was also introduced to Joseon.

Thus Joseon mathematics in the 18th century developed in two directions, i.e., the traditional and western ones.

For the traditional one, Hong JeongHa (洪正夏, 1684–?) completed the development once and for all in his work GuIlJib (九一集, 1724) [4, 5] whose main body was already written before 1713. We must note that Song–Yuan mathematics except YangHui Suanfa was completely lost in China during the 16–18th centuries and that Joseon mathematicians imported Siyuan Yujian (四元玉鑑, 1303) and Shushu Jiuzhang (數書九章, 1247) in the 19th century. Thus Hong's book does not contain the theory of systems of equations of higher orders and related topics, and the general theory of systems of linear congruences (大衍術). Except these, Hong JeongHa accomplished a complete restoration of Song–Yuan mathematics albeit he had references merely consisting of the four books mentioned above.

The purpose of this paper is to find the reasons for Hong JeongHa to achieve the above impossible task. Noticing different approaches in the four books and mathematical structural approaches in Suanxue Qimeng [13], Hong JeongHa recognized the importance of mathematical structures and structural approaches to mathematics. We will show that his perspective on mathematics based on structures single handedly contributed to his successful achievement.

In the first section, we study a history of Hong JeongHa and his family, and that of GuIlJib. In the second section, we deal with the development of Hong's perspective on mathematical structures. To do so, we also review briefly the related contents of GuIlJib.

In this paper, we use the revised romanization of Korean. We recall that Joseon and Hong JeongHa were known as Chosun and Hong JungHa respectively and suggest that the readers identify new versions with their Chinese characters.

The reader may find all the Chinese sources of this paper in ZhongGuo KeXue JiShu DianJi TongHui ShuXueJuan (中國科學技術典籍通彙 數學卷) [2] and hence they will not be numbered as an individual reference.

1 Hong JeongHa and GuIlJib

In this section, we include a brief history of Hong JeongHa and then a general review on his work GuIlJib.

From the beginning of Joseon Dynasty, the government made a law to select mathematical officials in HoJo (戶曹), Department of Home Affairs and Finances. The process was called JuHakChwiJae (籌學取才), where examination problems were chosen from Suanxue Qimeng, YangHui Suanfa and Xiangming Suanfa. Further, candidates for the examinations were restricted to members of JungIn (中人) families which were formed by descendants of a child born of noble class (兩班) men's concubines. JungIn means literally the people in the middle class between the noble class and commoners. There was another mathematical government position in the state observatory, called GwanSangGam (觀象監) whose officials were chosen by a national examination (雜科). Since the missions in two offices were disjoint, their officials did not have any significant communications.

We have the Record, JuHak IbGyeokAn (籌學入格案, [5]) of 1,626 mathematicians who passed the examination ChwiJae from the end of the 15th century to 1888 [7]. We do not have any information of Hong JeongHa except those in the Record. In the Record, his great grandfather is Hong InNam (洪仁南), presumably a noble class. There are all together 98 descendants of Hong InNam in the Record, starting from JeongHa's grandfather Hong SeoJu (敍疇, 1628–?) and SeoJu's brother SeoGu (敍九, 1609–?) passing the examination in 1646 and 1632 respectively, to the brothers UiGyeong (宜敬, 1874–?) and UiMin (宜敏, 1877–?) passing the examination in 1888. The total number 98 is the largest one among families of a single clan in the Record. JeongHa passed ChwiJae in 1706 and became HoeSa (會士, 從九品), HunDo (訓導, 正九品) and GyoSu (教授, 從六品) in 1706, 1718 and 1720, respectively.

Among JungIn mathematicians in Joseon, there are only three mathematicians who wrote mathematical works, namely MukSaJibSanBeob (默思集算法) by Gyung SeonJing (慶善徵, 1616–?), GuIlJib (九一集) by Hong JeongHa and ChaGeunBang-MongGu, SanSulGwanGyeon (算術管見, 1855), IkSan (翼算, 1868) by Lee SangHyeok (李尙爀, 1810–?) [5]. They are all related by marriages [14].

As is well known, JeongHa and his contemporary Yu SuSeok (劉壽錫) were chosen to discuss mathematics with the Qing delegates to Joseon, A Zhaitu (阿齊圖) and He Guozhu (何國柱) in 1713 when JeongHa served as the lowest official HoeSa in HoJo. The reason for the meeting with the delegates, was that JeongHa had already finished his book GuIlJib (九一集) before 1713. One can easily gather this fact through problems appeared in the discussion.

There are two versions of GuIlJib consisting of 9 books. The first one was republished in 1868 by Hong YeongSeok (洪永錫, 1814–?), JeongHa's great great great grandson. YeongSeok indicated the author's name with his correction as "五世孫男永錫校字" [4]. Nam ByeongGil (南秉吉, 1820–1869) wrote a preface and Lee SangHyeok a postscript in the book. Nam served as a minister and the head of GwanSangGam. Lee passed the national examination for astronomy in 1831 and ChwiJae in 1832, and then served later as an official in GwanSangGam. Nam and Lee became a very rare case of collaborators for their research on mathematics and astronomy in the history of Joseon mathematics. Further, the first father in law of SangHyeok's father, Lee ByeongCheol (李秉喆, 1782–?) is a nephew of JeongHa. In the postscript, SangHyeok said that his father had really appreciated GuIlJib and he recommended the book to Nam for the study of Song–Yuan mathematics and Qing mathematics. Nam also appreciated the book and encouraged the republication. In the preface, Nam was also astonished JeongHa's mathematics merely based on Suanxue Qimeng, Suanfa Tongzong and Yigu Yanduan (益古演段). Clearly Yigu Yanduan was mistakenly quoted instead of YangHui Suanfa. Nam and Lee both lamented that the book had not been widely available for more than two centuries and included that Hong JeongHa's mathematics is much more advanced than He Guozhu's mathematics.

The second version is without any preface and postscript [5]. We don't have any information which one is the transcription of the original book. As mentioned above, the book divides into 9 books. The two versions are exactly same in the book 1–8 which form a main body and include the preliminary remark (凡例) and miscellaneous records (雜錄). The records consist of a basic theory on astronomy and Chinese scales, Lulu (律呂) and a record of the mathematical discussion with the Qing delegates. The miscellaneous records of the second version were arranged in order of the above. In the first version, the book 9 begins with the record of discussion and Lee's postscript, the preliminary remark and the basic theory. Since YeongSeok is a direct descendant of JeongHa, it would be much difficult for him to change the contents. It is unusual that the preliminary remark begins with 3–7 Chafen (三七差分) and then the glossary and Zia Xian (賈憲) triangles which we will discuss later in detail. Thus it seems probable that the compiler of the second version may change the order of presentation.

2 Hong JeongHa's mathematical structures

As mentioned above, Hong JeongHa's references for GuIlJib are Zhu Shijie's Suanxue Qimeng, Yang Hui's YangHui Suanfa and Cheng Dawei's Suanfa Tongzong.

Before the publication of YangHui Suanfa, Yang Hui had already published Xiangjie Jiuzhang Suanfa (詳解九章算法, 1261). Yang Hui had revealed his structural approaches to mathematics in the book and its appendix Zuanlei (纂類) [13]. The first four chapters, Fangtian (方田), Xumi (粟米), Cuifen (衰分) and Shaoguang (少廣) were unfortunately lost in Xiangjie. The most important development of Song–Yuan mathematics in the history of Chinese mathematics is the theory of equations, the Tianyuanshu (天元術) and Zengcheng Kaifangfa (增乘開方法), and Yang Hui wrote the Xiangjie to include the era's mathematical development in the setting of mathematical structures, in particular those by Jia Xian and Liu Yi (劉益) as Guo Shuchun (郭書春) claimed [2, 3]. Thus Yang Hui compiled YangHui Suanfa with supplementary subjects to Xiangjie Jiuzhang Suanfa and hence became not that structural book as Xiangjie. We must note that Yang Hui's works contain polynomials represented by Tianyuanshu but he didn't include the process to get equations by Tianyuanshu. On the other hand, Yang included the procedures of the transition from Shisuo Kaifangfa (釋鎖開方法) to Zengcheng Kaifangfa for quadratic equations which gave the readers a complete picture for solving equations. Yang Hui's works were well transmitted in Jiuzhang Suanfa Bilei Daquan (九章算法比類大全, 1450) written by Wu Jing (吳敬). Although Cheng Dawei followed Wu's Daquan, he did not retain Yang's structural approaches as Wu did.

We have shown that Zhu Shijie emphasized decidedly mathematical structures in his Suanxue Qimeng [13] and that his theory of Division Algorithm substantiates his structural approaches to mathematics [12]. Further, the constructing process of equations by Tianyuanshu was the main feature of Qimeng but the method of solving equations was given by extractions of square and cube roots via Zengcheng Kaifangfa. Since their solutions are all double digit numbers, Zhu didn't give the cases of Zengcheng Kaifangfa for general equations with multi-digit numbers in Qimeng but he mentioned the cases of Fanfa (飜法) [8].

In the following, we will show that Hong JeongHa noticed the differences between the three references and built up his own structural approaches to mathematics.

We first discuss mathematical structures in the main body book 1–8. Book 1 deals with basic operations and their applications in the five sections JongHoengSeungJeMun (縱橫乘除門), ISeungDongJeMun (異乘同除門), JeonMuHyeongDanMun (田畝形段門), JeolByeonHoChaMun (折變互差門), and SangGongSuChukMun (商功修築門) and book 2 with GwiCheonChaBunMun (貴賤差分門), ChaDeungGyunBaeMun (差等均配門) and GwiCheonBanYulMun (貴賤反率門). The presentation is rather hasty and mixed for the author took these sections as easy topics. Although he used the titles of sections from those in the first two books of Suanxue Qimeng, his problems were chosen freely in various sections in Qimeng. In the first section, Hong included the problem

$$2^{30} = (2^5 \times 2^5) \times 2^{10} \times 2^{10} = ((2^5)^2)^2 \times (2^5)^2$$

where he used the exponential law and associative law.

In the first section GwiCheonChaBunMun of book 2, Hong obtained the characterization of least common multiples in problem 7–9 [10].

For the characterization, let d, l denote the greatest common divisor and least common multiple of natural numbers $a_1, a_2, \ldots, a_n$. He then observed that for $n = 2$, $dl = a_1 a_2$, $l = a_1 \frac{a_2}{d} = \frac{a_1}{d} a_2$ and $\frac{a_1}{d}, \frac{a_2}{d}$ are relatively prime.

Extending these, he has the following:

1) $\frac{a_1}{d}, \frac{a_2}{d}, \ldots, \frac{a_n}{d}$ are relatively prime.

2) There are relatively prime $c_1, c_2, \ldots, c_n$ with $l = a_i c_i (1 \leq i \leq n)$ and vice versa.

The process to reach the conclusion is as follows. The first problem deals with the case of $n = 2$ and the next one with the case of $n = 3$ where $\frac{a_i}{d} \times \frac{a_j}{d} (i \neq j)$ are relatively prime. Finally for the case of $n = 4$ where $\frac{a_i}{d} \times \frac{a_j}{d} \times \frac{a_k}{d}$ (i, j, k are distinct) are not relatively prime, Hong used another process to get relatively prime c_i. Here one can find Hong's inductive structural approach to the least common multiples.

The book 3 consists of JiBun JeDongMun (之分齊同門), MulBuJiChongMun (物不知總門) and YeongBuJokSulMun (盈不足術門). Although Hong JeongHa put JiBun JeDongMun this late as in Suanxue Qimeng, he did not fully understand Zhu's reason that Zhu presented mathematical results on the domain $\mathbb{Z}$ of integers and then introduced the fractions. Hong had freely used the structures of the field $\mathbb{Q}$ of rational numbers in the previous books 1–2 for he assumed that the readers were already familiar with operations. Thus he dealt very briefly with the theory of fractions. The second section concerned with systems of linear congruences and then included linear equations with conditions about totals (總). It is perhaps too early for Hong JeongHa to figure out the structure of Chinese Remainder Theorem.

The book 4 includes BangJeongJeongBuMun (方程正負門), GuCheokHaeEunMun (毬隻解隱門), BuByeongToeTaMun (缶甁堆垜門) and ChangDonJeokSokMun (倉囤積粟門). In the first section dealing with systems of linear equations, Hong JeongHa did not follow the presentations in Suanxue Qimeng but those in Suanfa Tongzong. He first presented systems of linear equations with two unknowns which are clearly the simplest ones and then extend to those with 3 unknowns. In these cases, systems were represented by matrices and solved by the eliminations. Mostly, the order of matrices in the process of eliminations was retained. But Cheng Dawei reduced the order once one unknown is eliminated. Hong JeongHa chose Cheng's method. This shows that Hong JeongHa didn't follow casually the others' methods but chose one according to mathematical structures. Further, the following example shows that he used his own result for a system.

今有絹一尺 紗二尺 羅三尺 綾四尺 共價八兩七錢二分 只云綾四尺羅五尺紗六尺
絹七尺之價適等 問四色價各若干

The problem says that there are 4 kinds of silk whose prices per a chi (尺) are x_1, x_2, x_3, x_4 with the conditions $x_1 + 2x_2 + 3x_3 + 4x_4 = 8.72$ and $4x_1 = 5x_2 = 6x_3 = 7x_4$. To solve the system, Hong noticed that $4, 5, 6, 7$ are relatively prime and hence the same number is precisely the least common multiple of x_1, x_2, x_3, x_4, say l. Then $x_i = \frac{l}{i+3}$ and then using the first condition, he solved the system.

The final two problems in the section, he chose the most important two problems in the section Fangcheng ZhengFuMen of Qimeng where the first example of Tianyuanshu was introduced. Indeed, the system has two equations with three unknowns. Hong changed the sides of a right triangle into monies so that they are indeterminate systems, but he put "此句股法" in his solution and hence assumed that monies satisfy the Pythagorean theorem.

The other three sections deal with volumes of spheres and polyhedrons and finite series. In the second section, Hong JeongHa introduced the Tianyuanshu to construct an equation as in the last section, Kaifang Geshumen (開方各術門) of Qimeng and then he used it freely whenever he needed it. We note that unlike Hong, Zhu Shijie did not use the Tianyuanshu for equations on finite series in Qimeng.

In the remaining four books, Hong JeongHa paid his attention exclusively to theory of equations. Indeed, the book 5 deals with solving right triangles in the section GuGoHaeEunMun (句股解隱門) and Liu Hui's Haidao Suanjing (海島算經) in the section MangHaeDoSulMun (望海島術門). The titles of remaining three books are all GaeBangGakSulMun (開方各術門, 上, 中, 下). Book 1–8 has 473 problems. Among them, the section GuGoHaeEunMun contains 78 problems and the book 6–8 166 problems whose total is 244 problems. It should be noted that in GuGoHaeEunMun, Hong studies the theory of equations in the setting of right triangles. The most important feature of these books in GuIlJib is that Hong approaches to problems through mathematical structures. As mentioned above, there were some weaknesses on the structural approaches in the previous book 1–4, but the second half of GuIlJib is completely different from the first half in this aspect. YangHui Suanfa and Suanfa Tongzong are completely devoid of the constructions by the Tianyuanshu so that Hong's reference for the second half is solely Suanxue Qimeng except the method of solving equations in YangHui Suanfa. Yang Hui introduced terminologies, Gougu Mingyi (句股名義) of various sums (和) and differences (較) in his Xiangjie Jiuzhang Suanfa which were retained in Suanfa Tongzong. Hong used the terminologies of only sums as Zhu did in Qimeng.

For the problems on right triangles, the early method to solve them in Jiuzhang Suanshu [1] is based on the identities given by geometric properties. In Jiuzhang, there is only one problem solved by a quadratic equation and Wang Xiaotong (王孝通, ca. 7 C.) first extended it to those of higher degrees in his Jigu Suanjing (緝

古算經). Both cases did not explain how they got the equations. In China, Zhang Dunren (張敦仁) published Jigu Suanjing Xicao (細艸, completed in 1803) in 1813 where Zhang explained how to construct equations by the Tianyuanshu. General quadratic equations in the east Asian literatures were given by conditions with the area of a rectangle and sum or difference of its two sides as in YangHui Suanfa. For the modern readers, we explain quadratic equations by the relation between roots and equations, or their coefficients instead of the traditional ones.

Hong's method to solve right triangles is mainly the Tianyuanshu and the relation between roots and coefficients. Unlike the others, Hong JeongHa recognized the needless differentiations between the base (句) and height (股) in solving right triangles and gave numerous pairs of problems which were made by interchanging the base and height. We quote an example, problem 46 which describes this case and is also solved by the relation between roots and coefficients.

今有句二十四尺 只云股乘弦得二千二百九十五尺 問股弦各若干

The problem is to find b, c under the given conditions $a = 24, bc = 2,295$, where a, b, c denote the base, height and hypotenuse in a right triangle, respectively. Noting that $a^2 = c^2 - b^2 = \alpha$ and $b^2c^2 = \beta$, Hong had a quadratic equation $x^2 - \alpha x - \beta = 0$ which has roots b^2 and $-c^2$. He had b^2 and b is its square root, and then he added the following remark:

若問 股若干 句乘弦若干之法 亦倣此 先得句冪也

It says that if the problem is given by interchanging the base and height, one can find a^2 by the exactly same way as above and then solve the problem. Incidentally, the problem of the latter kind was included as the last problem in Zhang's Jigu Suanjing Xicao. Moreover, the results in book 4–5 explain all the processes of constructing equations in Jigu Suanjing.

We note that the quartic equation in Jigu can be reduced to a quadratic equation for it does not contain any term of odd powers but in problem 47 with the condition $b - a = 21, ac = 1,224$, Hong has the equation $2x^4 + 42x^3 + 441x^2 - 1,498,176 = 0$ for a. Furthermore, one has to use $c = \dfrac{1,224}{a}$ in the process to have the equation and hence Hong may have some idea on Li Ye (李冶)'s Tianyuanshu which is a method to represent rational polynomials $\sum_{k=-m}^{n} a_k x^k (m, n \geq 0)$ [6].

The Tianyuan method is so well known that we don't need to discuss it any more. Hong JeongHa arranged the problems along the order of operations, sums, differences, multiplications and divisions of the sides of a right triangle. In some cases, he followed this arrangement so rigidly that he included even trivial cases as is shown later.

In the previous books, Hong just mentioned that he had the answer by solving the equation as "開之" after its construction. When a change of signs of coefficient occurs in a process of synthetic divisions of Zengcheng Kaifangfa, it is called Fanfa (飜法) and when the growth of coefficients in their absolute value with the same sign happens, it is called Yiji (益積). Hong included Fanfa Kaizhi (飜法開之) whenever it occurred. We note that Fanfa in Suanxue Qimeng occurs always at nonconstant terms [8]. Hong JeongHa introduced BeonJeokBeob (飜積法) and IkJeokBeob (益積法) which means Fanfa and Yiji occurring at the constant term (實) respectively. BeonJeokBeob is called Fanjishu (飜積術) in YangHui Suanfa and Huangu (換骨) in Shushu Jiuzhang, and IkJeokBeob Toutai (投胎) in Shushu Jiuzhang. Yang Hui introduced the terminology Yiji Kaifangshu in YangHui Suanfa which is simply an equation with a negative linear coefficient. Hong introduced first time Zengcheng Kaifangfa as well as IkJeokBeob in Problem 64, where the equation is $-0.5625x^2 - 18x - 81 = 0$ and the first guess (初商) of the solution is 30. In the first synthetic division, the constant term changes from -81 to -114.75 and Hong called it IkJeokBeob (此益積法). In the next synthetic division, Fanfa occurs and Hong called it Beongam (飜減). Finally, he put "餘皆倣此", i.e., the method of solving equations in the remaining part of the book will be the same as above. The next problem deals with BeonJeokBeob which is more familiar to the author but he put IkJeokBeob before BeonJeokBeob. Presumably Hong thought that he found first a bit unusual situation of IkJeokBeob. We note that Shushu Jiuzhang and Ceyuan HaiJing (1248) were brought into Joseon in the mid 19th century.

The problem is a trivial problem which is included by Hong's rigid arrangement and is to solve a right triangle under the conditions $c-b = 9, \frac{b}{c} = 0.8$. One can easily have b, c by solving a system of linear equations. But he constructed intentionally the equation $0.86x^2 - 18x + 81 = 0$ for c to show that in the first synthetic division, the constant term changes from 81 to -63, called BeonJeokBeob (此飜積法). With these two problems, Hong completed the basic principle of Zengcheng Kaifangfa which is exactly like the modern one [6].

Dealing with the Gougushu, one of the most important subject in the East Asian mathematics, Hong showed his exceptional structural approaches to Gougushu and obtained the most advanced results in the early 18th century East Asia [9].

The book 6 begins with the extraction of square roots to fifth roots. In this case, Hong constructed the equations $x^n - b = 0$ by the Tianyuanshu, solved them by Zengcheng Kaifangfa and then extended these to equations $ax^n - b = 0$ on circles, spheres and a part of a circle bounded by a chord and its arc, called Hushitian (弧矢田). These preliminaries lead to the general quadratic equations of the form $ax^2 + bx + c = 0$ and Hong obtained these equations from rectangles and squares. He divided them into two types, namely DaeJong (帶從) when $ab > 0$, and GamJong

(減從) when $ab < 0$. The terminologies come from the synthetic divisions in Zengcheng Kaifangfa. In this part, he included both BeonJeokBeob and BeonBeob. Combining squares and circles, various squares, and finally various circles, he produced quadratic equations. He also dealt with problems under the conditions of square roots, i.e., a system of equations which include an irrational equation and obtained general equations of higher degrees.

Hong JeongHa extended the above results to the solid figures, various frusta, cones with bases square and circle (方臺, 圓臺, 方錐, 圓錐) with the conditions of up to the 6th roots and obtains equations of degrees up to 8. Combining cubes, spheres and their inscribed cubes or spheres, various cubes, and various spheres, regular polyhedrons of dimensions up to 10, Hong constructed equations of degree up to 10. Here we can also easily find Hong JeongHa's structural approaches from simple figures to their counterpart of complex ones and he built up the theory of equations from simple equations to complex ones.

In The Annals of the Joseon Dynasty (朝鮮王朝實錄) [15], the article with the most important information on Joseon mathematics appears in the one at June 16, 6th year (1460) of King SeJo (世祖). The article says that mathematics in the era King SeJong (世宗) was really advanced but it was so much deteriorated in the 6th year of King SeJo that mathematical officials could not even extract the cube roots. The historian, compiler of the article lamented that it was completely futile to expect them to solve equations of degrees up to 10. More than two and a half centuries later, Hong JeongHa could finally fulfill the historian's wishes.

The topic MyungSeungBangSik (明乘方式) in the preliminary remark (凡例) describes how to get the Zia Xian's triangle, also known as the Pascal triangle. Let a, α be any numbers and n a natural number, then by the process of the successive synthetic divisions in Zengcheng Kaifangfa, Hong has

$$ay^n = b_n(y-\alpha)^n + b_{n-1}(y-\alpha)^{n-1} + \cdots + b_1(y-\alpha) + b_0.$$

Put $y = x + \alpha$, i.e., $x = y - \alpha$ and hence the above identity becomes

$$a(x+\alpha)^n = b_n x^n + b_{n-1}x^{n-1} + \cdots + x + b_0,$$

which is precisely the binomial expansion of $a(x+\alpha)^n$.

We note that Shisuo Kaifangfa depends on expansions of the terms $a_k(y+\alpha)^k$ for the first guess α and that Zengcheng Kaifangfa depends on the divisions by $x - \alpha$ [11]. We recall that in Yongle Dadian (永樂大典, 1408) Yang Hui obtained the binomial expansion of $(x+1)^n$ by the the successive synthetic divisions, which is retained in Jiuzhang Suanfa Bilei Daquan. Cheng Dawei copied Wu Jing's quotation without the commentary on the divisions. Thus Hong did find the method independently and extended Yang Hui's result to general cases. Moreover, Hong JeongHa begins with the following sentence:

置方面 —幾許— 於甲層 —平方則設四層 立方設五層 餘倣此—
置一算 —一段則一算 二段則二算 餘皆倣此— 於最下層

The words, GiHeo (幾許) and YeoGaeBangCha (餘皆倣此) mean that the process is true for any number α and for any a in the above. We have never seen so far the sentences like the above with the universal quantifier in any other East Asian mathematics books. Furthermore, Hong used "甲層" for any α to explain the process of successive synthetic divisions and appended the result of $(x+12)^3$. In other words, "甲層" is a symbol for α. He then included the triangle of $(x-1)^n, (1 \leq n \leq 12)$. This triangle indicates that Hong used the synthetic divisions by $(x-\alpha)$ for a negative α.

It is clear that without understanding the structures of mathematics and Kaifangfa, one can not have the above result. This also implies that Hong may write the above statements after his long study on the theory of equations in book 4–8 and then put the preliminary remark (凡例) after the book 8 as in Hong YeongSeok's version [4]. This gives another mathematical evidence that the compiler of the second version [5] changed the order as discussed before.

In all, Hong JeongHa was able to complete the theory of equations consisting of Tianyuanshu and Zengcheng Kaifangfa through the structural approach.

3 Conclusion

When Hong JeongHa finished the main body, book 1–8 of GuIlJib, he was not yet 30 years old and served as the lowest official in the ministry HoJo (戶曹) after he passed the examination for the post in 1706. Furthermore, he showed in the book that he had already studied Zhu Shijie's Suanxue Qimeng, Yang Hui's YangHui Suanfa and Cheng Dawei's Suanfa Tongzong and that he had built up his own mathematics according to the mathematical structures and logic.

Hong was rather hasty and showed some weakness in structures in the first three books because he took the topics of basic operations and their direct applications for granted and his aim to write the book is the theory of equations in the remaining book 4–8.

In the latter part, we show that Hong JeongHa emphasized the structural approaches and hence he could achieve the most important and advanced mathematics based on the mathematical structures in the history of Joseon mathematics.

References

1. Guo ShuChun tr. comm. *JiuZhang SuanShu YiZhu*, ShangHai GuJi Pub. Co., 2009. 郭書春 譯注, 《九章筭術 譯注》, 上海古籍出版社, 2009.

2. Guo ShuChun ed. *ZhongGuo KeXue JiShu DianJi TongHui* ShuXueJuan, HeNanJiaoYu Pub. Co., 1993. 郭書春 主編, 《中國科學技術典籍通彙》 數學卷 全五卷, 河南教育出版社, 1993.
3. Guo ShuChun, Li Zhaohua, *ZhongGuo KeXue JiShuShi*, Kexue Pub. Co., 2010. 郭書春 主編, 李兆華 副主編, 《中國科學技術史》 科學出版社, 2010.
4. Hong JeongHa, *GuIlJib* (1724), corrected by Hong YeongSeok, 1868, Tohoku University Library. 洪正夏, 《九一集》 (1724), 洪永錫 校字, 1868, 東北大學 圖書館.
5. Kim YongWoon ed. *HanGuk GwaHak GiSulSa JaRyo DaeGye*, SuHakPyeon, YeoGang Pub. Co., 1985. 金容雲 編, 《韓國科學技術史資料大系》 數學編, 驪江出版社, 1985.
6. Hong Sung Sa, Theory of Equations in the history of Chosun Mahtematics, Proceeding Book 2, The HPM Satellite Meeting of ICME-12, 719–731, 2012.
7. Hong Sung Sa, Hong Young Hee, Mathematics Teachers and Professors in Chosun Dynasty, *The Korean Journal for History of Mathematics*, 19(3) (2006), 1–20. 홍성사, 홍영희, 朝鮮의 算學訓導와 算學敎授, 한국수학사학회지, 19(3) (2006). 1–20.
8. Hong Sung Sa, Hong Young Hee, Chang Hye Won, History of Fanji and Yiji, *The Korean Journal for History of Mathematics*, 18(3) (2005), 39–54. 홍성사, 홍영희, 장혜원, 翻積과 益積의 歷史, 한국수학사학회지 18(3) (2005), 39–54.
9. Hong Sung Sa, Hong Young Hee, Kim Chang Il, GouGuShu in the 18th century Chosun, *The Korean Journal for History of Mathematics*, 22(2) (2009), 1–12. 홍성사, 홍영희, 김창일, 18世紀 朝鮮의 句股術, 한국수학사학회지 22(2) (2009), 1–12.
10. Hong Sung Sa, Hong Young Hee, Kim Chang Il, Hong JungHa's Number Theory, *The Korean Journal for History of Mathematics* 24(4) (2011), 1–6. 홍성사, 홍영희, 김창일, 洪正夏의 數論, 한국수학사학회지 24(4) (2011), 1–6.
11. Hong Sung Sa, Hong Young Hee, Kim Young Wook, Liu Yi and Hong JungHa's KaiFangShu, *The Korean Journal for History of Mathematics*, 24(1) (2011), 1–13. 홍성사, 홍영희, 김영욱, 劉益과 洪正夏의 開方術, 한국수학사학회지 24(1) (2011), 1–13.
12. Hong Sung Sa, Hong Young Hee, Lee Seung On, Division algorithm in Suanxue Qimeng, *Journal for History of Mathematics*, 26(5-6) (2013), 323–328.
13. Hong Sung Sa, Hong Young Hee, Lee Seung On, Mathematical Structures and Suanxue Qimeng, *Journal for History of Mathematics*, 26(2-3) (2013), 123–130.
14. Kim Chang Il, Hong Sung Sa, Hong Young Hee, Chosun Mathematician Hong JungHa's Genealogy, *The Korean Journal for History of Mathematics* 23(3) (2010), 1–20. 김창일, 홍성사, 홍영희, 朝鮮 算學者 洪正夏의 系譜, 한국수학사학회지 23(3) (2010), 1–20.
15. *The Annals of the Joseon Dynasty*. 朝鮮王朝實錄. http://sillok.history.go.kr