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Enumerations of Finite Topologies Associated with a Finite Simple Graph

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Abstract. The number of topologies (non-homeomorphic topologies) on a fixed finite set having n elements are now known up to $n = 18$ ($n = 16$ respectively) but still no complete formula yet. There are one to one correspondences among topologies, preorders and transitive digraphs on a given finite set. In this article, we enumerate topologies and non-homeomorphic topologies whose underlying graph is a given finite simple graph.

1. Introduction

A topology on a finite set, a *finite topology* is often used to demonstrate interesting phenomena and counterexamples to plausible sounding conjectures. Finite topology also plays a key role in the theory of image analysis [4, 5], the structures of molecular [7, 8], geometries of finite sets [10] and digital topology. For any positive integer $k \geq 2$, there exists a positive integer n and a topology T on an n-set such that $|T| = k$ and the minimum such number n has been studied [9]. One research problem on finite topology is to enumerate the number of topologies and non-homeomorphic topologies on a fixed finite set, we denote them by $\bar{\tau}(n)$ and $\hbar(n)$, respectively, where n is the cardinality of the finite set [1]. Due to many different works, these numbers are now known up to $n = 18$ and $n = 16$, respectively, but we do not know complete formulas yet [11].

It is not difficult to relate a topology on a finite set with a preorder on the finite set. It can be briefly explained as follows. Let X be a finite set. A relation R on X is called a *preorder* if it is reflexive and transitive. For a topology $\mathcal{T}$ on X, we define a relation $R(\mathcal{T})$ on X by a rule that $(x, y) \in R(\mathcal{T})$ if and only if every

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open set containing x also contains y, that is, $x \in \overline{\{y\}}^{\mathcal{T}}$, where $\overline{A}^{\mathcal{T}}$ is the closure of A with respect to the topology $\mathcal{T}$. Then $R(\mathcal{T})$ is a preorder on X. We call it *the preorder associated with the topology* $\mathcal{T}$. Conversely, for a preorder R on X, let $\mathcal{T}(R) = \{U \subset X : \forall x \in U, R(x) \subset U\}$, where $R(x) = \{y : (x, y) \in R\}$. Then $\mathcal{T}(R)$ is a topology on X and call it *the topology corresponding to the preorder* R. Notice that $\{R(x) : x \in X\}$ is a base for the topology $\mathcal{T}(R)$.

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. We use $|X|$ for the cardinality of a set X. Ever since the pioneer work of Evans, Harary and Lynn [2], counting such topologies can be done by counting digraphs as follows. For a preorder R on X, let $D(R)$ be the direct graph whose vertex set is X and arc set is $R \setminus \Delta(X) = \{(x, y) : x \neq y \text{ and } (x, y) \in R\}$. For a digraph D, *the underlying graph of* D is the graph whose vertex set is equal to that of $V(D)$ and edge set is $\{\{x, y\} : xy \text{ or } yx \text{ is an arc of } D\}$. Notice that there are $3^{|E(G)|}$ directed graphs (or digraphs) whose underlying graph is a given finite simple graph G. For a topology $\mathcal{T}$ on X, we say *the underlying graph of* $\mathcal{T}$ is that of the digraph corresponding to the preorder associated with $\mathcal{T}$. Notice that the digraph $D(R)$ corresponding to a preorder R is transitive, i.e., for any pair of distinct vertices a and c if ab and bc are arcs of $D(R)$, then ac is also an arc of $D(R)$. Example 2.1 can help to understand these relations if the reader is not familiar with graph theory. For terms in graph theory, we refer to [3].

There have been a few results on finite topology, preorder and digraphs. Evans *et.al* found a relation between labeled topologies on n points and the labeled transitive digraphs with n points [2]. Marijuan found a few useful properties on digraphs and topologies and relations between finite acyclic transitive digraphs and $\mathtt{T}_0$ topologies [6].

In the present article, we mainly focus on graphs and unlabeled(non-homeomorphic) topologies and we enumerate topologies whose underlying graph is a given finite simple graph G, in other word, we enumerate the number of topologies and non-homeomorphic topologies with respect to the underlying graphs instead of the number of vertices of graphs. For a given graph G, let $\mathrm{Top}\,(G)$ be the set of topologies having G as its underlying graph. Similarly, let $\bar{\tau}(G) = |\mathrm{Top}\,(G)|$ and let $\hbar(G)$ be the number of non-homeomorphic topologies whose underlying graph is G. Notice that $\bar{\tau}(G)$ is equal to the number of transitive digraphs whose underlying graph is G.

The outline of this article is as follows. First, we will provide precise definitions and some general formulae in Section 2. In Section 3, we study how $\bar{\tau}(G), \hbar(G)$ are related with graph operations. In Section 4, we find $\bar{\tau}(G), \hbar(G)$ for a few graphs.

2. General Formulae

For a finite set X, let $\mathrm{Top}\,(X)$ be the set of all topologies on X. Two topologies $\mathcal{T}_1$ on a set X and $\mathcal{T}_2$ on a set Y are equivalent if the two topological spaces $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are homeomorphic. For a natural number n, let N_n be the set

$\{1, 2, \ldots, n\}$. For convenience, let $\bar{\tau}(n) = |\mathrm{Top}\,(N_n)|$ and let $\hbar(n)$ be the number of non-homeomorphic topologies on N_n. It is clear that $\bar{\tau}(1) = 1$ and $\hbar(1) = 1$.

Example 2.1. Let $X = \{a, b\}$ and let $\mathcal{T} = \{\emptyset, \{a\}, X\}$. Then $\mathcal{T}$ is a topology on X. The preorder $R(\mathcal{T})$ associated with $\mathcal{T}$ is $\{(a, a), (b, b), (b, a)\}$ and the underlying graph of $\mathcal{T}$ is the complete graph K_2 on two vertices a and b. Let $R = \Delta(X) = \{(a, a), (b, b))\}$. Then R is a preorder on X. The topology $\mathcal{T}(R)$ associated with R is the discrete topology on X and the underlying graph of $\mathcal{T}(R)$ is the null graph $\mathcal{N}_2$ on two vertices a and b.

The following example demonstrates the topologies whose underlying graph is the complete graph K_2 with two vertices $\{a, b\}$.

Example 2.2. Let $X = \{a, b\}$. Then $\mathcal{T}_1 = \{\emptyset, X\}$, $\mathcal{T}_2 = \{\emptyset, \{a\}, X\}$, $\mathcal{T}_3 = \{\emptyset, \{b\}, X\}$ and $\mathcal{T}_4 = \{\emptyset, \{a\}, \{b\}, X\}$ are the list of all topologies on X. So, $\bar{\tau}(2) = 4$. Since $(X, \mathcal{T}_2)$ and $(X, \mathcal{T}_3)$ are homeomorphic, $\hbar(2) = 3$. It is clear that $\mathrm{Top}\,(K_2) = \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$. This implies that $\bar{\tau}(K_2) = 3$ and $\hbar(K_2) = 2$.

Some properties of finite topological spaces can be described by graph theoretical terminologies. The following lemma should be previously known, however, we could not find any exact references. Thus, we provide it in our language.

Lemma 2.3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two finite topological spaces. Then we have

(a) a function $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous if and only if $f : (X, R(\mathcal{T}_X)) \to (Y, R(\mathcal{T}_Y))$ preserves the relation, that is, f is a graph homomorphism between the two digraphs $D(R(\mathcal{T}_X))$ and $D(R(\mathcal{T}_Y))$,

(b) the number of components of the topological space $(X, \mathcal{T}_X)$ is equal to that of the underlying graph of $\mathcal{T}_X$. In particular, $(X, \mathcal{T}_X)$ is connected if and only if the underlying graph of $\mathcal{T}_X$ is connected.

Proof. (a) Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a continuous function. If $(x_1, x_2) \in R(\mathcal{T}_X)$ then $x_1 \in \overline{\{x_2\}}^{\mathcal{T}_X}$ and $f(x_1) \in f(\overline{\{x_2\}}^{\mathcal{T}_X})$. Since f is continuous, $f(\overline{\{x_2\}}^{\mathcal{T}_X}) \subset \overline{\{f(x_2)\}}^{\mathcal{T}_Y}$ and hence $f(x_1) \in \overline{\{f(x_2)\}}^{\mathcal{T}_Y}$. This implies that $(f(x_1), f(x_2)) \in R(\mathcal{T}_Y)$. Conversely, let A be a subset of X and let $a \in A$. If $x \in \overline{\{a\}}^{\mathcal{T}_X}$ then $(x, a) \in R(\mathcal{T}_X)$. Since f is a graph homomorphism, $(f(x), f(a)) \in R(\mathcal{T}_Y)$ and so $f(x) \in \overline{\{f(a)\}}^{\mathcal{T}_Y}$. This implies that $f(\overline{\{a\}}^{\mathcal{T}_X}) \subset \overline{\{f(a)\}}^{\mathcal{T}_Y}$. Since $\overline{A}^{\mathcal{T}_X} = \cup_{a \in A} \overline{\{a\}}^{\mathcal{T}_X}$, we have

$$\begin{aligned} f\left(\overline{A}^{\mathcal{T}_X}\right) &= f\left(\bigcup_{a \in A} \overline{\{a\}}^{\mathcal{T}_X}\right) = \bigcup_{a \in A} f\left(\overline{\{a\}}^{\mathcal{T}_X}\right) \subset \bigcup_{a \in A} \overline{\{f(a)\}}^{\mathcal{T}_Y} \\ &= \overline{\bigcup_{a \in A} \{f(a)\}}^{\mathcal{T}_Y} = \overline{f(A)}^{\mathcal{T}_Y}. \end{aligned}$$

(b) Let C be a component of the topological space $(X, \mathcal{T}_X)$. Since X is finite, every component of $(X, \mathcal{T}_X)$ is closed and open subset and so $C = \cup_{x\in C} R(x)$. For our convenience, let G be the underlying graph of $\mathcal{T}_X$. Suppose that the subgraph $G[C]$ of G induced by C is disconnected. Without loss of generality, we may assume that $G[C]$ is composed of two components H_1 and H_2. Then C is the union of two subsets $\cup_{x\in V(H_1)} R(x)$ and $\cup_{x\in V(H_2)} R(x)$. Since these two sets are open and disjoint, C is disconnected. This is a contradiction. Hence, the subgraph $G[C]$ of G induced by C is connected. Conversely, let H be a component of G. Suppose that $V(H)$ is a disconnected subset of $(X, \mathcal{T}_X)$. Then $V(H)$ is a disjoint union of two open subsets A and B. Since $A = \cup_{x\in A} R(x)$ and $B = \cup_{x\in B} R(x)$, there is no path from any fixed vertex a of A to any fixed vertex b of B in H. This contracts to the hypothesis that H is connected. □

It follows from Lemma 2.3 (a) that $\hbar(G)$ is equal to the number of isomorphism classes of transitive digraphs whose underlying graph is G.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on N_n. If they are the same, then their underlying graphs are also the same. So, two topologies having distinct underlying graphs can not be the same. For a graph G, let $\mathrm{Aut}\,(G)$ be the group of graph automorphisms of G. Now there are $\dfrac{|V(G)|!}{|\mathrm{Aut}\,(G)|}$ graphs on $V(G)$ that are isomorphic to G. The following proposition comes from this observation.

Proposition 2.4. Let n be a natural number. Then we have

$$\bar{\tau}(n) = \sum_{G} \frac{n!}{|\mathrm{Aut}\,(G)|} \bar{\tau}(G) \text{ and } \hbar(n) = \sum_{G} \hbar(G),$$

where G runs over all representatives of isomorphism classes of graphs of n vertices and $\mathrm{Aut}\,(G)$ is the group of all graph automorphisms of G.

In order to complete the computation $\bar{\tau}(n)$ and $\hbar(n)$, we need to compute $\bar{\tau}(G)$ and $\hbar(G)$ for a given graph G.

For a graph G, let $\mathcal{TD}(G)$ be the set of transitive digraphs whose underlying graph is G. Then $\mathrm{Aut}\,(G)$ acts on the set $\mathcal{TD}(G)$ and $\hbar(G) = |\mathcal{TD}(G)/\mathrm{Aut}\,(G)|$ by Lemma 2.3 (a). Now, the following lemma comes from the Burnside lemma.

Lemma 2.5. Let G be a connected graph and let $\mathcal{TD}(G)$ be the set of transitive digraphs whose underlying graph is G. Then

$$\hbar(G) = \frac{1}{|\mathrm{Aut}\,(G)|} \sum_{\sigma \,\in\, \mathrm{Aut}\,(G)} |\mathrm{Fix}_{\,\sigma}|,$$

where $\mathrm{Fix}_{\,\sigma} = \{D \in \mathcal{TD}(G) : \sigma(D) = D\}$.

It is easy to show that every vertex induced subgraph of a transitive digraph is transitive. From this, we can have the following lemma.

Lemma 2.6. For any graph G, $\bar{\tau}(G) = 0$ if and only if there exists a vertex induced subgraph H of G such that $\bar{\tau}(H) = 0$.

A graph G is said to be *triangle free* if G dose not contain any triangles. Let G be a triangle free graph having at most three vertices. Then every vertex of a transitive digraph having G as its underlying graph is a source or a sink. Since every source is adjacent to a sink and vise versa, $\bar{\tau}(G) \neq 0$ if and only if G is bipartite.

A bipartite graph G having vertex bipartition $X_1 \cup X_2$ is said to be *reflexible* if there exists a graph automorphism $f : G \to G$ such that $f(X_1) = X_2$ and $f(X_2) = X_1$. For example, complete bipartite graph $K_{m,n}$ is reflexible if and only if $m = n$. If G is a connected bipartite graph having two vertices, then G must be K_2. It is observed in Example 2.2 that $\bar{\tau}(K_2) = 3$ and $\hbar(K_2) = 2$. Let G be a connected bipartite graph having at least three vertices and let X_1 and X_2 be a vertex bipartition of G. Let D be a transitive digraph whose underlying graph is G. Then every vertex in X_1 is a source and every vertex in X_2 is a sink or vice versa. It implies that the number of transitive digraphs having G as its underlying graph is 2. Moreover, the two digraphs are isomorphic if and only if G is reflexible. We summarize this discussion as follows.

Theorem 2.7. *For a triangle free graph G, $\bar{\tau}(G) \neq 0$ if and only if G is bipartite. Moreover, for a connected bipartite graph G having at least two vertices, we have*

$$\bar{\tau}(G) = \begin{cases} 3 & \text{if } G = K_2, \\ 2 & \text{if } G \neq K_2, \end{cases}$$

$$\hbar(G) = \begin{cases} 1 & \text{if } G \neq K_2 \text{ and } G \text{ is reflexible}, \\ 2 & \text{otherwise}. \end{cases}$$

3. Topologies and Graph Operations

In this section, we will compute the number $\bar{\tau}(G \circledcirc H)$ and $\hbar(G \circledcirc H)$ when $\bar{\tau}(G)$, $\bar{\tau}(H)$, $\hbar(G)$, and $\hbar(H)$ are known, where $\circledcirc$ is either the disjoint union, the Cartesian product or the amalgamation of graphs.

The following lemma gives a computation formula for the graph that can be expressed by a disjoint union of some connected graphs.

Lemma 3.1. For a natural number ℓ, let $G_1, \dots, G_\ell$ be pairwise non-isomorphic ℓ connected graphs and let $n_1, \dots, n_\ell$ be ℓ natural numbers. Let $G = n_1 G_1 \cup n_2 G_2 \cup \cdots \cup n_\ell G_\ell$. Then

$$\bar{\tau}(G) = \prod_{i=1}^{\ell} \bar{\tau}(G_i)^{n_i} \quad \text{and} \quad \hbar(G) = \prod_{i=1}^{\ell} \hbar(n_i G_i) = \prod_{i=1}^{\ell} \binom{\hbar(G_i) + n_i - 1}{n_i},$$

where mH stands for the disjoint union of m copies of H.

Proof. Let H_1 and H_2 be two graphs having disjoint vertex sets. Then $\bar{\tau}(H_1 \cup H_2) = \bar{\tau}(H_1)\bar{\tau}(H_2)$. It implies that $\bar{\tau}(G) = \prod_{i=1}^{\ell} \bar{\tau}(G_i)^{n_i}$. To prove the second statement, we firstly show that $\hbar(nH) = \binom{\hbar(H)+n-1}{n}$. Let us identify a topology $\mathcal{T} \in \mathrm{Top}\,(nH)$ with a finite sequence $(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n)$ of length n with $\mathcal{T}_i \in \mathrm{Top}\,(H)$ for each $i = 1, 2, \ldots, n$. Then two topologies $(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n)$ and $(\mathcal{T}'_1, \mathcal{T}'_2, \ldots, \mathcal{T}'_n)$ are equivalent if and only if there exists a bijection $\sigma : N_n \to N_n$ such that $\mathcal{T}_i$ and $\mathcal{T}'_{\sigma(i)}$ are equivalent for each $i = 1, 2, \ldots, n$. It implies that the number $\hbar(nH)$ is equal to the number of selections with repetitions of n objects chosen from $\hbar(H)$ types of objects, i.e., $\hbar(nH) = \binom{\hbar(H)+n-1}{n}$. For given two non-isomorphic graphs H_1 and H_2, it is not hard to show that $\hbar(H_1 \cup H_2) = \hbar(H_1)\hbar(H_2)$. This completes the proof. $\square$

For two graphs G and H, the Cartesian product $G\square H$ is a graph such that $V(G\square H) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if ($u_1 = u_2$ and $v_1v_2 \in E(H)$) or ($u_1u_2 \in V(G)$ and $v_1 = v_2$). We aim to compute $\bar{\tau}(G\square H)$ and $\hbar(G\square H)$.

Lemma 3.2. For any odd number $n \geq 3$, $\bar{\tau}(K_2\square C_n) = 0$.

Proof. If n is greater than or equal to 5, then it is not hard to show that there is no transitive digraph whose underlying graph is C_n. Since $K_2\square C_n$ has an induced subgraph isomorphic to C_n, $\bar{\tau}(K_2\square C_n) = 0$ by Lemma 2.6.

Let $n = 3$ and let D be a digraph whose underlying graph is $K_2\square C_3$. Let $V(K_2) = \{u_1, u_2\}$ and $V(C_3) = \{v_1, v_2, v_3\}$. One can check that there are $u_i \in V(K_2)$ and $v_j, v_k \in V(C_3)$ such that both $((u_i, v_j), (u_{3-i}, v_j))$ and $((u_{3-i}, v_j), (u_{3-i}, v_k))$ are directed edges in D. Since ((u_i, v_j) and (u_{3-i}, v_k)) are not adjacent in $K_2\square C_3$, D is not transitive. Hence $\bar{\tau}(K_2\square C_3) = 0$. $\square$

Note that for two graphs G and H, $G\square H$ is bipartite if and only if both G and H are bipartite. Furthermore, for bipartite graph $G\square H$, one can check that $G\square H$ is reflexible if and only if either G or H is reflexible.

Theorem 3.3. *For any two connected graphs G and H,*

$$\bar{\tau}(G\square H) = \begin{cases} 0 & \text{if } G \text{ or } H \text{ is not bipartite,} \\ 2 & \text{otherwise,} \end{cases}$$

and

$$\hbar(G\square H) = \begin{cases} 0 & \text{if } G \text{ or } H \text{ is not bipartite,} \\ 1 & \text{if } G \text{ and } H \text{ are bipartite and } G \text{ or } H \text{ is reflexible,} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Assume that G or H is not bipartite. Then $G\Box H$ is not bipartite and $G\Box H$ contains an induced subgraph isomorphic to $K_2\Box C_n$ for some odd n. By Lemmas 2.6 and Lemma 3.2, we have $\bar{\tau}(G\Box H) = 0$.

Suppose G and H are bipartite. Now $G\Box H$ is bipartite and hence $\bar{\tau}(G\Box H) = 2$ by Theorem 2.7. Furthermore, $\hbar(G\Box H)$ is 1 if either G or H is reflexible; 2 otherwise. □

Let v be a cut vertex of a graph G. Then v is a sink or a source in every transitive digraph having G as its underlying graph. Let G and H be two graphs. For two vertices $u \in V(G)$ and $v \in V(H)$, a graph $G *_{u=v} H$ is obtained from G and H by identifying the vertices u and v. We call it *the amalgamation of G and H along the vertices u and v*. We note that u (or v) is a cut vertex of $G *_{u=v} H$. For a graph G and a vertex $u \in V(G)$, let $\bar{\tau}_{si}(G, u)$ ($\bar{\tau}_{so}(G, u)$, respectively) be the number of topologies having G as its underlying graph and v as a sink(source, respectively) in the digraphs corresponding them. Similarly, we define $\hbar_{si}(G, u)$ and $\hbar_{so}(G, u)$. For any transitive digraph D, if we change direction of every directed edge in D, then the resulting digraph is also a transitive digraph. Moreover the correspondence is bijective and hence $\bar{\tau}_{si}(G, u) = \bar{\tau}_{so}(G, u)$ and $\hbar_{si}(G, u) = \hbar_{so}(G, u)$. Note that for any preorder R, the digraph obtained by changing direction of every edge in $D(R)$ corresponds to the topology composed of all closed sets in $\mathcal{T}(R)$.

Now the following theorem comes from the fact that v is a cut vertex of $G*_{u=v}H$.

Theorem 3.4. *Let G and H be two graphs.*

(a) *If v is a cut vertex of G, then v is a sink or a source in every transitive digraph having G as its underlying graph and hence $\bar{\tau}(G) = 2\bar{\tau}_{si}(G, v)$. Moreover, if v is the unique cut vertex, then $\hbar(G) = 2\hbar_{si}(G, v)$.*

(b) *If $u \in V(G)$ and $v \in V(H)$, then $\bar{\tau}(G *_{u=v} H) = 2\bar{\tau}_{si}(G, u)\bar{\tau}_{si}(H, v)$. Moreover, if neither G nor H has cut vertex, then*

$$\hbar(G *_{u=v} H) = \begin{cases} (\hbar_{si}(G, u) + 1)\, \hbar_{si}(G, u) & \text{if } u \simeq_f v, \\ 2\hbar_{si}(G, u)\, \hbar_{si}(H, v) & \text{otherwise,} \end{cases}$$

where $u \simeq_f v$ means that there exists a graph isomorphism $f : G \to H$ such that $f(u) = v$.

From Proposion 2.4 and Lemma 3.1, we can see that the computation of $\bar{\tau}(n)$ and $\hbar(n)$ can be completed if we can compute $\bar{\tau}(G)$ and $\hbar(G)$ for any connected graph with n vertices. In the next section, we compute $\bar{\tau}(G)$ and $\hbar(G)$ for some special classes of connected graphs.

4. Topologies Having a Fixed Underlying Graph

In this section, we compute $\bar{\tau}(G)$ and $\hbar(G)$ when G is a cycle, a wheel or a complete graph.

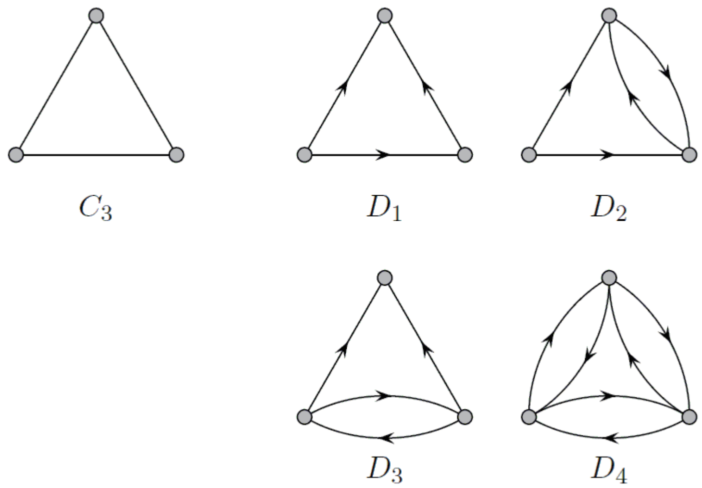

Figure 1: C_3 and four non-isomorphic transitive digraphs whose underlying graph is C_3.

First, we aim to compute $\bar{\tau}(C_n)$ and $\hbar(C_n)$ for a natural number $n \geq 3$. Notice that C_3 is the complete graph K_3 on three vertices. There are four non-isomorphic transitive digraphs whose underlying graph is K_3 as illustrated in Figure 1 and the following is the list of all representatives of them; $D_1 = \{1|2|3\}$, $D_2 = \{1|2,3\}$, $D_3 = \{1,2|3\}$, $D_4 = \{1,2,3\}$, where $D_3 = \{1,2|3\}$ stands for the digraph with vertex set $\{1,2,3\}$ and arc set $\{12, 21, 13, 23\}$. Hence $\hbar(C_3) = 4$. For convenience, let α_i be the number of digraphs that are isomorphic to D_i, Then $\alpha_1 = 3! = 6$, $\alpha_2 = \alpha_3 = \dfrac{3!}{2!} = 3$, $\alpha_4 = \dfrac{3!}{3!} = 1$. Hence, $\bar{\tau}(C_3) = \sum_{i=1}^{4} \alpha_i = 6 + 6 + 1 = 13$. Since there is no transitive digraphs whose underlying graph is C_n when n is odd greater than 3, $\bar{\tau}(C_n) = 0 = \hbar(C_n)$ if n is odd and $n \geq 5$.

Now, the following corollary comes from Lemma 2.6, Theorem 2.7 and the fact that every cycle of even length is reflexible.

Corollary 4.1. For a natural number $n \geq 3$,

$$\bar{\tau}(C_n) = \begin{cases} 13 & \text{if } n = 3, \\ 0 & \text{if } n \text{ is odd and } n \geq 5, \\ 2 & \text{otherwise,} \end{cases}$$

and

$$\hbar(C_n) = \begin{cases} 4 & \text{if } n = 3, \\ 0 & \text{if } n \text{ is odd and } n \geq 5, \\ 1 & \text{otherwise.} \end{cases}$$

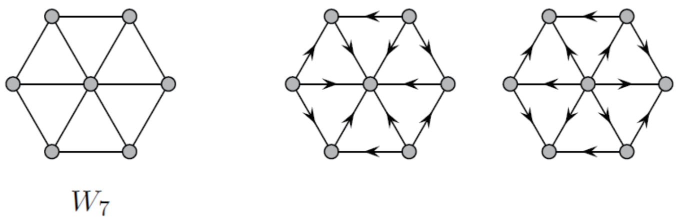

Figure 2: W_7 and two non-isomorphic transitive digraphs whose underlying graph is W_7.

Next, we will compute $\bar{\tau}(G)$ when G is the wheel graph. For a natural number $n \geq 4$, *the wheel* W_n is a graph with n vertices which contains a cycle C_{n-1} as an induced subgraph, and every vertex in the cycle is adjacent to one other vertex. Note that W_4 is the complete graph K_4 on four vertices. A wheel graph W_7 and two non-isomorphic transitive digraphs whose underlying graph is W_7 are depicted in Figure 2. There are eight non-isomorphic transitive digraphs whose underlying graph is K_4 and the following is the list of all representatives of them; $D_1 = \{1|2|3|4\}$, $D_2 = \{1|2|3,4\}$, $D_3 = \{1|2,3|4\}$, $D_4 = \{1,2|3|4\}$, $D_5 = \{1|2,3,4\}$, $D_6 = \{1,2,3|4\}$, $D_7 = \{1,2|3,4\}$, and $D_8 = \{1,2,3,4\}$. For convenience, let β_i be the number of digraphs that are isomorphic to D_i. Then $\beta_1 = 24$, $\beta_2 = \beta_3 = \beta_4 = 12$, $\beta_5 = \beta_6 = 4$, $\beta_7 = 6$, and $\beta_8 = 1$. Hence $\bar{\tau}(K_4) = \sum_{i=1}^{8} \beta_i = 24 + 36 + 8 + 6 + 1 = 75$. Now, the following comes from Lemma 2.6 and a simple computation.

Theorem 4.2. *For a natural number $n \geq 4$, let W_n be the wheel graph. Then we have*

$$\bar{\tau}(W_n) = \begin{cases} 4 & \text{if } n \text{ is odd and } n \geq 7, \\ 0 & \text{if } n \text{ is even and } n \geq 6, \\ 75 & \text{if } n = 4, \\ 8 & \text{if } n = 5, \end{cases}$$

and

$$\hbar(W_n) = \begin{cases} 2 & n \text{ is odd and } n \geq 7, \\ 0 & n \text{ is even and } n \geq 6, \\ 8 & \text{if } n = 4, \\ 4 & \text{if } n = 5. \end{cases}$$

Finally, we will compute $\bar{\tau}(K_n)$ and $\hbar(K_n)$ for the complete graph K_n on n vertices. Let $S(n,k)$ be the number of ways of partitions of N_n into exactly k nonempty parts which is known as the *Stirling number of the second kind* [12].

Theorem 4.3. *For a natural number n, we have*

$$\bar{\tau}(K_n) = \sum_{k=1}^{n} Surj(n,k) = \sum_{k=1}^{n} S(n,k)k! \quad \text{and} \quad \hbar(K_n) = 2^{n-1},$$

where $Surj(n,k)$ is the number of surjections from N_n to N_k.

Proof. Let R be a preorder on N_n whose underlying graph is the complete graph K_n. Then $(i,j) \in R$ or $(j,i) \in R$ for any two distinct elements i and j in N_n. We define another relation $E(R)$ on N_n by (x,y) in $E(R)$ if and only if both (x,y) and (y,x) are in R. Then $E(R)$ is an equivalence relation on N_n. Let $\widetilde{R}$ be a relation on $N_n/E(R)$ defined by $([x],[y]) \in \widetilde{R}$ if and only if $(x,y) \in R$. By the transitivity of R, $\widetilde{R}$ is a well defined total order on $N_n/E(R)$. For convenience, let $N_n/E(R) = \{[i_1],[i_2],\ldots,[i_k]\}$ and $([i_s],[i_t]) \in \widetilde{R}$ if and only if $s \le t$. We define $f_R : N_n \to N_k$ by $f_R(i) = s$ if $(i,i_s) \in E(R)$. Then f_R is a surjection. Conversely, for a given surjection $f : N_n \to N_k$ we define a relation R_f on N_n by $(i,j) \in R_f$ if and only if $f(i) \le f(j)$. Then R_f is a preorder on N_n whose underlying graph is K_n. Since $R_{f_R} = R$ and $f_{R_f} = f$, the correspondence is one-to-one and hence $\bar{\tau}(K_n) = \sum_{k=1}^{n} Surj(n,k)$. Since $Surj(n,k) = S(n,k)k!$, $\bar{\tau}(K_n) = \sum_{k=1}^{n} Surj(n,k) = \sum_{k=1}^{n} S(n,k)k!$.

For a proof of the second equation, let $f : N_n \to N_h$ and $g : N_n \to N_k$ be two surjections. Then $\mathcal{T}(R_f)$ and $\mathcal{T}(R_r)$ are equivalent if and only if $h = k$ and $|f^{-1}(i)| = |g^{-1}(i)|$ for all $i = 1,2,\ldots,h = k$ (by Lemma 2.3). For a k-tuple $(n_1,n_2,\ldots,n_k)$ of natural numbers such that $n = n_1 + n_2 + \cdots + n_k$, we define a surjection $\varphi : N_n \to N_k$ such that $\varphi^{-1}(1) = \{1,2,\cdots,n_1\}$ and

$$\varphi^{-1}(i) = \left\{ \left(\sum_{t=1}^{i-1} n_t\right) + 1, \left(\sum_{t=1}^{i-1} n_t\right) + 2, \ldots, \left(\sum_{t=1}^{i-1} n_t\right) + n_i - 1, \left(\sum_{t=1}^{i} n_t\right) \right\}$$

for each $i = 2,\ldots,k$. Then the topology corresponding to φ is a representative of the equivalence class of topologies corresponding to all surjections f satisfying $|f^{-1}(i)| = n_i$ for all $i = 1,2,\ldots,k$. It is clear that two different k-tuples represent two different equivalence topologies. So, the number of equivalence classes of topologies corresponding to the set of all surjections from $N_n \to N_k$ is equal to the number of ways to choose $k-1$ positions among $n-1$ positions between the n numbers $1,2,\ldots,n$. Hence we have $\hbar(K_n) = \displaystyle\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$. □

Remark 4.4. We already know that $\bar{\tau}(1) = 1$, $\hbar(1) = 1$, $\bar{\tau}(2) = 4$ and $\hbar(2) = 3$. In order to compute $\bar{\tau}(3)$ and $\hbar(3)$, we list all representatives of isomorphism classes of graphs on three vertices as follows; the null graph $\mathcal{N}_3$, H, the path P_2 of length 2, and the complete graph K_3, where H is the disjoint union of K_2 and the null graph $\mathcal{N}_1$. It is not hard to show that $\bar{\tau}(\mathcal{N}_3) = 1$, $\hbar(\mathcal{N}_3) = 1$, $\bar{\tau}(H) = 3$, $\hbar(H) = 2$, $\bar{\tau}(P_2) = 2$, $\hbar(P_2) = 2$, $\bar{\tau}(K_3) = 13$, and $\hbar(K_3) = 4$. Now, it comes from Proposition 2.4 that $\bar{\tau}(3) = 1 + 3 \times 3 + 2 \times 3 + 13 = 29$ and $\hbar(3) = 1 + 2 + 2 + 4 = 9$. Similarly, we can see that $\bar{\tau}(4) = 355$ and $\hbar(4) = 33$.

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