

HALF LIGHTLIKE SUBMANIFOLDS OF A SEMI-RIEMANNIAN SPACE FORM WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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ABSTRACT. In this paper, we study screen quasi-conformal irrotational half lightlike submanifolds M of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection, whose structure vector field ζ is tangent to M . The main result is a classification theorem for such Einstein half lightlike submanifolds of a Lorentzian space form admitting a semi-symmetric non-metric connection.

1. INTRODUCTION

The theory of lightlike submanifolds is indeed important for both the geometry of submanifolds to mathematics and its applications to physics. The study of such notion was initiated by Duggal and Bejancu [3] and later studied by many authors (see up-to date in [4, 5]). The notion of a semi-symmetric non-metric connection on a Riemannian manifold was introduced by Ageshe and Chafle [1]. Although now we have lightlike version of a large variety of Riemannian submanifolds, the geometry of lightlike submanifolds of semi-Riemannian manifolds admitting semi-symmetric non-metric connections has been few known. Recently Yasar, Cöken and Yücesan [15] and Jin [6, 7] studied lightlike hypersurfaces in a semi-Riemannian manifold admitting a semi-symmetric non-metric connection. Jin [10] and Jin-Lee [11] studied general lightlike submanifolds and half lightlike submanifolds of a semi-Riemannian manifold with a semi-symmetric non-metric connection.

The objective of this paper is to study screen quasi-conformal irrotational half lightlike submanifolds M of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection, whose structure vector field ζ of $\widetilde{M}(c)$ is tangent to M but it does not belongs to $S(TM)$. The reason for this geometric restriction on

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M is due to the fact that such a class admits an integrable screen distribution and a symmetric induced Ricci tensor of M . Our main result is a classification theorem for such Einstein half lightlike submanifolds M of a Lorentzian space form admitting a semi-symmetric non-metric connection.

2. SEMI-SYMMETRIC NON-METRIC CONNECTION

Let $(\widetilde{M}, \widetilde{g})$ be a semi-Riemannian manifold. A connection $\widetilde{\nabla}$ on \widetilde{M} is called a *semi-symmetric non-metric connection* [1] if $\widetilde{\nabla}$ and its torsion tensor \widetilde{T} satisfy

$$(2.1) \quad (\widetilde{\nabla}_X \widetilde{g})(Y, Z) = -\pi(Y)\widetilde{g}(X, Z) - \pi(Z)\widetilde{g}(X, Y),$$

$$(2.2) \quad \widetilde{T}(X, Y) = \pi(Y)X - \pi(X)Y,$$

for any vector fields X, Y and Z on \widetilde{M} , where π is a 1-form associated with a non-vanishing vector field ζ , which is called the *structure vector field*, by

$$\pi(X) = \widetilde{g}(X, \zeta).$$

A submanifold (M, g) of codimension 2 is called *half lightlike submanifold* if the radical distribution $Rad(TM) = TM \cap TM^\perp$ is a vector subbundle of the tangent bundle TM and the normal bundle TM^\perp of M , with rank 1. In this case, there exists complementary non-degenerate distributions $S(TM)$ and $S(TM^\perp)$ of $Rad(TM)$ in TM and TM^\perp respectively, which are called the *screen* and *co-screen distributions* on M respectively, such that

$$(2.3) \quad TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^\perp = Rad(TM) \oplus_{orth} S(TM^\perp),$$

where \oplus_{orth} denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M , by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M and by $(2.3)_i$ the i -th equation of (2.3). We use same notations for any others. Choose $L \in \Gamma(S(TM^\perp))$ as a spacelike unit vector field, without loss of generality, i.e., $\widetilde{g}(L, L) = 1$. We call L the *canonical normal vector field* of M . Consider the orthogonal complementary vector bundle $S(TM)^\perp$ to $S(TM)$ in $T\widetilde{M}$. Certainly $Rad(TM)$ and $S(TM^\perp)$ are vector subbundles of $S(TM)^\perp$. As $S(TM^\perp)$ is non-degenerate, we have

$$S(TM)^\perp = S(TM^\perp) \oplus_{orth} S(TM^\perp)^\perp,$$

where $S(TM^\perp)^\perp$ is the orthogonal complementary to $S(TM^\perp)$ in $S(TM)^\perp$. It is well-known [3] that, for any null section ξ of $Rad(TM)$ on a coordinate neighborhood

$\mathcal{U} \subset M$, there exists a uniquely defined lightlike vector bundle $ltr(TM)$ and a null vector field N of $ltr(TM)$ on \mathcal{U} satisfying

$$\tilde{g}(\xi, N) = 1, \tilde{g}(N, N) = \tilde{g}(N, X) = \tilde{g}(N, L) = 0, \forall X \in \Gamma(S(TM)).$$

We call N , $ltr(TM)$ and $tr(TM) = S(TM^\perp) \oplus_{orth} ltr(TM)$ the *lightlike transversal vector field*, *lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to $S(TM)$ respectively [11]. Then $T\tilde{M}$ is decomposed as

$$(2.4) \quad \begin{aligned} T\tilde{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^\perp). \end{aligned}$$

In the entire discussion of this article, we shall assume that the structure vector field ζ of \tilde{M} to be spacelike unit tangent vector field of M . In the sequel, we take $X, Y, Z, W \in \Gamma(TM)$, unless otherwise specified. Let P be the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.3)₁. Then the local Gauss and Weingarten formulas of M and $S(TM)$ are given respectively by

$$(2.5) \quad \tilde{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L,$$

$$(2.6) \quad \tilde{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L,$$

$$(2.7) \quad \tilde{\nabla}_X L = -A_L X + \phi(X)N;$$

$$(2.8) \quad \nabla_X PY = \nabla_X^* PY + C(X, PY)\xi,$$

$$(2.9) \quad \nabla_X \xi = -A_\xi^* X - \tau(X)\xi,$$

where ∇ and ∇^* are induced linear connections on TM and $S(TM)$ respectively, B and D are called the *local second fundamental forms* of M , C is called the *local second fundamental form* on $S(TM)$. A_N , A_ξ^* and A_L are linear operators on TM , which are called the *shape operators*, and τ , ρ and ϕ are 1-forms on TM . We say that $h(X, Y) = B(X, Y)N + D(X, Y)L$ is the *global second fundamental form tensor* of M . Using (2.1), (2.2) and (2.5), we have

$$(2.10) \quad \begin{aligned} (\nabla_X g)(Y, Z) &= B(X, Y)\eta(Z) + B(X, Z)\eta(Y) \\ &\quad - \pi(Y)g(X, Z) - \pi(Z)g(X, Y), \end{aligned}$$

$$(2.11) \quad T(X, Y) = \pi(Y)X - \pi(X)Y$$

and B and D are symmetric on TM , where T is the torsion tensor with respect to the induced connection ∇ and η is a 1-form on TM such that

$$\eta(X) = \tilde{g}(X, N).$$

From the facts $B(X, Y) = \tilde{g}(\tilde{\nabla}_X Y, \xi)$ and $D(X, Y) = \tilde{g}(\tilde{\nabla}_X Y, L)$, we know that B and D are independent of the choice of $S(TM)$ and satisfy

$$(2.12) \quad B(X, \xi) = 0, \quad D(X, \xi) = -\phi(X).$$

The above three local second fundamental forms M and $S(TM)$ are related to their shape operators by

$$(2.13) \quad g(A_\xi^* X, Y) = B(X, Y), \quad \tilde{g}(A_\xi^* X, N) = 0,$$

$$(2.14) \quad g(A_L X, Y) = D(X, Y) + \phi(X)\eta(Y), \quad \tilde{g}(A_L X, N) = \rho(X),$$

$$(2.15) \quad g(A_N X, PY) = C(X, PY) - fg(X, PY) - \eta(X)\pi(PY), \\ \tilde{g}(A_N X, N) = -f\eta(X),$$

where f is the smooth function given by $f = \pi(N)$. From (2.12) and (2.13), we show that A_ξ^* is $S(TM)$ -valued self-adjoint and satisfies

$$(2.16) \quad A_\xi^* \xi = 0.$$

Denote by \tilde{R} , R and R^* the curvature tensors of the semi-symmetric non-metric connection $\tilde{\nabla}$ of \tilde{M} , the induced connection ∇ on M and the induced connection ∇^* on $S(TM)$. Using the Gauss-Weingarten formulas for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$(2.17) \quad \begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \\ &\quad + D(X, Z)A_L Y - D(Y, Z)A_L X \\ &\quad + \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &\quad + B(Y, Z)[\tau(X) - \pi(X)] - B(X, Z)[\tau(Y) - \pi(Y)] \\ &\quad + D(Y, Z)\phi(X) - D(X, Z)\phi(Y)\}N \\ &\quad + \{(\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) + B(Y, Z)\rho(X) \\ &\quad - B(X, Z)\rho(Y) - D(Y, Z)\pi(X) + D(X, Z)\pi(Y)\}L, \end{aligned}$$

$$(2.18) \quad \begin{aligned} \tilde{R}(X, Y)N &= -\nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X, Y] \\ &\quad + \tau(X)A_N Y - \tau(Y)A_N X + \rho(X)A_L Y - \rho(Y)A_L X \\ &\quad + \{B(Y, A_N X) - B(X, A_N Y) + 2d\tau(X, Y) \\ &\quad + \phi(X)\rho(Y) - \phi(Y)\rho(X)\}N \\ &\quad + \{D(Y, A_N X) - D(X, A_N Y) + 2d\rho(X, Y) \\ &\quad + \rho(X)\tau(Y) - \rho(Y)\tau(X)\}L, \end{aligned}$$

$$\begin{aligned}
 (2.19) \quad \widetilde{R}(X, Y)L &= -\nabla_X(A_L Y) + \nabla_Y(A_L X) + A_L[X, Y] \\
 &+ \phi(X)A_N Y - \phi(Y)A_N X \\
 &+ \{B(Y, A_L X) - B(X, A_L Y) + 2d\phi(X, Y) \\
 &+ \tau(X)\phi(Y) - \tau(Y)\phi(X)\}N \\
 &+ \{D(Y, A_L X) - D(X, A_L Y) + \rho(X)\phi(Y) - \rho(Y)\phi(X)\}L;
 \end{aligned}$$

$$\begin{aligned}
 (2.20) \quad R(X, Y)PZ &= R^*(X, Y)PZ + C(X, PZ)A_\xi^* Y - C(Y, PZ)A_\xi^* X \\
 &+ \{(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\
 &+ C(X, PZ)[\tau(Y) + \pi(Y)] - C(Y, PZ)[\tau(X) + \pi(X)]\}\xi
 \end{aligned}$$

$$\begin{aligned}
 (2.21) \quad R(X, Y)\xi &= -\nabla_X^*(A_\xi^* Y) + \nabla_Y^*(A_\xi^* X) + A_\xi^*[X, Y] \\
 &+ \tau(Y)A_\xi^* X - \tau(X)A_\xi^* Y \\
 &+ \{C(Y, A_\xi^* X) - C(X, A_\xi^* Y) - 2d\tau(X, Y) \\
 &+ \rho(X)\phi(Y) - \rho(Y)\phi(X)\}\xi.
 \end{aligned}$$

A complete simply connected semi-Riemannian manifold \widetilde{M} of constant curvature c is called a *semi-Riemannian space form* and denote it by $\widetilde{M}(c)$. For any $X, Y, Z \in \Gamma(T\widetilde{M})$, the curvature tensor \widetilde{R} of $\widetilde{M}(c)$ is given by

$$(2.22) \quad \widetilde{R}(X, Y)Z = c\{\widetilde{g}(Y, Z)X - \widetilde{g}(X, Z)Y\}.$$

Taking the scalar product with ξ and L to (2.22), we get

$$(2.23) \quad \widetilde{g}(\widetilde{R}(X, Y)Z, \xi) = \widetilde{g}(\widetilde{R}(X, Y)Z, L) = 0, \quad \forall X, Y, Z \in \Gamma(TM).$$

From this results and (2.17), for all $X, Y, Z \in \Gamma(TM)$, we obtain

$$\begin{aligned}
 (2.24) \quad \widetilde{R}(X, Y)Z &= R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \\
 &+ D(X, Z)A_L Y - D(Y, Z)A_L X.
 \end{aligned}$$

3. CHARACTERIZATION THEOREMS

Definition. A half lightlike submanifold M of a semi-Riemannian manifold \widetilde{M} is said to be *irrotational* [12] if $\widetilde{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$.

From (2.5) and (2.12), we show that the above definition is equivalent to the condition: $D(X, \xi) = 0 = \phi(X)$ for all $X \in \Gamma(TM)$.

Lemma 1 ([8, 11]). *Let M be an irrotational half lightlike submanifold of a semi-Riemannian manifold \widetilde{M} admitting a semi-symmetric non-metric connection such*

that the structure vector field ζ of \widetilde{M} is tangent to M . Then ζ is conjugate to any vector field X on M , i.e., ζ satisfies $h(X, \zeta) = 0$.

Note that $h(X, \zeta) = 0$ is equivalent to the following two equations:

$$(3.1) \quad B(X, \zeta) = \pi(A_\xi^* X) = 0, \quad D(X, \zeta) = \pi(A_L X) = 0, \quad \forall X \in \Gamma(TM).$$

Definition. A half lightlike submanifold M of a semi-Riemannian manifold \widetilde{M} admitting a semi-symmetric non-metric connection is called *screen quasi-conformal* [9, 13] if the second fundamental forms B and C satisfy

$$(3.2) \quad C(X, PY) = \varphi B(X, Y) + \eta(X)\pi(PY),$$

where φ is a non-vanishing function on a coordinate neighborhood \mathcal{U} in M .

Due to (2.13) and (2.15), we show that M is screen quasi-conformal if and only if the shape operators A_N and A_ξ^* are related by

$$(3.3) \quad A_N X = \varphi A_\xi^* X - fX.$$

We quote the following results for irrotational screen quasi-conformal half lightlike submanifold due to Jin [9]:

Theorem 3.1. *Let M be an irrotational screen quasi-conformal half lightlike submanifolds M of a semi-Riemannian space form $\widetilde{M}(c)$ admitting a semi-symmetric non-metric connection. If the structure vector field ζ is tangent to M but it does not belong to $S(TM)$, then we have $c = 1$.*

Let \widetilde{Ric} be the Ricci curvature tensor of \widetilde{M} and $R^{(0,2)}$ the induced Ricci type tensor on M given respectively by

$$\begin{aligned} \widetilde{Ric}(X, Y) &= \text{trace}\{Z \rightarrow \widetilde{R}(Z, X)Y\}, \quad \forall X, Y \in \Gamma(T\widetilde{M}), \\ R^{(0,2)}(X, Y) &= \text{trace}\{Z \rightarrow R(Z, X)Y\}, \quad \forall X, Y \in \Gamma(TM). \end{aligned}$$

Consider a quasi-orthonormal frame field $\{\xi; W_a\}$ on M , where $Rad(TM) = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$ and let $E = \{\xi, N, W_a\}$ be the corresponding frame field on \widetilde{M} . Using this quasi-orthonormal frame field, we obtain

$$\begin{aligned} R^{(0,2)}(X, Y) &= \widetilde{Ric}(X, Y) + B(X, Y)tr A_N + D(X, Y)tr A_L \\ &\quad - g(A_N X, A_\xi^* Y) - g(A_L X, A_L Y) + \rho(X)\phi(Y) \\ &\quad - \widetilde{g}(\widetilde{R}(\xi, Y)X, N) - \widetilde{g}(\widetilde{R}(L, X)Y, L), \end{aligned}$$

This shows that $R^{(0,2)}$ is not symmetric. The tensor field $R^{(0,2)}$ is called its *induced Ricci tensor* [4, 5], denoted by Ric , of M if it is symmetric. It is known [11] that

$R^{(0,2)}$ is an induced Ricci tensor of M if and only if the 1-form τ is closed, i.e., $d\tau = 0$, for any coordinate neighborhood $\mathcal{U} \subset M$.

Remark 1. If $R^{(0,2)}$ is symmetric, then there exists a null pair $\{\xi, N\}$ such that the corresponding 1-form τ satisfies $\tau = 0$ [11], which called a *canonical null pair* of M . Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^\sharp = TM/Rad(TM)$ [12]. This implies that all screen distribution are mutually isomorphic. For this reason, in case $d\tau = 0$ we consider only lightlike hypersurfaces M endow with the canonical null pair.

We say that M is an *Einstein manifold* if the Ricci tensor of M satisfies

$$(3.4) \quad Ric = \kappa g.$$

It is well-known that if $\dim M > 2$, then κ is a constant. For $\dim M = 2$, any manifold M is Einstein but κ is not necessarily constant.

In case the ambient manifold \widetilde{M} is a space form $\widetilde{M}(c)$, $R^{(0,2)}$ is given by

$$(3.5) \quad \begin{aligned} R^{(0,2)}(X, Y) &= mcg(X, Y) + B(X, Y)trA_N + D(X, Y)trA_L \\ &\quad - g(A_N X, A_\xi^* Y) - g(A_L X, A_L Y) + \rho(X)\phi(Y). \end{aligned}$$

Taking the scalar product with ξ to (2.17) and using (2.22), we have

$$(3.6) \quad \begin{aligned} &(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &= B(Y, Z)\{\pi(X) - \tau(X)\} - B(X, Z)\{\pi(Y) - \tau(Y)\}. \end{aligned}$$

Definition. A vector field X on \widetilde{M} is said to be *conformal Killing* [8] if

$$\widetilde{\mathcal{L}}_X \widetilde{g} = -2\delta \widetilde{g}$$

for any non-vanishing smooth function δ , where $\widetilde{\mathcal{L}}$ denotes the Lie derivative on \widetilde{M} , that is, for all $Y, Z \in \Gamma(T\widetilde{M})$,

$$(\widetilde{\mathcal{L}}_X \widetilde{g})(Y, Z) = X(\widetilde{g}(Y, Z)) - \widetilde{g}([X, Y], Z) - \widetilde{g}(Y, [X, Z]).$$

In particular, if $\delta = 0$, then X is called a *Killing vector field* on \widetilde{M} .

Theorem 3.2 ([8, 11]). *Let M be a half lightlike submanifold of \widetilde{M} admitting a semi-symmetric non-metric connection. If the canonical normal vector field L is conformal Killing, then L is a Killing vector field.*

Proof. Using (2.1) and (2.2), for any $X, Y, Z \in \Gamma(T\widetilde{M})$, we have

$$(\widetilde{\mathcal{L}}_X \widetilde{g})(Y, Z) = \widetilde{g}(\widetilde{\nabla}_Y X, Z) + \widetilde{g}(Y, \widetilde{\nabla}_Z X) - 2\pi(X)\widetilde{g}(Y, Z).$$

As L is conformal Killing, we have $\tilde{g}(\tilde{\nabla}_X L, Y) = -D(X, Y)$ by (2.9) and (2.16). This implies $(\tilde{\mathcal{L}}_L \tilde{g})(X, Y) = -2D(X, Y)$ for any $X, Y \in \Gamma(TM)$. Thus we have

$$D(X, Y) = \delta g(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

Taking $X = Y = \zeta$ to this and using (3.1)₂, we get $\delta = 0$ and L is Killing. \square

Theorem 3.3 ([11]). *Let M be a half lightlike submanifold of a semi-Riemannian manifold \tilde{M} admitting a semi-symmetric metric connection. Then the following assertions are equivalent:*

- (1) *The screen distribution $S(TM)$ is an integrable distribution.*
- (2) *C is symmetric, i.e., $C(X, Y) = C(Y, X)$ for all $X, Y \in \Gamma(S(TM))$.*
- (3) *The shape operator A_N is self-adjoint with respect to g , i.e.,*

$$g(A_N X, Y) = g(X, A_N Y), \quad \forall X, Y \in \Gamma(S(TM)).$$

Remark 2. Just as in the well-known case of locally product Riemannian or semi-Riemannian manifolds [3, 4, 5, 14], if $S(TM)$ is an integrable distribution, then M is locally a product manifold $\mathcal{C} \times M^*$ where \mathcal{C} is a null curve tangent to $Rad(TM)$ and M^* is a leaf of the integrable distribution $S(TM)$.

Theorem 3.4. *Let M be a screen quasi-conformal irrotational Einstein half lightlike submanifold of a Lorentzian space form $\tilde{M}(c)$ with a semi-symmetric non-metric connection. If ζ is tangent to M but it does not belong to $S(TM)$, the canonical normal vector field is conformal Killing and the mean curvature of M is constant, then M is locally a product manifold $M = \mathcal{C} \times M_1 \times M_2$, where \mathcal{C} is a null curve, M_1 is an Euclidean space and M_2 is a totally umbilical Riemannian space.*

Proof. As L is Killing, we get $D = \phi = 0$ and $g(A_L X, Y) = 0$ for any $X, Y \in \Gamma(TM)$. From (3.3), (3.5) and the fact A_ξ^* is self-adjoint, we show that $R^{(0,2)}$ is a symmetric induced Ricci tensor Ric and $S(TM)$ is an integrable distribution. As $g(A_\xi^* \zeta, X) = B(\zeta, X) = 0$ and $S(TM)$ is non-degenerate, we have

$$(3.7) \quad A_\xi^* \zeta = 0.$$

Using (2.13), (3.3), (3.4) and the fact $c = 1$, from (3.5) we have

$$(3.8) \quad g(A_\xi^* X, A_\xi^* Y) - \alpha g(A_\xi^* X, Y) + \varphi^{-1}(\kappa - m)g(X, Y) = 0,$$

for all $X, Y \in \Gamma(TM)$ due to $c = 1$, where $\alpha = tr A_\xi^* - f m \varphi^{-1}$. Taking $X = Y = \zeta$ to (3.8) and using (3.7), we have $\kappa = m$. Thus (3.8) becomes

$$(3.9) \quad g(A_\xi^* X, A_\xi^* Y) - \alpha g(A_\xi^* X, Y) = 0.$$

As \widetilde{M} is Lorentzian manifold, $S(TM)$ is a Riemannian. Since ξ is an eigenvector field of A_ξ^* corresponding to the eigenvalue 0 due to (2.16) and A_ξ^* is $S(TM)$ -valued real self-adjoint operator, A_ξ^* have m real orthonormal eigenvector fields in $S(TM)$ and is diagonalizable. Consider a frame field of eigenvectors $\{\xi, E_1, \dots, E_m\}$ of A_ξ^* such that $\{E_1, \dots, E_m\}$ is an orthonormal frame field of $S(TM)$ and $A_\xi^* E_i = \lambda_i E_i$. Put $X = Y = E_i$ in (3.9), each eigenvalue λ_i is a solution of the equation

$$x^2 - \alpha x = 0.$$

As this equation has at most two distinct solutions 0 and α , there exists $p \in \{0, 1, \dots, m\}$ such that $\lambda_1 = \dots = \lambda_p = 0$ and $\lambda_{p+1} = \dots = \lambda_m = \alpha (\neq 0)$, by renumbering if necessary. As $\text{tr} A_\xi^* = 0p + (m-p)\alpha$, we have

$$(m-p-1)\alpha = fm\varphi^{-1}.$$

Consider four distributions D_o, D_α, D_o^s and D_α^s on $S(TM)$ given by

$$\begin{aligned} D_o &= \{X \in \Gamma(TM) \mid A_\xi^* X = 0\}, & D_o^s &= D_o \cap S(TM), \\ D_\alpha &= \{U \in \Gamma(TM) \mid A_\xi^* U = \alpha PU\}, & D_\alpha^s &= D_\alpha \cap S(TM). \end{aligned}$$

Clearly we show that $D_o \cap D_\alpha = \text{Rad}(TM)$, $D_o^s \cap D_\alpha^s = \{0\}$ as $\alpha \neq 0$ and $D_o^s = PD_o$, $D_\alpha^s = D_\alpha$. In the sequel, we take the vector fields $X, Y \in \Gamma(D_o)$, $U, V \in \Gamma(D_\alpha)$ and $Z, W \in \Gamma(TM)$. Denote $X^* = PX$, $Y^* = PY$, $U^* = PU$ and $V^* = PV$. Then $X^*, Y^* \in \Gamma(D_o^s)$ and $U^*, V^* \in \Gamma(D_\alpha^s)$. Since X^* and U^* are eigenvector fields of the real self-adjoint operator A_ξ^* corresponding to the different eigenvalues 0 and α respectively, $X^* \perp U^*$ and $g(X, U) = g(X^*, U^*) = 0$, that is, $D_o \perp_g D_\alpha$. Also, since $B(X, U) = g(A_\xi^* X, U) = 0$, we show that $D_\alpha \perp_B D_o$. Since $\{E_i\}_{1 \leq i \leq p}$ and $\{E_a\}_{p+1 \leq a \leq m}$ are vector fields of D_o^s and D_α^s respectively and D_o^s and D_α^s are mutually orthogonal, D_o^s and D_α^s are non-degenerate distributions of rank p and rank $(m-p)$ respectively. Thus $S(TM)$ is decomposed as $S(TM) = D_\alpha^s \oplus_{\text{orth}} D_o^s$.

From (3.9), we get $A_\xi^*(A_\alpha^* - \alpha P) = 0$. Let $W \in \text{Im} A_\xi^*$. Then there exists $Z \in \Gamma(TM)$ such that $W = A_\xi^* Z$. Then $(A_\xi^* - \alpha P)W = 0$ and $W \in \Gamma(D_\alpha)$. Thus $\text{Im} A_\xi^* \subset \Gamma(D_\alpha)$. By duality, we have $\text{Im}(A_\xi^* - \alpha P) \subset \Gamma(D_o)$.

Applying ∇_X to $B(Y, U) = 0$ and using (2.13) and $A_\xi^* Y = 0$, we obtain

$$(\nabla_X B)(Y, U) = -g(A_\xi^* \nabla_X Y, U).$$

Substituting this into (3.6) and using (2.11) and $A_\xi^* X = A_\xi^* Y = 0$, we get

$$g(A_\xi^*[X, Y], U) = 0.$$

As $\text{Im } A_\xi^* \subset \Gamma(D_\alpha)$ and D_α is non-degenerate, we get $A_\xi^*[X, Y] = 0$. Thus $[X, Y] \in \Gamma(D_o)$ and D_o is integrable. This result implies $[X^*, Y^*] \in \Gamma(D_o)$. On the other hand, since $S(TM)$ is integrable, $[X^*, Y^*] \in \Gamma(S(TM))$. Thus $[X^*, Y^*] \in \Gamma(D_o^s)$. Thus D_o^s is also an integrable distribution.

Applying ∇_V to $B(U, Y) = 0$ and using $A_\xi^*Y = 0$ and $A_\xi^*U = \alpha PU$, we get

$$(\nabla_V B)(U, Y) = -\alpha g(\nabla_V Y, U).$$

Substituting this into (3.6) and using the fact $\alpha \neq 0$, we obtain

$$g(\nabla_V Y, U) = g(V, \nabla_U Y).$$

Applying ∇_V to $g(Y, U) = 0$ and using (2.10), we have

$$\pi(Y)g(U, V) - B(V, U)\eta(Y) - g(\nabla_V Y, U) = g(Y, \nabla_V U).$$

Taking the skew-symmetric part of this equation and using (2.11), we have

$$g([V, U], Y) = 0, \quad \forall Y \in \Gamma(D_o) \text{ and } U, V \in \Gamma(D_\alpha).$$

From this, we get $g([V^*, U^*], Y^*) = 0$ for all $Y^* \in \Gamma(D_o^s)$ and $U^*, V^* \in \Gamma(D_\alpha^s)$. As D_o^s and D_α^s are mutually orthogonal non-degenerate distributions, we show that $[V^*, U^*] \in \Gamma(D_\alpha^s)$. Thus D_α^s is also an integrable distribution.

Applying ∇_U to $B(X, Y) = 0$ and ∇_X to $B(U, Y) = 0$, we have

$$(\nabla_U B)(X, Y) = 0, \quad (\nabla_X B)(U, Y) = -\alpha g(\nabla_X Y, U).$$

Substituting this two equations into (3.6), we have $\alpha g(\nabla_X Y, U) = 0$. As

$$g(A_\xi^* \nabla_X Y, U) = B(\nabla_X Y, U) = \alpha g(\nabla_X Y, U) = 0$$

and $\text{Im } A_\xi^* \subset \Gamma(D_\alpha)$ and D_α is non-degenerate, we get $A_\xi^* \nabla_X Y = 0$. This implies $\nabla_X Y \in \Gamma(D_o)$. Thus D_o is an auto-parallel distribution on $S(TM)$. This implies that $\nabla_{X^*} Y^* \in \Gamma(D_o)$ for any $X^*, Y^* \in \Gamma(D_o^s)$. As $C(X^*, Y^*) = \varphi B(X^*, Y^*) + \eta(X^*)\pi(Y^*) = 0$, we have $\nabla_{X^*} Y^* = \nabla_{X^*}^* Y^* \in \Gamma(S(TM))$. Thus $\nabla_{X^*} Y^* \in \Gamma(D_o^s)$ and D_o^s is also an auto-parallel distribution.

As $A_\xi^* \zeta = 0$, ζ belongs to D_o . Thus $\pi(U) = 0$ for any $U \in \Gamma(D_\alpha)$. Applying ∇_X to $g(U, Y) = 0$ and using (2.10) and the fact D_o is auto-parallel, we get $g(\nabla_X U, Y) = 0$. This implies $\nabla_X U \in \Gamma(D_\alpha)$.

Assume that the mean curvature vector field

$$\mu = \frac{1}{m} g(A_\xi^* E_a, E_a) = \frac{m-p}{m} \alpha$$

of M is constant. Then α is a constant. Applying ∇_X to $B(U, V) = \alpha g(U, V)$ and ∇_U to $B(X, V) = 0$ and using the fact α is constant, we have

$$(\nabla_X B)(U, V) = 0, \quad (\nabla_U B)(X, V) = -\alpha g(\nabla_U X, V).$$

Substituting this two equations into (3.6) and using $D_o \perp_B D_\alpha$, we have

$$g(\nabla_U X, V) = \pi(X)g(U, V).$$

Applying ∇_U to $g(X, V) = 0$ and using (2.10), we obtain

$$g(X, \nabla_U V) = 0.$$

From this, we get $g(X^*, \nabla_{U^*} V^*) = 0$ for all $X^* \in \Gamma(D_o^s)$ and $U^*, V^* \in \Gamma(D_\alpha^s)$. As D_o^s and D_α^s are mutually orthogonal non-degenerate distributions, we show that $\nabla_{U^*} V^* \in \Gamma(D_\alpha^s)$. Thus D_α^s is auto-parallel distribution.

Since the leaf M^* of $S(TM)$ is a Riemannian manifold and $S(TM) = D_\alpha^s \oplus_{orth} D_o^s$, where D_α^s and D_o^s are auto-parallel distributions of M^* , by the decomposition theorem of de Rham [2] we have $M^* = M_1 \times M_2$, where M_1 is a totally geodesic leaf of D_o^s and M_2 is a totally umbilical leaf of D_α^s . Consider the frame field of eigenvectors $\{\xi, E_1, \dots, E_m\}$ of A_ξ^* such that $\{E_i\}_i$ is an orthonormal frame field of $S(TM)$, then $B(E_i, E_j) = C(E_i, E_j) = 0$ for $1 \leq i < j \leq m$ and $B(E_i, E_i) = C(E_i, E_i) = 0$ for $1 \leq i \leq m - 1$. From (2.17) and (2.20), we have $\tilde{g}(\tilde{R}(E_i, E_j)E_j, E_i) = g(R^*(E_i, E_j)E_j, E_i) = 0$. Thus the sectional curvature K of M_2 is given by

$$K(E_i, E_j) = \frac{g(R^*(E_i, E_j)E_j, E_i)}{g(E_i, E_i)g(E_j, E_j) - g^2(E_i, E_j)} = 0.$$

Thus M is a locally product $\mathcal{C} \times M_1 \times M_2$, where \mathcal{C} is a null curve, M_1 is an Euclidean space and M_2 is a totally umbilical Riemannian space. \square

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