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On Almost Pseudo Conharmonically Symmetric Manifolds

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Abstract. The object of the present paper is to study almost pseudo conharmonically symmetric manifolds. Some geometric properties of almost pseudo conharmonically symmetric manifolds have been studied under certain curvature conditions. Finally, we give three examples of almost pseudo conharmonically symmetric manifolds.

1. Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by Cartan [5], who, in particular, obtained a classification of those spaces.
Let $(M^n, g), (n = dimM)$ be a Riemannian manifold, i.e., a manifold M with the Riemannian metric g and let ∇ be the Levi-Civita connection of (M^n, g). A Riemannian manifold is called locally symmetric [5] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [18]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature. During the last six decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent such as conformally symmetric manifolds by Chaki and Gupta [6], recurrent manifolds introduced by Walker [25], conformally recurrent manifolds by Adati and Miyazawa [2], conformally symmetric Ricci-recurrent spaces by Roter [22], pseudo symmetric manifolds introduced by Chaki [7] etc. The notion of weakly symmetric and weakly projective symmetric manifolds were introduced by Tamassy and Binh [24] and later Binh [4] studied decomposable weakly symmetric manifolds. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called weakly symmetricif the curvature tensor R of type (1,3) satisfies the condition:

$$(\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W + B(Y)R(X, Z)W + D(Z)R(Y, X)W$$

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$$+E(W)R(Y,Z)X + g(R(Y,Z)W,X)P,$$

where ∇ denotes the Levi-Civita connection on (M^n, g) and A, B, D, E and P are 1-forms and a vector field, respectively which are non-zero simultaneously. Such a manifold is denoted by $(WS)_n$. Weakly symmetric manifolds have been studied by several authors ([8], [9],[20], [21]) and many others.

A non-flat Riemannian manifold $(M^n, g), (n > 2)$ is said to be a pseudo symmetric manifold [7] if its curvature tensor satisfies the condition

$$(\nabla_X R)(Y,Z)W = 2A(X)R(Y,Z)W + A(Y)R(X,Z)W$$
$$+A(Z)R(Y,X)W + A(W)R(Y,Z)X + g(R(Y,Z)W,X)P, \tag{1.1}$$

where A is a non-zero 1-form, P is a vector field defined by

$$g(X,P) = A(X), \tag{1.2}$$

for all X and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. The 1-form A is called the associated 1-form of the manifold. If $A = 0$, then the manifold reduces to a symmetric manifold in the sense of Cartan. An n-dimensional pseudo symmetric manifold is denoted by $(PS)_n$. This is to be noted that the notion of pseudo symmetric manifold studied in particular by Deszcz [15] is different from that of Chaki [7].

In a recent paper, De and Gazi [10] introduced the notion of almost pseudo symmetric manifolds. A Riemannian manifold$(M^n, g), (n > 2)$ is said to be almost pseudo symmetric if its curvature tensor $\acute{R}$ of type (0,4) satisfies the condition:

$$(\nabla_X \acute{R})(Y,Z,U,V) = [A(X) + B(X)]\acute{R}(Y,Z,U,V) + A(Y)\acute{R}(X,Z,U,V)$$
$$+A(Z)\acute{R}(Y,X,U,V) + A(U)\acute{R}(Y,Z,X,V) + A(V)\acute{R}(Y,Z,U,X), \tag{1.3}$$

where A, B are non-zero 1-forms defined by $g(X,P) = A(X)$, $g(X,Q) = B(X)$, for all vector fields X, ∇ denotes the operator of covariant differentiation with respect to the metric g, $\acute{R}$ is defined by $\acute{R}(X,Y,Z,W) = g(R(X,Y)Z,W))$, where R is the curvature tensor of type (1,3). The 1-forms A and B are called the associated 1-forms of the manifold . Such a manifold is denoted by $(APS)_n$. Here the vector fields P and Q are called the basic vector fields of the manifold corresponding to the associated 1-forms A and B respectively.

If in the above equation $B = A$, then the $(APS)_n$ reduces to a$(PS)_n$. In subsequent papers ([11], [12]) De and Gazi studied almost pseudo conformally symmetric manifolds and conformally flat almost pseudo Ricci symmetric manifolds.

It may be mentioned that $(PS)_n$ is a particular case of an $(APS)_n$, but$(WS)_n$ is not a particular case of an $(APS)_n$.

Let M be a semi-Riemannian manifold of dimension n equipped with two metric g and $\bar{g}$ related by

$$\bar{g}(X,Y) = \sigma^2 g(X,Y), \tag{1.4}$$

where σ is a positive smooth function on M. If a transformation of M does not change the angle between two tangent vectors at a point with respect to g and $\bar{g}$, then such a transformation is said to be a conformal transformation of the metrics on the manifold [3].

It is known that a harmonic function is defined as a function whose Laplacian vanishes. A harmonic function is not invariant, in general. The condition under which a harmonic function remains invariant have been studied by Ishii [17] who introduced the conharmonic trasformation as a subgroup of the conformal transformation group satisfying the condition

$$\sigma^i{}_{,i} + \sigma_{,i}\,\sigma^i, = 0, \tag{1.5}$$

where comma denotes the covariant differentiation with respect to the metric g.
A (0,4) type tensor $\tilde{C}(Y,Z,U,V)$ which remains invariant under conharmonic transformation for an n-dimensional Riemannian manifold M^n is given by

$$\begin{aligned} \tilde{C}(Y,Z,U,V) = \acute{R}(Y,Z,U,V) - \frac{1}{n-2}&[S(Z,U)g(Y,V) - S(Y,U)g(Z,V) \\ &+S(Y,V)g(Z,U) - S(Z,V)g(Y,U)], \end{aligned} \tag{1.6}$$

where $\acute{R}$ denotes the Riemannian curvature tensor of type (0,4) defined by

$$\acute{R}(Y,Z,U,V) = g(R(Y,Z)U,V), \tag{1.7}$$

where R is the Riemannian curvature tensor of type (1,3) and S denotes the Ricci tensor of type (0,2) respectively.

The curvature tensor defined by (1.6) is known as conharmonic curvature tensor. A manifold whose conharmonic curvature vanishes at every point of the manifold is called conharmonically flat manifold. Thus this tensor represents the deviation of the manifold from conharmonic flatness. It satisfies all the symmetric properties of the Riemannian curvature tensor $\acute{R}$. There are many physical applications of the tensor $\tilde{C}$. For example, in [1], Abdussattar and Dwivedi showed that sufficient condition for a space-time to be conharmonic to a flat space-time is that the tensor $\tilde{C}$ vanishes identically. A conharmonically flat space-time is either empty in which case it is flat or filled with a distribution represented by energy momentum tensor T possessing the algebraic structure of an electromagnetic field and conformal to a flat space-time [1]. Also he described the gravitational field due to a distribution of pure radiation in presence of disordered radiation by means of spherically symmetric conharmonically flat space-time. Conharmonic curvature tensor have been studied by Siddiqui and Ahsan [23], *Özgür* [19], and many others. The object of the present paper is to study a type of non-flat Riemannian manifold$(M^n, g), (n > 2)$ whose conharmonic curvature tensor $\tilde{C}$ of type (0,4) satisfies the condition:

$$\begin{aligned} (\nabla_X \tilde{C})(Y,Z,U,V) = {} & [A(X) + B(X)]\tilde{C}(Y,Z,U,V) \\ & +A(Y)\tilde{C}(X,Z,U,V) + A(Z)\tilde{C}(Y,X,U,V) \\ & +A(U)\tilde{C}(Y,Z,X,V) + A(V)\tilde{C}(Y,Z,U,X), \end{aligned} \tag{1.8}$$

where A and B are non-zero 1-forms called associated 1-forms, defined by

$g(X,P) = A(X)$, $g(X,Q) = B(X)$ and P , Q are called the basic vector fields of the manifold corresponding to the associated 1-forms A and B respectively. Such a manifold shall be called an almost pseudo conharmonically symmetric manifold and an n-dimensional manifold of this kind shall be denoted by $(AP\tilde{C}S)_n)$. If $A = B$, then $(AP\tilde{C}S)_n$ reduces to a $(P\tilde{C}S)_n$. In a recent paper De and Mallick [14] studied almost pseudo concircularly symmetric manifolds. In this connection it may be mentioned that De and De [13] studied almost pseudo-conformally symmetric Ricci-recurrent manifolds with application to relativity.
Motivated by the above studies in the present paper we have studied a type of non-flat Riemannian manifold. The paper is organized as follows:
After preliminaries in Section 2, we study $(AP\tilde{C}S)_n(n > 2)$ satisfying Codazzi type of Ricci tensor. Section 4 is devoted to study Einstein $(AP\tilde{C}S)_n$. In Section 5, we study Ricci symmetric $(AP\tilde{C}S)_n(n > 2)$. Finally, non-trivial examples of $(AP\tilde{C}S)_n$ have been constructed.

2. Preliminaries

In this section, some formulas are derived, which will be useful to the study of $(AP\tilde{C}S)_n$. Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \le i \le n$.

Now from (1.6) we have

$$\Sigma_{i=1}^n \tilde{C}(Y,Z,e_i,e_i) = 0 = \Sigma_{i=1}^n \tilde{C}(e_i,e_i,U,V) \tag{2.1}$$

and

$$\begin{aligned}\Sigma_{i=1}^n \tilde{C}(e_i,Z,U,e_i) &= \Sigma_{i=1}^n \tilde{C}(Z,e_i,e_i,U)\\ &= -\frac{r}{(n-2)} g(Z,U),\end{aligned} \tag{2.2}$$

where $r = \Sigma_{i=1}^n S(e_i,e_i)$ is the scalar curvature. Also from (1.6) it follows that

$$\begin{aligned}&\text{(i)}\;\; \tilde{C}(X,Y,Z,U) = -\tilde{C}(Y,X,Z,U),\\ &\text{(ii)}\;\; \tilde{C}(X,Y,Z,U) = -\tilde{C}(X,Y,U,Z),\\ &\text{(iii)}\;\; \tilde{C}(X,Y,Z,U) = \tilde{C}(Z,U,X,Y),\\ &\text{(iv)}\;\; \tilde{C}(X,Y,Z,U) + \tilde{C}(Y,Z,X,U) + \tilde{C}(Z,X,Y,U) = 0.\end{aligned} \tag{2.3}$$

3. An Almost Pseudo Conharmonically Symmetric Manifold of Dimension $n,(n > 2)$ with Codazzi Type of Ricci Tensor

A.Gray [16] introduced two classes of Riemannian manifolds determined by the covariant differentiation of Ricci tensor. The class $\mathcal{A}$ consisting of all Riemannian manifolds whose Ricci tensor S is a Codazzi tensor, that is,

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z).$$

The class $\mathcal{B}$ consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel, that is,

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0$$

Suppose that the manifold under consideration satiafies Codazzi type of Ricci tensor. That is,

$$(\nabla_Y S)(Z,W) = (\nabla_Z S)(Y,W). \tag{3.1}$$

Now from (1.6) we get

$$\begin{aligned}(\nabla_X \tilde{C})(Y,Z,U,V) = (\nabla_X \acute{R})(Y,Z,U,V) - \frac{1}{n-2}[(\nabla_X S)(Z,U)g(Y,V) \\ -(\nabla_X S)(Y,U)g(Z,V) + (\nabla_X S)(Y,V)g(Z,U) \\ -(\nabla_X S)(Z,V)g(Y,U)].\end{aligned} \tag{3.2}$$

Then using (3.2) we get

$$\begin{aligned}&(\nabla_X \tilde{C})(Y,Z,U,V) + (\nabla_Y \tilde{C})(Z,X,U,V) + (\nabla_Z \tilde{C})(X,Y,U,V) \\ &= [(\nabla_X \acute{R})(Y,Z,U,V) + (\nabla_Y \acute{R})(Z,X,U,V) + (\nabla_Z \acute{R})(X,Y,U,V)] \\ &\quad -\frac{1}{n-2}[\{(\nabla_X S)(Z,U)g(Y,V) - (\nabla_X S)(Y,U)g(Z,V) \\ &\quad +(\nabla_X S)(Y,V)g(Z,U) - (\nabla_X S)(Z,V)g(Y,U)\} \\ &\quad +\{(\nabla_Y S)(X,U)g(Z,V) - (\nabla_Y S)(Z,U)g(X,V) \\ &\quad +(\nabla_Y S)(Z,V)g(X,U) - (\nabla_Y S)(X,V)g(Z,U)\} \\ &\quad +\{(\nabla_Z S)(Y,U)g(X,V) - (\nabla_Z S)(X,U)g(Y,V) \\ &\quad +(\nabla_Z S)(X,V)g(Y,U) - (\nabla_Z S)(Y,V)g(X,U)\}].\end{aligned} \tag{3.3}$$

We also have 2nd Bianchi identity for (0,4) Riemannian curvature tensor $\acute{R}$ as follows:

$$(\nabla_X \acute{R})(Y,Z,U,V) + (\nabla_Y \acute{R})(Z,X,U,V) + (\nabla_Z \acute{R})(X,Y,U,V) = 0. \tag{3.4}$$

Using (3.1) and (3.4) in (3.3) we get

$$(\nabla_X \tilde{C})(Y,Z,U,V) + (\nabla_Y \tilde{C})(Z,X,U,V) + (\nabla_Z \tilde{C})(X,Y,U,V) = 0. \tag{3.5}$$

Hence we have the following theorem

Theorem 3.1. *In an almost pseudo conharmonically symmetric manifold of dimension $n(n > 2)$ satisfying Codazzi type of Ricci tensor, the conharmonic curvature tensor satisfies Bianchi's 2nd identity.*

It is well known that the conharmonic curvature tensor satisfies the condition

$$\tilde{C}(X,Y)Z + \tilde{C}(Y,Z)X + \tilde{C}(Z,X)Y = 0, \tag{3.6}$$

and

$$\tilde{C}(X,Y)Z = -\tilde{C}(Y,X)Z. \tag{3.7}$$

But, in general, the conharmonic curvature tensor $\tilde{C}(Y,Z,U,V)$ does not satisfy Bianchi's 2nd identity.

$$(\nabla_X \tilde{C})(Y,Z,U,V) + (\nabla_Y \tilde{C})(Z,X,U,V) + (\nabla_Z \tilde{C})(X,Y,U,V) = 0. \tag{3.8}$$

We suppose that the condition (3.8) holds in the investigated almost pseudo conharmonically symmetric manifold. Now using (1.8), (3.6), (3.7) in (3.8) we get

$$[A(X) - B(X)]\tilde{C}(Y,Z,U,V) + [A(Y) - B(Y)]\tilde{C}(Z,X,U,V) + [A(Z) \\ -B(Z)]\tilde{C}(X,Y,U,V) = 0$$

which implies

$$\alpha(X)\tilde{C}(Y,Z,U,V) + \alpha(Y)\tilde{C}(Z,X,U,V) \\ +\alpha(Z)\tilde{C}(X,Y,U,V) = 0, \tag{3.9}$$

where $\alpha(X) = A(X) - B(X)$ and ρ is a vector field defined by

$$g(X,\rho) = \alpha(X). \tag{3.10}$$

Contracting over Y and V in (3.9) we get

$$\alpha(X)[-\frac{r}{n-2}g(Z,U)] + \alpha(\tilde{C}(Z,X)U) + \alpha(Z)[\frac{r}{n-2}g(X,U)] = 0$$

or,

$$(\frac{r}{n-2})[-\alpha(X)g(Z,U) + \alpha(Z)g(X,U)] + \alpha(\tilde{C}(Z,X)U) = 0. \tag{3.11}$$

Again contracting over Z and U we get

$$(\frac{r}{n-2})[-n\alpha(X) + \alpha(X)] + \frac{r}{n-2}\alpha(X) = 0$$

or,

$$\frac{r}{n-2}[2\alpha(X) - n\alpha(X)] = 0$$

or,

$$r\alpha(X) = 0. \tag{3.12}$$

Hence either $r = 0$ or $\alpha(X) = 0$. Hence we have the following theorem:

Theorem 3.2. *If in an almost pseudo conharmonically symmetric manifold with non-zero scalar curvature, the conharmonic curvature tensor $\tilde{C}$ satisfies Bianchi's 2nd identity, then the manifold is a pseudo conharmonically symmetric manifold.*

From Theorem 3.1 and Theorem 3.2 we can state the following:

Corollary 3.1. An almost pseudo conharmonically symmetric manifold with non-zero scalar curvature satisfying Codazzi type of Ricci tensor is a pseudo conharmonically symmetric manifold.

4. Einstein $(AP\tilde{C}S)_n (n > 2)$

This section deals with an $(AP\tilde{C}S)_n$ defined by (1.8) which is Einstein manifold. If an $(AP\tilde{C}S)_n$ is Einstein, then

$$S(X, Y) = \frac{r}{n} g(X, Y), \tag{4.1}$$

where r is a constant.
Therefore,

$$(\nabla_Z S)(X, Y) = 0 \tag{4.2}$$

and

$$dr = 0. \tag{4.3}$$

Using (4.1) and (4.2) we get from (1.6)

$$(\nabla_X \tilde{C})(Y, Z, U, V) = (\nabla_X \acute{R})(Y, Z, U, V). \tag{4.4}$$

which implies

$$\begin{aligned} [A(X) + B(X)]\tilde{C}(Y, Z, U, V) + A(Y)\tilde{C}(X, Z, U, V) \\ + A(Z)\tilde{C}(Y, X, U, V) + A(U)\tilde{C}(Y, Z, X, V) \\ + A(V)\tilde{C}(Y, Z, U, X) = [A(X) + B(X)]\acute{R}(Y, Z, U, V) \\ A(Y)\acute{R}(X, Z, U, V) + A(Z)\acute{R}(Y, X, U, V) \\ + A(U)\acute{R}(Y, Z, X, V) + A(V)\acute{R}(Y, Z, U, X). \end{aligned} \tag{4.5}$$

Again using (4.1) in (1.6) we get

$$\begin{aligned} \tilde{C}(Y, Z, U, V) = \acute{R}(Y, Z, U, V) - \frac{2r}{n(n-2)}[g(Z, U)g(Y, V) \\ - g(Z, V)g(Y, U)]. \end{aligned} \tag{4.6}$$

Using (4.6) in (4.5) we get

$$\frac{2r}{n(n-2)}[\{A(X)+B(X)\}\{g(Z,U)g(Y,V)-g(Y,U)g(Z,V)\} \\ +A(Y)\{g(Z,U)g(X,V)-g(X,U)g(Z,V)\} \\ +A(Z)\{g(X,U)g(Y,V)-g(Y,U)g(X,V)\} \\ +A(U)\{g(Z,X)g(Y,V)-g(Y,X)g(Z,V)\} \\ +A(V)\{g(Z,U)g(Y,X)-g(Y,U)g(Z,X)\}=0,$$

that is,

$$r[\{A(X)+B(X)\}\{g(Z,U)g(Y,V)-g(Y,U)g(Z,V)\} \\ +A(Y)\{g(Z,U)g(X,V)-g(X,U)g(Z,V)\} \\ +A(Z)\{g(X,U)g(Y,V)-g(Y,U)g(X,V)\} \\ +A(U)\{g(Z,X)g(Y,V)-g(Y,X)g(Z,V)\} \\ +A(V)\{g(Z,U)g(Y,X)-g(Y,U)g(Z,X)\}]=0. \tag{4.7}$$

Putting $Y = V = e_i$ in (4.7), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$r[(n-1)\{A(X)+B(X)\}g(Z,U)+A(X)g(Z,U) \\ -A(Z)g(X,U)+(n-1)\{A(Z)g(X,U)+A(U)g(Z,X)\} \\ +A(X)g(Z,U)-A(U)g(Z,X)]=0$$

or,

$$r[\{(n+1)A(X)+(n-1)B(X)\}g(Z,U) \\ +(n-2)\{A(Z)g(X,U)+A(U)g(Z,X)\}]=0. \tag{4.8}$$

Again, putting $Z = U = e_i$ in (4.8) where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$r[n(n+1)A(X)+n(n-1)B(X)+2(n-2)A(X)]=0$$

or,

$$r(n-1)[(n+4)A(X)+nB(X)]=0$$

or,

$$r[(n+4)A(X)+nB(X)]=0. \tag{4.9}$$

Again putting $X = Z = e_i$ in (4.8) ,where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and then taking summation over i , $1 \leq i \leq n$ we get

$$r[(n+1)A(U)+(n-1)B(U)+(n-2)A(U)+n(n-2)A(U)]=0$$

or,

$$r[(n+1)A(U)+B(U)]=0. \tag{4.10}$$

Replacing U by X we get

$$r[(n+1)A(X)+B(X)]=0. \tag{4.11}$$

Again putting $X=U=e_i$ in (4.8) , where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and then taking summation over i, $1 \le i \le n$ we get

$$r[(n+1)A(Z)+(n-1)B(Z)+n(n-2)A(Z)+(n-2)A(Z)]=0$$

or,

$$r[(n+1)A(Z)+B(Z)]=0. \tag{4.12}$$

Replacing Z by X we get

$$r[(n+1)A(X)+B(X)]=0. \tag{4.13}$$

Now adding (4.9),(4.11) and (4.13) we get

$$r[(3n+6)A(X)+(n+2)B(X)]=0$$

or,

$$r(n+2)[3A(X)+B(X)]=0.$$

Therefore , either $r=0$ or $3A(X)+B(X)=0$. Hence we have the following:

Theorem 4.1. *If an Einstein almost pseudo conharmonically symmetric manifold* $(n>2)$ *is an almost pseudo symmetric manifold, then the scalar curvature of the manifold vanishes, provided that* $3A(X)+B(X)\neq 0$.

Again , if in an Einstein almost pseudo conharmonically symmetric manifold $r=0$, then using (1.8) and (4.6) in (4.4) we get

$$\begin{aligned}(\nabla_X \acute{R})(Y,Z,U,V) = [A(X)+B(X)]\acute{R}(Y,Z,U,V)\\ +A(Y)\acute{R}(X,Z,U,V)+A(Z)\acute{R}(Y,X,U,V)\\ +A(U)\acute{R}(Y,Z,X,V)+A(V)\acute{R}(Y,Z,U,X),\end{aligned}$$

which is the defining condition of an almost pseudo symmetric manifold. Hence we can state the following:

Theorem 4.2. *If in an Einstein* $(AP\tilde{C}S)_n(n>2)$ *the scalar curvature vanishes, then it is an almost pseudo symmetric manifold.*

Next we suppose that in an Einstein almost pseudo conharmonically symmetric manifold the vector field ρ defined in (3.10) is parallel. Then we have

$$\nabla_X \rho = 0 \text{ for all X.} \tag{4.14}$$

Making use of Ricci identity we get

$$\acute{R}(X, Y, \rho, U) = 0, \tag{4.15}$$

from which it follows that

$$S(Y, \rho) = 0. \tag{4.16}$$

From (4.16) and (3.10) we get

$$rg(Y, \rho) = 0.$$

This implies $r = 0$ if $\| \rho \|^2 \neq 0$. Again , if $r = 0$ then from (4.4) using (1.8) and (4.6) it follows that the manifold is an almost pseudo symmetric manifold. Thus we can state the following:

Theorem 4.3. *If the vector field defined by $g(X, \rho) = A(X) - B(X)$ is a parallel vector field in an Einstein almost pseudo conharmonically symmetric manifold $(n > 2)$, then it is an almost pseudo symmetric manifold provided $\| \rho \| \neq 0$.*

5. Ricci Symmetric $(AP\tilde{C}S)_n$

A Riemannian manifold is said to be Ricci symmetric if its Ricci tensor S of type (0,2) satisfies the condition

$$\nabla S = 0 \tag{5.1}$$

and

$$dr = 0. \tag{5.2}$$

Then using (5.1) in (3.2) we get

$$(\nabla_X \tilde{C})(Y, Z, U, V) = (\nabla_X \acute{R})(Y, Z, U, V). \tag{5.3}$$

Hence

$$\begin{aligned} &(\nabla_X \tilde{C})(Y, Z, U, V) + (\nabla_Y \tilde{C})(Z, X, U, V) + (\nabla_Z \tilde{C})(X, Y, U, V) \\ &= (\nabla_X \acute{R})(Y, Z, U, V) + (\nabla_Y \acute{R})(Z, X, U, V) + (\nabla_Z \acute{R})(X, Y, U, V). \end{aligned} \tag{5.4}$$

Using (3.4) in (5.4) we get

$$(\nabla_X \tilde{C})(Y, Z, U, V) + (\nabla_Y \tilde{C})(Z, X, U, V) + (\nabla_Z \tilde{C})(X, Y, U, V) = 0.$$

Hence we have the following:

Theorem 5.1. *In a Ricci symmetric almost pseudo conharmonically symmetric manifold of dimension $n(n > 2)$, the conharmonic curvature tensor satisfies the Bianchi's 2nd identity.*

Now contracting (3.2) over X and V we get

$$(div\tilde{C})(Y,Z)U = \frac{(n-3)}{(n-2)}[(\nabla_Y S)(Z,U) - (\nabla_Z S)(Y,U)] - \frac{1}{2(n-2)}[g(Z,U)dr(Y) - g(Y,U)dr(Z)]. \tag{5.5}$$

Again contracting (1.8) over X and V we get

$$(div\tilde{C})(Y,Z)U = 2A(\tilde{C}(Y,Z)U) + B(\tilde{C}(Y,Z)U) - \frac{r}{(n-2)}[A(Y)g(Z,U) - A(Z)g(Y,U)]. \tag{5.6}$$

Using (5.5) in (5.6) we get

$$\begin{aligned} &2A(\tilde{C}(Y,Z)U) + B(\tilde{C}(Y,Z)U) \\ &- \frac{r}{(n-2)}[A(Y)g(Z,U) - A(Z)g(Y,U)] \\ &= (\frac{n-3}{n-2})[(\nabla_Y S)(Z,U) - (\nabla_Z S)(Y,U)] \\ &- \frac{1}{2(n-2)}[g(Z,U)dr(Y) - g(Y,U)dr(Z)]. \end{aligned} \tag{5.7}$$

Using (5.1) and (5.2) in (5.7) we get

$$2A(\tilde{C}(Y,Z)U) + B(\tilde{C}(Y,Z)U) - \frac{r}{(n-2)}[A(Y)g(Z,U) - A(Z)g(Y,U)] = 0. \tag{5.8}$$

Now contracting over Z, U we get

$$-\frac{2r}{(n-2)}A(Y) - \frac{r}{(n-2)}B(Y) - \frac{r}{(n-2)}(n-1)A(Y) = 0.$$

or,

$$-\frac{r}{(n-2)}[(n+1)A(Y) + B(Y)] = 0.$$

Therefore, $B(Y) = -(n+1)A(Y)$, provided $r \neq 0$. Hence we have the following:

Theorem 5.2. *In a Ricci symmetric n-dimensional $(n > 2)$ almost pseudo conharmonically symmetric manifold , the vector fields corresponding to the 1-forms A and B are in opposite direction.*

6. Examples of an $(AP\tilde{C}S)_n$

Example 6.1. Let $(\mathbb{R}^4, g)$ be a 4-dimensional Riemannian manifold endowed with the Riemannian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (1+2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{6.1}$$

where $(i, j = 1, 2, 3, 4)$ and $q = \frac{e^{x^1}}{k^2}$ and k is a non-zero constant.

Here the only non-vanishing components of the Christoffel symbols and the curvature tensors are respectively :

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{q}{1+2q}, \quad \Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = -\frac{q}{1+2q},$$

$$R_{1221} = R_{1331} = R_{1441} = \frac{q}{1+2q}, \quad R_{2332} = R_{2442} = R_{3443} = \frac{q^2}{1+2q}$$

and the components obtained by the symmetry propertes. The non-vanishing components of the Ricci tensors are:

$$R_{11} = \frac{3q}{(1+2q)^2}, \quad R_{22} = R_{33} = R_{44} = \frac{q}{1+2q},$$

and the non-vanishing conharmonic curvature tensors and their covariant derivatives are:

$$\tilde{C}_{1221} = \tilde{C}_{1331} = \tilde{C}_{1441} = \tilde{C}_{2332} = \tilde{C}_{3443} = \tilde{C}_{2442} = -\frac{q(1+q)}{(1+2q)},$$

$$\tilde{C}_{1221,1} = \tilde{C}_{1331,1} = \tilde{C}_{1441,1} = \tilde{C}_{2332,1} = \tilde{C}_{3443,1} = \tilde{C}_{2442,1} = \frac{2q^2(1+q)}{(1+2q)^2}.$$

Let us choose the associated 1-forms A_i and B_i as follows:

$$A_i(x) = \begin{cases} 0 & \text{for i=1} \\ q & \text{otherwise,} \end{cases} \tag{6.2}$$

$$B_i(x) = \begin{cases} -\frac{2q}{1+2q} & \text{for i=1} \\ -q & \text{otherwise,} \end{cases} \tag{6.3}$$

at any point $x \in \mathbb{R}^4$.

Now the equation (1.8) reduces to the equations

$$\tilde{C}_{1221,1} = [A_1 + B_1]\tilde{C}_{1221} + A_1\tilde{C}_{1221} + A_2\tilde{C}_{1121} + A_2\tilde{C}_{1211} + A_1\tilde{C}_{1221}, \tag{6.4}$$

(6.5) $$\tilde{C}_{1331,1} = [A_1 + B_1]\tilde{C}_{1331} + A_1\tilde{C}_{1331} + A_3\tilde{C}_{1131} + A_3\tilde{C}_{1311} + A_1\tilde{C}_{1331},$$

(6.6) $$\tilde{C}_{1441,1} = [A_1 + B_1]\tilde{C}_{1441} + A_1\tilde{C}_{1441} + A_4\tilde{C}_{1141} + A_4\tilde{C}_{1411} + A_1\tilde{C}_{1441},$$

(6.7) $$\tilde{C}_{2332,1} = [A_1 + B_1]\tilde{C}_{2332} + A_2\tilde{C}_{1332} + A_3\tilde{C}_{2132} + A_3\tilde{C}_{2312} + A_2\tilde{C}_{2331},$$

(6.8) $$\tilde{C}_{2442,1} = [A_1 + B_1]\tilde{C}_{2442} + A_2\tilde{C}_{1442} + A_4\tilde{C}_{2142} + A_4\tilde{C}_{2412} + A_2\tilde{C}_{2441},$$

(6.9) $$\tilde{C}_{3443,1} = [A_1 + B_1]\tilde{C}_{3443} + A_3\tilde{C}_{1443} + A_4\tilde{C}_{3143} + A_4\tilde{C}_{3413} + A_3\tilde{C}_{3441}.$$

Using (6.2) and (6.3) we get from (6.4)

$$\begin{aligned}
\text{R.H.S. of (6.4)} &= [A_1 + B_1]\tilde{C}_{1221} + A_1\tilde{C}_{1221} + A_1\tilde{C}_{1221} \\
&= 3A_1\tilde{C}_{1221} + B_1\tilde{C}_{1221} \\
&= 3(0)(-\frac{q(1+q)}{(1+2q)}) + (-\frac{2q}{1+2q})(-\frac{q(1+q)}{(1+2q)}) \\
&= \frac{2q^2(1+q)}{(1+2q)^2} \\
&= \tilde{C}_{1221,1} \\
&= \text{L.H.S. of (6.4).}
\end{aligned}$$

By similar argument it can be shown that (6.5), (6.6), (6.7), (6.8) and (6.9) are all true. So, $\mathbb{R}^4$ is an $(AP\tilde{C}S)_n$ which is neither conharmonically flat nor conharmonically symmetric.

Example 6.2.

In a recent paper [11] De and Gazi proved a theorem as follows:

If $V_n (n \geq 4)$ be a Riemannian space with the metric of the form

(6.10) $$ds^2 = \phi(dx^1)^2 + K_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n,$$

where $[K_{\alpha\beta}]$ is a symmetric and non-singular matrix cosisting of constant and ϕ is a function of $x^1, x^2, ..., x^{n-1}$ and independent of x^n. Here $K_{\alpha\beta}$ considered as Kronecker symbol $\delta_{\alpha\beta}$ and $\phi = (M_{\alpha\beta} + \delta_{\alpha\beta})x^\alpha x^\beta e^{(x^1)^2}$, and $M_{\alpha\beta}$ are constant and satisfy the relations

$$\begin{aligned}
M_{\alpha\beta} &= 0, \quad for \;\; \alpha \neq \beta \\
&\neq 0, \quad for \;\; \alpha = \beta
\end{aligned}$$

and

(6.11) $$\Sigma_{\alpha=2}^{n-1} M_{\alpha\alpha} = 0,$$

then V_n is an almost pseudo conformally symmetric space with zero scalar curvature which is neither conformally flat nor conformally symmetric.
From this theorem we see that the scalar curvature $r = 0$.

Therefore , the conformal curvature tensor C reduces to conharmonic curvature tensor $\tilde{C}$. Thus V_n is an almost pseudo conharmonically symmetric space.

Example 6.3. Let $(\mathbb{R}^4, g)$ be a 4-dimensional Riemannian manifold endowed with the Riemannian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2, \tag{6.12}$$

where $(i, j = 1, 2, 3, 4)$ and x^4 is non-zero and non-constant.

Then the only non-vanishing components of the Christoffel symbols, the curvature tensors and the Ricci tensors are:

$$\Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = -\frac{2}{3}(x^4)^{\frac{1}{3}}, \quad \Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4},$$

$$R_{1221} = R_{1331} = R_{2332} = \frac{4}{9}(x^4)^{\frac{2}{3}}, \quad R_{1441} = R_{2442} = R_{3443} = -\frac{2}{9(x^4)^{\frac{2}{3}}},$$

$$R_{11} = R_{22} = R_{33} = \frac{2}{3(x^4)^{\frac{2}{3}}}, \quad R_{44} = -\frac{2}{3(x^4)^2}.$$

It can be easily shown that the scalar curvature r of the resulting manifold $(\mathbb{R}^4, g)$ is $\frac{4}{3(x^4)^2}$, which is non-vanishing and non-constant. The non-vanishing components of conharmonic curvature tensors and their covariant derivatives are:

$$\tilde{C}_{1221} = \tilde{C}_{1331} = \tilde{C}_{2332} = -\frac{2}{9}(x^4)^{\frac{2}{3}}, \quad \tilde{C}_{1441} = \tilde{C}_{2442} = \tilde{C}_{3443} = -\frac{2}{9}(x^4)^{-\frac{2}{3}},$$

$$\tilde{C}_{1221,4} = \tilde{C}_{1331,4} = \tilde{C}_{2332,4} = -\frac{4}{27(x^4)^{\frac{1}{3}}}, \quad \tilde{C}_{1441,4} = \tilde{C}_{2442,4} = \tilde{C}_{3443,4} = \frac{4}{27(x^4)^{\frac{5}{3}}}.$$

Let us choose the associated 1-forms as follows:

$$A_i(x) = \begin{cases} -\frac{2}{3x^4} & \text{for i=4} \\ 0 & \text{otherwise,} \end{cases} \tag{6.13}$$

$$B_i(x) = \begin{cases} \frac{4}{3x^4} & \text{for i=4} \\ 0 & \text{otherwise,} \end{cases} \tag{6.14}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.8) reduces to the equations

$$\tilde{C}_{1221,4} = [A_4 + B_4]\tilde{C}_{1221} + A_1\tilde{C}_{4221} + A_2\tilde{C}_{1421} + A_2\tilde{C}_{1241} + A_1\tilde{C}_{1224}. \tag{6.15}$$

$$\tilde{C}_{1331,4} = [A_4 + B_4]\tilde{C}_{1331} + A_1\tilde{C}_{4331} + A_3\tilde{C}_{1431} + A_3\tilde{C}_{1341} + A_1\tilde{C}_{1334}. \tag{6.16}$$

(6.17) $$\tilde{C}_{2332,4} = [A_4 + B_4]\tilde{C}_{2332} + A_2\tilde{C}_{4332} + A_3\tilde{C}_{2432} + A_3\tilde{C}_{2342} + A_2\tilde{C}_{2334}.$$

(6.18) $$\tilde{C}_{1441,4} = [A_4 + B_4]\tilde{C}_{1441} + A_1\tilde{C}_{4441} + A_4\tilde{C}_{1441} + A_4\tilde{C}_{1441} + A_1\tilde{C}_{1444}.$$

(6.19) $$\tilde{C}_{2442,4} = [A_4 + B_4]\tilde{C}_{2442} + A_2\tilde{C}_{4442} + A_4\tilde{C}_{2442} + A_4\tilde{C}_{2442} + A_2\tilde{C}_{2444}.$$

(6.20) $$\tilde{C}_{3443,4} = [A_4 + B_4]\tilde{C}_{3443} + A_3\tilde{C}_{4443} + A_4\tilde{C}_{3443} + A_4\tilde{C}_{3443} + A_3\tilde{C}_{3444}.$$

Using (6.13) and (6.14) we get from (6.15)

$$\begin{aligned} \text{R.H.S. of (6.15)} &= [A_4 + B_4]\tilde{C}_{1221} \\ &= [(-\frac{2}{3x^4}) + (\frac{4}{3x^4})][-\frac{2}{9}(x^4)^{\frac{2}{3}}] \\ &= [\frac{2}{3x^4}][-\frac{2}{9}(x^4)^{\frac{2}{3}}] \\ &= -\frac{4}{27}(x^4)^{-\frac{1}{3}} \\ &= \text{L.H.S. of (6.15)}. \end{aligned}$$

By similar arguement it can be shown that (6.16), (6.17), (6.18), (6.19) and (6.20) are all true. So, $\mathbb{R}^4$ is an $(AP\tilde{C}S)_n$ which is neither conharmonically flat nor conharmonically symmetric with non-zero and non-constant scalar curvature $r = \frac{4}{3(x^4)^2}$.

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