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Special Right Jacobson Radicals for Right Near-rings

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Abstract. In this paper three more right Jacobson-type radicals, $J^r_{g_\nu}$, are introduced for near-rings which generalize the Jacobson radical of rings, $\nu \in \{0, 1, 2\}$. It is proved that $J^r_{g_\nu}$ is a special radical in the class of all near-rings. Unlike the known right Jacobson semisimple near-rings, a $J^r_{g_\nu}$-semisimple near-ring R with DCC on right ideals is a direct sum of minimal right ideals which are right R-groups of type-g_ν, $\nu \in \{0, 1, 2\}$. Moreover, a finite right g_2-primitive near-ring R with eRe a non-ring is a near-ring of matrices over a near-field (which is isomorphic to eRe), where e is a right g_2-primitive idempotent in R.

1. Introduction

Special radicals for near-rings are introduced in [1] by G. L. Booth and N. J. Groenewald using equiprime near-rings. Among the known left Jacobson-type radicals, $J_3, J_{3(0)}$ are the only special radicals in the class of zero-symmetric near-rings and in the class of all near-rings respectively.

Srinivasa Rao and Siva Prasad [6, 7] introduced and studied J^r_ν, the right Jacobson radical type-ν, $\nu \in \{0, 1, 2\}$. In [9, 10] Srinivasa Rao and Siva Prasad along with T. Srinivas showed that J^r_ν is a Kurosh-Amitsur radical in the Fuchs variety $\mathcal{F}$ of all near-rings R in which the constant part R_c of R is an ideal of R, $\nu \in \{0, 1, 2\}$. But J^r_ν is not s-hereditary in the class of all zero-symmetric near-rings and hence it is not an ideal-hereditary radical in that class, $\nu \in \{0, 1, 2\}$.

Also in [5]([11]) Srinivasa Rao and Siva Prasad (along with T. Srinivas) intro-

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duced and studied the right Jacobson type of radical $J^r_{\nu(e)}$, $\nu \in \{1, 2\}$ ($J^r_{0(e)}$) and showed that it is a Kurosh-Amitsur radical in the class of all near-rings and is an ideal hereditary Kurosh-Amitsur radical in the class of all zero-symmetric near-rings. Moreover, they are special radicals in the class of all near-rings.

In this paper we introduce three more right Jacobson radicals, $J^r_{g_\nu}$, $\nu \in \{0, 1, 2\}$. We show that they are special radicals in the class of all near-rings. So, in the class of all near-rings, they are Kurosh-Amitsur radicals, their semisimple classes are hereditary and radicals classes are c-hereditary. Unlike the known right Jacobson semisimple near-rings, a $J^r_{g_\nu}$-semisimple near-ring R with DCC on right ideals is a direct sum of right ideals which are right R-groups of type-g_ν, $\nu \in \{0, 1.2\}$. A finite right g_2-primitive near-ring R with eRe a non-ring is a near-ring of matrices over a near-field (which is isomorphic to eRe), where e is a right g_2-primitive idempotent in R.

Near-rings considered are right near-rings (not necessarily zero-symmetric) and R is a near-ring. Now we present some definitions and results of [6] and [7].

A group $(G, +)$ is called a *right R-group* if there is a mapping $((g, r) \to gr)$ of $G \times R$ into G such that (1) $(g + h)r = gr + hr$, (2) $g(rs) = (gr)s$ for all $g, h \in G$ and $r, s \in R$. A subgroup (normal subgroup) H of a right R-group G is called an *R-subgroup (ideal)* of G if $hr \in H$ for all $h \in H$ and $r \in R$.

Let G be a right R-group. An element $g \in G$ is called a *generator* of G if $gR = G$ and $g(r + s) = gr + gs$ for all $r, s \in R$. G is said to be *monogenic* if G has a generator. G is said to be *simple* if $G \neq \{0\}$ and G, and $\{0\}$ are the only ideals of G.

A monogenic right R-group G is said to be a *right R-group of type-*0 if G is simple.

A right R-group G of type-0 is said to be of *type-*1 if G has exactly two R-subgroups, namely $\{0\}$ and G.

A right R-group G of type-0 is said to be of *type-*2 if $gR = G$ for all $0 \neq g \in G$.

Note that a right R-group of type-2 is of type-1 and a right R-group of type-1 is of type-0.

Let $\nu \in \{0, 1, 2\}$. A right modular right ideal K of R is called *right ν-modular* if R/K is a right R-group of type-ν.

An ideal P of R is called *right ν-primitive* if P is the largest ideal of R contained in a right ν-modular right ideal of R. R is called a *right ν-primitive near-ring* if $\{0\}$ is a right ν-primitive ideal of R.

Now we present some definitions of [11] and [5].

Let G be a right R-group of type-ν, $\nu \in \{0, 1, 2\}$. Suppose that $G0 = \{0\}$ for $\nu = 0$ and P is the largest ideal of R contained in $(0 : G) = \{r \in R \mid Gr = \{0\}\}$. Then G is said to be a *right R-group of type-$\nu(e)$* if $0 \neq g \in G, r_1, r_2 \in R$ and $gxr_1 = gxr_2$ for all $x \in R$ implies $r_1 - r_2 \in P$.

A right modular right ideal K of R is called *right $\nu(e)$-modular* if R/K is a right R-group of type-$\nu(e)$.

Let G be a right R-group of type-$\nu(e)$. Then $(0 :$ G) is an ideal of R and is called a *right $\nu(e)$-primitive ideal* of R.

A near-ring R is called *right $\nu(e)$-primitive* if $\{0\}$ is a right $\nu(e)$-primitive ideal of R.
A near-ring R is called an *equiprime near-ring* [2] if $0 \neq a \in R$, x, y $\in$ R and arx = ary for all r $\in$ R, implies x = y. An ideal I of R is called *equiprime* if R/I is an equiprime near-ring. Moreover, an equiprime near-ring is zero-symmetric.

It is known that a near-ring R is equiprime if and only if ([2])
1. x, y $\in$ R and xRy = $\{0\}$ implies x = 0 or y = 0.
2. If $\{0\} \neq$ I is an invariant subnear-ring of R, x, y $\in$ R and ax = ay for all a $\in$ I implies x = y.

In [1], G. L. Booth and N. J. Groenewald defined special radicals for near-rings. A class $\mathcal{E}$ consisting of equiprime near-rings is called a *special class* if it is hereditary and closed under left invariant essential extensions. If $\mathcal{R}$ is the upper radical in the class of all near-rings determined by a special class of near-rings, then $\mathcal{R}$ is called a special radical. A class of near-rings $\mathcal{E}$ is said to satisfy condition F_l if $J \lhd I \lhd R$ and I is left invariant in R and $I/J \in \mathcal{E}$ implies $J \lhd R$. We need the following theorem:

Theorem 1.1. ([12]) Let $\mathcal{E}$ be a class of zero-symmetric near-rings. If $\mathcal{E}$ is regular, closed under essential left invariant extensions and satisfies condition (F_l), then $\mathcal{R}$:= $\mathcal{U}\mathcal{E}$ is a c-hereditary radical class in the variety of all near-rings, $\mathcal{S}\mathcal{R} = \overline{\mathcal{E}}$ and $\mathcal{S}\mathcal{R}$ is hereditary. So, $\mathcal{R}(R) = \cap \{I \lhd R \mid R/I \in \mathcal{E}\}$ for any near-ring R.

2. Right Jacobson Radicals of Type-g_ν

Let G be a right R-group and T be a subset of G. Then $(0 : T) := \{r \in R \mid tr = 0$ for all $t \in T\}$. By Proposition 3.7 of [11], if G is a right R-group of type-0 and $G0 = \{0\}$, then there is a largest ideal of R contained in $(0 : G)$. Moreover, by Proposition 3.1 of [5], if G is a right R-group of type-ν, then $G0 = \{0\}$, $\nu \in \{1, 2\}$.
Definition 2.1. Let $\nu \in \{0, 1, 2\}$. Let G be right R-group of type-ν and $G0 = \{0\}$ for $\nu = 0$, and T be the set of all generators of the right R-group G. Then G is said to be a right R-group of type-g_ν if $(0 : T) = P$, where P is the largest ideal of R contained in $(0 : G)$.

We present an example of a right R-group of type-g_0 which is not of type-g_1.

Example 2.2. Let $(G, +)$ be a finite non-abelian simple group. Since $\{0\}$ is the maximal normal subgroup of $(G, +)$, $\{0\}$ is the maximal right ideal of $M_0(G)$ and hence $M_0(G)$ is a right $M_0(G)$-group of type-0. This example was considered in [7] and it was shown that $M_0(G)$ is not a right $M_0(G)$-group of type-1. Each $0 \neq h \in G$ give rise to the inner automorphism t_h of G defined by $t_h(x) = h + x - h$ for all $x \in G$. Clearly, a generator of the right $M_0(G)$-group $M_0(G)$ is an automorphism of $(G, +)$. Let T be the set of all automorphisms of G. Suppose that for some $t \in M_0(G)$ and $0 \neq h \in G$, $t_h t = 0$. Now $0 = (t_{-h})t_h t = (t_h)^{-1} t_h t = t$. Therefore $\{0\} = (0 : t_h) = (0 : T)$. Since the largest ideal of $M_0(G)$ contained in $(0 : M_0(G))$ is $\{0\}$, $M_0(G)$ is a right $M_0(G)$-group of type-g_0 but not of type-g_1.

Now we present an example of a right R-group of type-g_1 which is not of type-g_2.

Example 2.3. Let $(G,+)$ be a finite cyclic group of prime order p, where $p \neq 2$. Since $\{0\}$ is the only proper subgroup of G, $\{0\}$ is the only proper right $M_0(G)$-subgroup of $M_0(G)$. Therefore, $M_0(G)$ is a right $M_0(G)$-group of type-1. Clearly, $M_0(G)$ is not a right $M_0(G)$-group of type-2, as $M_0(G)$ is not a near-field. This example was considered in [7]. A a generator of the right $M_0(G)$-group $M_0(G)$ an is automorphism $(G,+)$. We know that G has $p-1$ automorphisms. Let T be the set of all these automorphisms. Suppose that for some $s \in M_0(G)$ and $t \in T$, $ts = 0$. Now $0 = (t^{-1})ts = s$. So $\{0\} = (0 : t) = (0 : T)$. Since the largest ideal of $M_0(G)$ contained in $(0 : M_0(G))$ is $\{0\}$, $M_0(G)$ is a right $M_0(G)$-group of type-g_1 but not of type-g_2.

The following are examples of right R-groups of type-g_2.

Example 2.4. Let R be a near-field. Then R is a right R-group of type-2. Clearly, R is also a right R-group of type-g_2.

Example 2.5. Let $(R,+)$ be a group and let K be a subgroup of $(R,+)$ of index 2. The trivial multiplication on $(R,+)$ determined by $R\backslash K$ is given by $a.b = a$ if $b \in R\backslash K$ and 0 if $b \in K$. Now $(R,+,.)$ is a near-ring. It is clear that K is a maximal (right) ideal of R. Let $a \in R\backslash K$. Now $R = K \cup a+K$. It can be easily verified that $a+K$ is the generator of the right R-group R/K. So R/K is a right R-group of type-2 and $(0 : a+K) = (0 : R/K)$ is the largest ideal of R contained in $(0 : R/K)$. Hence R/K is a right R-group of type-g_2.

Now we introduce some notions related to the right R-groups of type-g_ν.

Definition 2.6. Let $\nu \in \{0,1,2\}$ and K be a right modular right ideal of R. Then K is said to be right g_ν-modular right ideal of R if R/K is a right R-group of type-g_ν.

Definition 2.7. Let $\nu \in \{0,1,2\}$. An ideal P of R is called a right g_ν-primitive ideal of R if P is the largest ideal of R contained in $(0 : G) := \{r \in R \mid Gr = \{0\}\}$ for some right R-group G of type-g_ν.

Definition 2.8. Let $\nu \in \{0,1,2\}$. A near-ring R is called a *right g_ν-primitive near-ring* if $\{0\}$ is a right g_ν-primitive ideal of R.

Definition 2.9. Let $\nu \in \{0,1,2\}$. The intersection of all right g_ν-primitive ideals of R is called the *right Jacobson radical of R of type-g_ν* and is denoted by $J^r_{g_\nu}(R)$. If R has no right g_ν-primitive ideals, then $J^r_{g_\nu}(R)$ is defined to be R.

Note that if R is a ring then $J^r_{g_\nu}(R) = J(R)$, where J is the Jacobson radical of R.
By Proposition 3.1 of [11], for a right R-group G, $G0 = \{0\}$ if and only if $GR_c = \{0\}$. Since for a right R-group G of type-g_ν, $G0 = \{0\}$, R_c is contained in $(0 : g)$ for every generator g of G. So $R_c \subseteq P$ for every right g_ν-primitive ideal P of R. Hence a right g_ν-primitive ideal P of R is invariant. This shows that a right g_ν-primitive near-ring is zero-symmetric.

Proposition 2.10. Let $\nu \in \{0, 1, 2\}$. An ideal P of R is a right g_ν-primitive ideal of R if and only if P is the largest ideal of R contained in a right g_ν-modular right ideal of R.

Proof. Let P be a right g_ν-primitive ideal of R. There is a right R-group G of type-g_ν such that P is the largest ideal of R contained in $(0 : G)$. Let g_0 be a generator of the right R-group G. The mapping $r \to g_0 r$ is a right R-homomorphism of R on to G with kennel $K := (0 : g_0)$. So R/K is right R-isomorphic to G (as right R-groups). Now K is a right g_ν-modular right ideal of R and P is contained in K. Let Q is the largest ideal of R contained in K. Now $GQ = \{0\}$, that is, $Q \subseteq (0 : G)$ as $RQ \subseteq Q$, Q being invariant ideal of R. Since P is the largest ideal of R contained in $(0 : G)$, $Q \subseteq P$. Now $P \subseteq Q$ as Q is the largest ideal of R contained in K. Therefore $P = Q$, that is, P is the largest ideal of R contained in K. On the other hand suppose that P is the largest ideal of R contained in a right g_ν-modular right ideal K of R. Now $G := R/K$ is a right R-group of type-g_ν. We have $(0 : G) = (0 : R/K) = (K : R)$ and $RP \subseteq P$ as P is an invariant ideal of R. So $P \subseteq (K : R)$. Let T be the largest ideal of R contained in $(K : R) = \{r \in R \mid Rr \subseteq K\}$. Since P is an invariant ideal of R, and $P \subseteq T$, T is an invariant ideal of R. So $RT \subseteq T$. Let K be right modular by e. Now $r - er \in K$ for all $r \in R$. We have $t - et \in K$ for all $t \in T$. Since $RT \subseteq T, T \subseteq K$. Since P is the largest ideal of R contained in K, $T \subseteq P$. So $T = P$. Now P is the largest ideal of R contained in $(K : R)$ and hence P is a right g_ν-primitive ideal of R. □

Proposition 2.11. Let $\nu \in \{0, 1, 2\}$. An ideal P of R is a right g_ν-primitive ideal of R if and only if R/P is a right g_ν-primitive near-ring.

Proof. Let $\nu \in \{0, 1, 2\}$ and P be an ideal of R. Suppose that P is a right g_ν-primitive ideal of R. So, we get a right g_ν- modular right ideal M of R such that P is the largest ideal of R contained in M. Now M/P is a right g_ν-modular right ideal of R/P. Since P is the largest ideal of R contained in M, the zero ideal of R/P is the largest ideal of R/P contained in M/P. Therefore, R/P is a right g_ν-primitive near-ring. Suppose now that R/P is a right g_ν-primitive near-ring. So, we get a right g_ν-modular right ideal M/P of R/P such that the zero ideal of R/P is the largest ideal of R/P contained in M/P. Clearly, M is a right g_ν-modular right ideal of R. Since the zero ideal of R/P is the largest ideal of R/P contained in M/P, P is the largest ideal of R contained in M. Therefore, P is a right g_ν-primitive ideal of R. □

Proposition 2.12. $J^r_{g_\nu}$ is the Hoehnke radical determined by the class of all right g_ν-primitive near-rings, $\nu \in \{0, 1, 2\}$.

Theorem 2.13. *Let G be a right R-group of type-g_ν and S be an invariant subnear-ring (and right ideal for $\nu = 0$) of R with $GS \neq \{0\}$. Then G is a right S-group of type-g_ν, $\nu \in \{0, 1, 2\}$.*

Proof. If G is a right R-group of type-0 and S is an invariant subnear-ring and right ideal of R with $GS \neq \{0\}$, then under the restriction of G to S, by Theorem 3.2 of [9], G is a right S-group type-0. Also if G be a right R-group of type-ν and S is an invariant subnear-ring of R with $GS \neq \{0\}$, then under the restriction of G to S, by Theorems 3.1 and 3.2 of [10], G is a right S-group type-ν, where $\nu \in \{1, 2\}$. Therefore G is a right S-group of type-ν, $\nu \in \{0, 1, 2\}$. Let A be the set of generators of the right R-group G and P be the largest ideal of R contained in $(0 : G)_R := \{r \in R \mid Gr = \{0\}\}$. A generator of the right R-group G is also a generator of the right S-group G. From the proof of Theorem 3.10 of [9] (and Theorems 3.9 and 3.10 of [10] for $\nu \in \{1, 2\}$) as the extension of G from S to R coincides with the action of G on R, it follows that a generator of the right S-group G is also a generator of the right R-group G. So A is the set of generators of the right S-group G. We have $P = (0 : A) = \{r \in R \mid ar = 0 \text{ for all } a \in A\}$. Now $P \cap S = (0 : A) \cap S = \{s \in S \mid As = \{0\}\}$. Let Q be the largest ideal of S contained in $(0 : G)_S := \{s \in S \mid Gs = \{0\}\} = (0 : G) \cap S$. Clearly $P \cap S \subseteq (0 : G)_S$. By the definition of Q, $P \cap S \subseteq Q$. Since $AQ = \{0\}$, $Q \subseteq P$. So $Q \subseteq P \cap S$. Therefore $Q = P \cap S$. Hence G is a right S-group of type-g_ν. □

Proposition 2.14. A right R-group of type-g_ν is an R-group of type-$\nu(e)$, $\nu \in \{0, 1, 2\}$.

Proof. Let G be a right R-group of type-g_ν, $\nu \in \{0, 1, 2\}$. So G is a right R-group of type-ν. In view of Remark 3.9 of [11] G is a right$R$-group of type-$\nu(e)$ if $r, s \in R$ and $gr = gs$ for all $g \in G$, then $r - s \in P$ where P is the largest ideal of R contained in $(0 : G) := \{r \in R \mid Gr = \{0\}\}$. Let $gr = gs$ for all $g \in G$, $r, s, \in R$ and P be the largest ideal of R contained in $(0 : G)$. Let A be the set of all generators of the right R-group G. Now $ar = as$ for all $a \in A$. Since each $a \in A$ is distributive, $a(r - s) = 0$ for all $a \in A$. Therefore $r - s \in P$ as $P = (0 : A)$. Hence G is a right R-group of type-$\nu(e)$. □

Remark 2.15. If G is a right R-group of type-$\nu(e)$, then by Proposition 3.12 of [11], $(0 : G) := \{r \in R \mid Gr = \{0\}\}$ is an ideal of R. Also, by Theorem 3.24 of [11], a right g_ν-primitive near-ring is an equiprime near-ring.

Definition 2.16. Let G be a right R-group of type-g_ν, $\nu \in \{0, 1, 2\}$. Then G is called faithful if $(0 : G) = \{0\}$.

Theorem 2.17. *Let G be a faithful right S-group of type-g_ν and S be an essential left invariant ideal of R. Then G is a faithful right R-group of type-g_ν, $\nu \in \{0, 1, 2\}$.*

Proof. Let h_0 be a generator of the right S-group G. From the proof of Theorem 3.10 of [9], for $h \in H, r \in R$ the operation defined by $hr := h_0(sr)$ if $h = h_0 s, s \in S$, makes G a right R-group and is an extension the action of G on S to R. Moreover, Theorem 3.10 of [9] and Theorems 3.9 and 3.10 of [10], G is a right R-group of type-ν, for $\nu \in \{1, 2\}$. Since G is a right R-group of type-$\nu(e)$, by Theorem 3. 33 of [11] and Theorem of [5], G is a faithful R-group of type-$\nu(e)$. Let A be the set of all generators of the right S-group G. Now $(0 : G)_S := \{s \in S \mid Gs = \{0\}\} = \{0\}$. We have $\{0\} = (0 : A)_S := \{s \in S \mid As = \{0\}\}$. Since G is a faithful right R-group, $(0 : G)_R := \{r \in R \mid Gr = \{0\}\} = \{0\}$. From the proof of Theorem 3.10 of [9], it can be easily seen that a generator of the right S-group G is also a generator of the right R-group G. So A is the set of generators of the right R-group G. Suppose that $r \in (0 : A)$. Now $Ar = \{0\}$. So $\{0\} = (Ar)S = A(rS)$ and hence $rS = \{0\}$ as $rS \subseteq S$. Since S is an ideal, $KS = \{0\}$ and S is a prime near-ring, we have $K = \{0\}$, where K is the ideal of R generated by r. Therefore $r = 0$ and hence $(0 : A)_R = \{0\}$. So G is a faithful right R-group of type-g_ν. □

From the above theorem we have:

Theorem 2.18. *The class of all right g_ν-primitive near-rings is closed under essential left invariant extensions, $\nu \in \{0, 1, 2\}$.*

In view of Theorem 1.1, we have the following:

Theorem 2.19. *Let $\nu \in \{0, 1, 2\}$. Let $\mathcal{E}$ be the class of all right g_ν-primitive near-rings and $\mathcal{U}\mathcal{E}$ be the upper radical class determined by $\mathcal{E}$. Then $\mathcal{U}\mathcal{E}$ is a c-hereditary Kurosh-Amitsur radical class in the variety of all near-rings with hereditary semisimple class $\mathcal{S}\mathcal{U}\mathcal{E} = \mathcal{E}$. So, $J^r_{g_\nu}$ is a Kurosh-Amitsur radical in the class of all near-rings and for any ideal I of R, $J^r_{g_\nu}(I) \subseteq J^r_{g_\nu}(R) \cap I$ with equality, if I is left invariant.*

Theorem 2.20. *$J^r_{g_\nu}$ is an ideal-hereditary Kurosh-Amitsur radical in the class of all zero-symmetric near-rings.*

Theorem 2.21. *$J^r_{g_\nu}$ is a special radical in the class of all near-rings.*

3. Examples

In this section we present some examples of near-rings R and their right R-groups to show that the present right Jacobson radicals are distinct from the known right Jacobson radicals of near-rings. Now we present an example of a right R-group of type-$\nu(e)$ which is not of type-g_ν, $\nu \in \{0, 1, 2\}$.

Proposition 3.1. If G be a finite group and G has a subgroup of index two, then $\mathrm{M_0(G)}$ is a right 2(e)-primitive near-ring.

Proof. Let G be a finite group and H be a subgroup of G of index 2. So H is a normal subgroup of G. Let $\mathrm{R = M_0(G)}$. Then R/K is a right R-group of type-2(e), where $\mathrm{K = (H : G) = \{r \in R \mid r(g) \in H, \text{ for all } g \in G\}}$. To show this we consider the two distinct cosets H and H + a of H in G. Now $\mathrm{G = H \cup H + a}$, H and H + a are disjoint sets. K is a right ideal of R which is right modular by the identity element of R. So R/K is a monogenic right R-group. Now we show that R/K is a right R-group of type-2. Let $\mathrm{0 \neq r + K \in R/K}$. $\mathrm{(r + K)R = R/K}$ if and only if there is an s ∈ R such that (r + K)s = 1 + K, that is, 1 - rs ∈ K. Let P_1 = {x ∈ G | r(x) ∈ H} and P_2 = {x ∈ G | r(x) ∈ H + a}. Let b ∈ P_2 and r(b) = $h^{'}$ + a, $h^{'}$ ∈ H. Define s : G → G by s(g) = b, if g ∈ H + a, and 0, if g ∈ H. We have s ∈ R. For y ∈ H, (1 - rs)(y) = y - r(s(y)) = y - r(0) = y ∈ H and for z = h + a ∈ H + a, (1 - rs)(z) = z - r(s(z)) = z - r(b) = (h + a) - ($h^{'}$ + a) = h - $h^{'}$ ∈ H. Therefore, 1 - rs ∈ (H : G) = K and hence R/K is a right R-group of type-2. Since R is simple, {0} is the largest ideal of R contained in (0 : R/K) = (K : R) = {t ∈ R | Rt ⊆ K}. Let u, v ∈ R and (t + K)u = (t + K)v for all t + K ∈ R/K. Now tu - tv ∈ K, for all t ∈ R. Suppose that g ∈ G and u(g) ≠ v(g). We can choose a t ∈ R such that (tu)(g) - (tv)(g) ∈ H + a, a contradiction to the fact that tu - tv ∈ K. Therefore, u = v and hence R/K is a right R-group of type-2(e). Since R is simple, it is a right 2(e)-primitive near-ring. □

Example 3.2. Let G be the non-abelian group of order 6. Let N be the subgroup of G of order 3. By Proposition 3.1, $M_0(G)/(N : G)$ is a right $M_0(G)$-group of type-2(e) and $M_0(G)$ is a right 2(e)-primitive near-ring. Since N is the maximal (normal) subgroup of G, $(N : G)$ is the only proper (maximal) right ideal of $M_0(G)$. So a right $M_0(G)$-group of type-0 is $M_0(G)$-isomorphic to $M_0(G)/(N : G)$. Therefore, if $f + (N : G)$ is a generator of the right $M_0(G)$-group $M_0(G)/(N : G)$, then $(0 : f + (N : G)) = (N : G) \neq \{0\}$. Note that as $M_0(G)$ is a simple near-ring, $\{0\}$ is the largest ideal of $M_0(G)$ contained in $(0 : M_0(G)/(N : G))$. Hence $M_0(G)/(N : G)$ is not a right $M_0(G)$-group of type-g_ν, $\nu \in \{0, 1, 2\}$.

Now we present another example to show that there are right R-groups of type-$\nu(e)$ which are not of type-g_ν. The following example was considered in [3] and [11].

Example 3.3. Consider $G := Z_8$, the group of integers under addition modulo 8. Now $T : G \to G$ defined by $T(g) = 5g$ for all $g \in G$ is an automorphism of G. T fixes $0, 2, 4, 6$ and maps 1to 5, 5 to 1, 3 to 7 and 7 to 3. Now $A := \{I, T\}$ is an automorphism group of G and $\{0\}, \{2\}, \{4\}, \{6\}, \{1, 5\}$ and $\{3, 7\}$ are the orbits. Let R be the centralizer near-ring $M_A(G)$, the near-ring of all self maps of G which fix 0 and commute with T. An element of R is completely determined by its action on $\{1, 2, 3, 4, 6\}$. Note that for $f \in R$ we have $f(2), f(4), f(6)$ are arbitrary in $2G$ and $f(1), f(3)$ are arbitrary in G. In [3] shown that $I := (0 : 2G) = \{f \in R \mid f(h) = 0$, for all $h \in 2G\}$ is the only non-trivial ideal of R. Let $K := (2G : G) = \{t \in R \mid t(G) \subseteq 2G\} \neq R$. Let t_0 be the identity element in R. Now $t_0 + K$ is a generator of the right R-group R/K. Let $h \in R - K$. We show now that $(h+K)R = R/K$. Since $h \notin K$, there is an $a \in G - 2G$ such that $b := h(a) \notin 2G$. We construct an element $s \in R$ such that $s(1) = s(3) = a$, so that $s(5) = s(7) = a+4$, and $s = 0$ on $2G$. Since s maps $G - 2G$ to $G - 2G$, we get that $t_0 - hs \in K$ and hence $(h + K)s = t_0 + K$. So $(h + K)R = R/K$. Therefore, R/K is a right R-group of type-ν. Moreover, $(R/K)I \neq \{K\}$. Therefore, $\{0\}$ is the largest ideal of R contained in $(K : R)$ and hence $J^r_\nu(R) = \{0\}$. Consider s_1, $s_1 \in R$, where $s_1(1) = 1$ and 0 on $G - \{1, 5\}$ and

$s_2(1) = 5$ and 0 on $G - \{1, 5\}$. Clearly $(h + K)s_1 = (h + K)s_2$ for all $h \in R$ as $h(1) - h(5) \in 2G$ for all $h \in R$. But $s_1 - s_2 \notin \{0\}$. Therefore, R/K is not a right R-group of type-$\nu(e)$.

Proposition 3.4. Let R be the near-ring considered in the Example 3.3 and let K be a right ideal of R. Then $H_1 := \{f(g) \mid f \in K, g \in G\} \subseteq G$ and $H_2 := \{f(g) \mid f \in K, g \in 2G\} \subseteq 2G$ are (normal) subgroups of G and 2G respectively.

Proof. We show that H_1 is a subgroup of G. Since $0 \in H_1$, H_1 is non-empty. Let $h_1, h_2 \in H_1$. We get $f_1, f_2 \in K$ and $g_1, g_2 \in G$ such that $h_1 = f_1(g_1)$ and $h_2 = f_2(g_2)$. Clearly, $-h_1 = (-f_1)(g_1) \in H_1$ as $-f_1 \in K$. Suppose that one of the g_i is in G - 2G. With out loss of generality, suppose that $g_1 \in G - 2G$. We get $f_3 \in R$ such that $f_3(g_1) = g_2$. Now $f_1 - f_2f_3 \in K$ and $h_1 - h_2 = (f_1 - f_2f_3)(g_1) \in H_1$. Assume now that $g_1, g_2 \in 2G$. So, $h_1, h_2 \in 2G$. If $g_1 = 0$, then $h_1 - h_2 = -h_2 \in H_1$. Suppose that $g_1 \neq 0$. So, we get $f_4 \in R$ such that $f_4(g_1) = g_2$. Now $f_1 - f_2f_4 \in K$ and $h_1 - h_2 = (f_1 - f_2f_4)(g_1) \in H_1$. Therefore, H_1 is a subgroup of G. Similarly, we get that H_2 is a subgroup of 2G. □

Proposition 3.5. Let R, K, H_1 and H_2 be as defined in Proposition 3.4. If $H_1 = G$ and $H_2 = 2G$, then K = R.

Proof. Suppose that $H_1 = G$ and $H_2 = 2G$. We have $1, 3 \in H_1$. So, for $i \in \{1, 3\}$, we get $f_i \in K$ such that $f_i(g_i) = i$, where $g_i \in \{1, 3, 5, 7\} = G - 2G$. For i = 1, 3 we also get $m_i \in R$ such that $m_i(i) = g_i$, so that $m_i(i + 4) = g_i + 4$ and $m_i = 0$ on G - {i, i + 4}. Now $f_im_i \in K$, i = 1, 3. Clearly, $f_1m_1 + f_3m_3$ fixes all the elements of G - 2G and maps all the elements of 2G to 0. We have $2, 4, 6 \in H_2 = 2G = \{0, 2, 4, 6\}$. For i = 2, 4, 6 we get $f_i \in K$ such that $f_i(g_i) = i$, $g_i \in 2G$. So, for i = 2, 4, 6 we get $m_i \in R$ such that $m_i(i) = g_i$ and m_i is 0 on G - {i}. Now $f_im_i \in K$, i = 2, 4, 6. $f_2m_2 + f_4m_4 + f_6m_6$ fixes all the elements of 2G and maps all the elements of G -2G to 0. Therefore, the identity map I of G can be expressed as $I = f_1m_1 + f_2m_2 + f_3m_3 + f_4m_4 + f_6m_6 \in K$. Hence, K = R. □

Proposition 3.6. Let R, K, H_1 and H_2 be as defined in Proposition 3.4. If K is a maximal right ideal of R, then $K = (2G : G) = \{f \in R \mid f(G) \subseteq 2G\}$ or $(4G : 2G) = \{f \in R \mid f(2G) \subseteq 4G\}$

Proof. Suppose that K is a maximal right ideal of R. Clearly, if H and T are (normal) subgroups of G and 2G respectively, then $(H : G) = \{f \in R \mid f(G) \subseteq H\}$ and $(T : 2G) = \{f \in R \mid f(2G) \subseteq T\}$ are right ideals of R. Now 2G and 4G are the maximal (normal) subgroups of G and 2G respectively. We have $K \subseteq (H_1 : G)$ and $K \subseteq (H_2 : 2G)$. Since K is a maximal right ideal of R, by Proposition 3.5, either $H_1 \neq G$ or $H_2 \neq 2G$.
Case(i) Suppose that $H_2 \neq 2G$. Since K is a maximal right ideal of R and $K \subseteq (H_2 : 2G) \neq R$, we get that $H_2 = 4G$ and $K = (4G : 2G)$.
case(ii) Suppose that $H_1 \neq G$. Since K is a maximal right ideal of R and $K \subseteq (H_1 : G) \neq R$, we get that $H_1 = 2G$ and $K = (2G : G)$.
Therefore, either $K = (2G : G)$ or $(4G : 2G)$. □

Proposition 3.7. Let R be the near-ring considered in the Example 3.3. Let U = (4G : 2G) = {f ∈ R | f(2G) ⊆ 4G}. Then U is a maximal right ideal of R and R/U is a right R-group of type-2(e).

Proof. Clearly, U is a right ideal of R. Consider the right R-group R/U. We prove that R/U is a right R-group of type-2. Since R has identity I, I + U is a generator of the right R-group R/U and hence R/U is a monogenic right R-group. Let 0 ≠ f + U ∈ R/U. So, f ∉ U. We get 0 ≠ a ∈ 2G such that b := f(a) ∉ 4G. So, 2G = {0, b, 2b, 3b} as 2 and 6 are generators of 2G. Construct r ∈ R by r(b) = a, r(2b) = 0, r(3b) = a and r = 0 on G - {0, 1, 3, 5, 7}. Now (I - fr)(x) ∈ 4G for all x ∈ 2G. Therefore, I - fr ∈ U and hence (f + U)r = I + U. This shows that (f + U)R = R/U. So, R/U is a right R-group of type-2. We know that P := (0 : 2G) is the only non-trivial ideal of R. Therefore, P is the largest ideal of R contained in U = (4G : 2G) and hence P is the largest ideal of R contained in (0 : R/U) = (U : R) = {f ∈ R | Rf ⊆ U}. Let 0 ≠ s + U ∈ R/U and f, h ∈ R. Suppose that (s + U)rf = (s + U)rh for all r ∈ R. So, srf - srh ∈ U for all r ∈ R. We show that f - h ∈ P. If possible, suppose that f - h ∉ P. We get 0 ≠ a ∈ 2G such that (f - h)(a) = f(a) - h(a) ≠ 0 with h(a) ≠ 0. Let s(c) ∉ {0, 4} for some c ∈ 2G. Choose r ∈ R such that r(f(a)) = 0 and r(h(a)) = c. Now (srf)(a) = 0 and (srh)(a) = s(c). So, (srf - srh)(a) = 0 - s(c) ∉ {0, 4}, a contradiction to the fact that srf - srh ∈ U. Therefore, f(a) = h(a) for all a ∈ 2G. Hence f - h ∈ P. So, R/U is a right R-group of type-2(e). □

Proposition 3.8. Let R be the near-ring considered in Example 3.3. Then J^r_ν(R) = {0} and $J^r_{\nu(e)}$(R) = (0 : 2G) ≠ {0}.

Proof. We know that {0} and I := (0 : 2G) = {f ∈ R | f(2G) ={0}} are the only proper ideals of R. Let K_1 := (2G : G) = {f ∈ R | f(G) ⊆ 2G} and K_2 := (4G : 2G) = {f ∈ R | f(2G) ⊆ 4G}. By Proposition 3.6, a maximal right ideal of R is either K_1 or K_2. So, a right R-group of type-0 is isomorphic to R/K_1 or R/K_2. By Example 3.3, R/K_1 is a right R-group of type-2 but not of type-2(e). Since {0} is the largest ideal of R contained in K_1, {0} is a right 2-primitive ideal of R but not a right 2(e)-primitive ideal of R. By Proposition 3.7, R/K_2 is a right R-group of type-2(e). Since I = (0 : 2G) is the largest ideal of R contained in K_2, I is a right 2(e)-primitive ideal of R. Therefore, J^r_ν(R) = {0} and $J^r_{\nu(e)}(R) = (0 : 2G)$. □

Proposition 3.9. Let R be the near-ring considered in Example 3.3. Then $J^r_{g_\nu}(R) = R$, $\nu \in \{0, 1, 2\}$.

Proof. Let R be the near-ring considered in the Example 3.3 and $K = (2G : G)$, $U = (4G : 2G)$. As seen above K, U are the only maximal right ideals of R and R/K is a right R-group of type-2 but not of type-2(e), where as R/U is a right R-group of type-2(e). If $f + K$ is a generator of the right R-group R/K, then the maximal right ideal $(0 : f + K)$ must be either K or U. Since $0(K) = 2^{10} \neq 2^9 = 0(U)$, and $R/(0 : f + K)$ is right R-isomorphic R/K, $(0 : f + K) = K$. Hence R/K is not a right R-group of type-g_ν as {0}, $(0 : 2G)$ and R are the only ideals of R. By a similar argument we get that R/U is not a right R-group of type-g_ν. So $J^r_{g_\nu}(R) = R$. □

4. $J^r_{g_\nu}$-semisimple Near-rings, $\nu \in \{0,1,2\}$

In this section we present structure theorems for $J^r_{g_\nu}$-semisimple near-rings.

Proposition 4.1. Let R $(\neq \{0\})$ be a $J^r_{g_\nu}$-semisimple near-rings satisfying DCC on right ideals of R, $\nu \in \{0,1,2\}$. Then R is a finite direct sum of minimal right ideals which are right R-groups of type-g_ν.

Proof. Let $P_i, i \in I$ be the collection of right g_ν-primitive ideals of R. Since R is a $J^r_{g_\nu}$-semisimple near-ring, $\cap\{P_i \mid i \in I\} = \{0\}$. We get a right R-group G_i of type-g_ν such that $P_i = (0 : G_i) := \{r \in R \mid G_i r = \{0\}\}, i \in I$. Let A_i be the set of generators of $G_i, i \in I$. Now $P_i = (0 : A_i) := \{r \in R \mid A_i r = \{0\}\}$. Note that for each $a \in A_i, (0 : a) := \{r \in R \mid ar = 0\}$ is a right g_ν-modular right ideal of R and the right R-group $R/(0 : a)$ is right R-isomorphic to $G_i, i \in I$. Since each P_i is an intersection of right g_ν-modular right ideal of R and $\cap\{P_i \mid i \in I\} = \{0\}$, the intersection of all right g_ν-modular right ideal of R is zero. We get a finite number of right g_ν-modular right ideals $K_1, K_2, ..., K_n$ of R such that $\cap\{K_j \mid j = 1, 2, ..., n\} = \{0\}$. Let $T_i := K_1 \cap K_2 \cap ... \cap K_{i-1} \cap K_{i+1} \cap ... \cap K_n, i = 1, 2, ..., n$. We may assume that $T_i \neq \{0\}$ for all $i = 1, 2, ..., n$. Now by Proposition 3.12[(2)] of [8], $R = T_1 \oplus T_2 \oplus ... \oplus T_n$, a direct sum of minimal right ideals T_i of R which are right R-groups of type-g_ν. □

In [8](Definition 3.5), if R is a direct sum of n minimal right ideals of R, then the *dimension of* R is defined as n and is denoted by *dim* R.

Definition 4.2. A distributive idempotent e of R is called *right g_ν-primitive* if eR is a right R-group of type-$g_\nu, \nu \in \{0,1,2\}$.

Theorem 4.3. *Let R be a right g_ν-primitive near-rings satisfying DCC on right ideals of R, $\nu \in \{0,1,2\}$. Then R is a simple near-ring with identity and R has a subnear-ring which is isomorphic to the matrix near-ring $M_n(S)$, where $S = eRe$, e is a right g_ν-primitive idempotent and $n = dim\ R$. If, in addition, R is distributively generated, then R isomorphic to $M_n(S)$.*

Proof. R satisfies the hypothesis of Theorem 4.3 of [8] and hence the conclusion follows from it. □

Theorem 4.4. *Let R be a finite right g_2-primitive near-ring and eRe be a non-ring. Then R is (isomorphic to) the matrix near-ring $M_n(F)$, where $n = dim\ R$, $F := eRe$ is a near-field and e is a right g_2-primitive idempotent in R.*

Proof. Proof follows from Theorem 4.16 of [8]. □

Theorem 4.5. *Let R $(\neq \{0\})$ be a $J^r_{g_\nu}$-semisimple near-rings satisfying DCC on right ideals of R, $\nu \in \{0,1,2\}$. Then R is a direct sum of minimal ideals which are simple right g_ν-primitive near-rings with identity.*

Proof. Let $P_i, i \in I$ be the collection of right g_ν-primitive ideals of R, $\nu \in \{0, 1, 2\}$. Now $\cap\{P_i \mid i \in I\} = \{0\}$. Since R has DCC on right ideals of R, we get a finite number of right g_ν-primitive ideals of $P_1, P_2, ..., P_n$ of R such that $P_1 \cap P_2 \cap ... \cap P_n = \{0\}$. We may assume that $K_j := P_1 \cap P_2 \cap ... \cap P_{j-1} \cap P_{j+1} \cap ... \cap P_n \neq \{0\}, j = 1, 2, ..., n$. By Theorem 4.3, R/P_i is a simple near-ring with identity as R/P_i is a right g_ν-primitive near-ring with DCC on right ideals. Now by Theorem 2.50 of Pilz [4], $R = K_1 \oplus K_2 \oplus ... \oplus K_n$, K_i are minimal ideals of R and are simple right g_ν-primitive near-rings with identity. □

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