

On Some New Nonlinear Integral Inequalities of Gronwall-Bellman Type

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ABSTRACT. In this paper, we establish some new nonlinear integral inequalities of Gronwall-Bellman type. These inequalities generalize some famous inequalities which can be used in applications as handy tools to study the qualitative as well as quantitative properties of solutions of some nonlinear ordinary differential and integral equations. More accurately we extend certain results which have been proved in A. Abdeldaim and M. Yakout [1] and H. El-Owaidy, A. A. Ragab, A. Abdeldaim [7] too.

1. Introduction

It is well known that, the differential and integral inequalities of one variable which provide explicit bounds on unknown functions, occupy a very privileged position in the development of the theory of linear and nonlinear ordinary differential and integral equations see for example [2, 8, 9, 10, 11]. In the qualitative theory of differential and Volterra integral equations, Gronwall type inequalities of one variable for the real functions play a very important role. In the recent years, these inequalities have been greatly enriched by the recognition of their potential and

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Received December 28, 2012; accepted April 15, 2013.

2010 Mathematics Subject Classification: 54C60, 54H25.

Key words and phrases: Gronwall-Bellman inequality, integral inequalities, ordinary differential equations, integral equations.

intrinsic worth in many applications of the applied sciences. The first use of Gronwall inequality to establish boundedness and stability is due to R. Bellman, for the ideas and the methods of R. Bellman see [5]. In 1943, Bellman [6] proved the fundamental lemma (see Theorem 1.1) named Gronwall-Bellman's inequality which plays a vital role in studying stability and asymptotic behavior of solutions of differential and integral equations see for instance [4, 5]. After the discovery of the Gronwall-Bellman's inequality, the inequalities of this type are known in the literature as Bellman's inequality, Bellman-Gronwall's inequality or Gronwall-Bellman's inequality [2, 11]. In view of the important applications of the Gronwall-Bellman's inequality see [2, 3], in the past few years, Pachpatte in [11], established new generalizations of the Gronwall-Bellman's inequality which can be used as powerful tools in the study of certain classes of differential and integral equations. In 1999 El-Owaidy et al [7] obtained several new integral inequalities of Gronwall-Bellman inequality type. These inequalities are directly useful in studying some properties of solutions of ordinary differential equations. The aim of this paper is to extend certain results which have been proved in [1] and [7] to obtain new generalizations for some former famous inequalities, which can be used as handy tools to study the qualitative as well as the quantitative properties of solutions of some nonlinear ordinary differential and integral equations.

Theorem 1.1. (Gronwall-Bellman's inequality [6]) *let $x(t)$ and $f(t)$ be non-negative continuous functions defined on $I = [0, \infty)$, for which the inequality*

$$x(t) \leq x_0 + \int_0^t f(s)x(s)ds, \quad \forall t \in I,$$

holds, where $x_0 \geq 0$ is a constant.

Then

$$x(t) \leq x_0 \exp \left[\int_0^t f(\lambda)d\lambda \right], \quad \forall t \in I.$$

Theorem 1.2. (A. Abdeldaim and M. Yakout's inequality [1]) *Let $x(t)$, $f(t)$ and $h(t)$ be non-negative real-valued continuous functions defined on $I = [0, \infty)$, and satisfy the inequality*

$$x^p(t) \leq x_0 + \int_0^t f(s)x^p(s)ds + \int_0^t h(s)x^q(s)ds, \quad \forall t \in I,$$

where $p > q \geq 0$, are constants. Then

$$x(t) \leq \exp \left(\frac{1}{p} \int_0^t f(s)ds \right) \left[x_0^{p_1} + p_1 \int_0^t h(s) \exp \left(-p_1 \int_0^s f(\lambda)d\lambda \right) ds \right]^{\left[\frac{1}{p-q} \right]},$$

for all $t \in I$, where $p_1 = \left[\frac{p-q}{p} \right]$.

2. Main Results

In this section, we state and prove some new integral inequalities of Gronwall-Bellman type, which can be used in applications as handy tools, and in the analysis of various problems in the theory of the nonlinear ordinary differential and integral equations.

Theorem 2.1. *Let $x(t)$ and $h(t)$ be real-valued non-negative continuous functions defined on $I = [0, \infty)$, and $n(t)$ be a positive monotonic non-decreasing continuous function defined on $I = [0, \infty)$ and satisfy the inequality*

$$(2.1) \quad x^p(t) \leq n^p(t) + \int_0^t h(s)x^q(s)ds, \quad \forall t \in I,$$

where $p > q \geq 0$. Then

$$(2.2) \quad x(t) \leq n(t) \left[1 + \left[\frac{p-q}{p} \right] \int_0^t h(s)n^{-[p-q]}(s)ds \right]^{\left[\frac{1}{p-q} \right]}, \quad \forall t \in I.$$

Proof. Since $n(t)$ is a positive, monotonic non-decreasing function, we observe from (2.1) that

$$\left[\frac{x(t)}{n(t)} \right]^p \leq 1 + \int_0^t h(s)n^{-[p-q]}(s) \left[\frac{x(s)}{n(s)} \right]^q ds, \quad \forall t \in I.$$

Let

$$(2.3) \quad m(t) = \frac{x(t)}{n(t)}, \quad m(0) \leq 1,$$

hence

$$m^p(t) \leq 1 + \int_0^t h(s)n^{-[p-q]}(s)m^q(s)ds, \quad \forall t \in I.$$

Using Theorem 1.2 at $f(t) = 0$, we have

$$(2.4) \quad m(t) \leq \left[1 + \left[\frac{p-q}{p} \right] \int_0^t h(s)n^{-[p-q]}(s)ds \right]^{\left[\frac{1}{p-q} \right]}, \quad \forall t \in I.$$

The required inequality in (2.2) follows from (2.3) and (2.4).

The proof is complete. □

Remark 2.1. It is interesting to note that when $q = 1$ Theorem 2.1 reduces to Theorem 2 in [7].

Now we will give an inequality, which is circulated to the previous inequalities in Theorems 1.2, 2.1, and many other famous inequalities in different papers, also it has many applications.

Theorem 2.2. Let $x(t)$ and $f(t)$ be non-negative real-valued continuous functions defined on $I = [0, \infty)$ and $n(t)$ be a positive monotonic non-decreasing continuous function defined also on I and satisfy the inequality

$$(2.5) \quad x^p(t) \leq n^p(t) + \int_0^t f(s)x^p(s)ds + \int_0^t h(s)x^q(s)ds, \quad \forall t \in I,$$

where $p > q \geq 0$. Then

$$(2.6) \quad x(t) \leq n(t)k_1(t), \quad \forall t \in I,$$

where

$$(2.7) \quad k_1(t) = \exp\left(\frac{1}{p} \int_0^t f(s)ds\right) \times \left[1 + \left[\frac{p-q}{p}\right] \int_0^t h(s)n^{-[p-q]}(s) \exp\left(\frac{-[p-q]}{p} \int_0^s f(\lambda)d\lambda\right) ds\right]^{\left[\frac{1}{p-q}\right]},$$

for all $t \in I$.

Proof. Since $n(t)$ is a positive monotonic non-decreasing function, we observe from (2.5) that

$$\left[\frac{x(t)}{n(t)}\right]^p \leq 1 + \int_0^t f(s) \left[\frac{x(s)}{n(s)}\right]^p ds + \int_0^t h(s)n^{-[p-q]}(s) \left[\frac{x(s)}{n(s)}\right]^q ds, \quad \forall t \in I.$$

Let

$$(2.8) \quad m(t) = \frac{x(t)}{n(t)}, \quad m(0) \leq 1, \quad \forall t \in I.$$

Hence

$$m^p(t) \leq 1 + \int_0^t f(s)m^p(s)ds + \int_0^t h(s)n^{-[p-q]}(s)m^q(s)ds, \quad \forall t \in I.$$

Now, we have from Theorem 1.2.

$$(2.9) \quad m(t) \leq k_1(t), \quad \forall t \in I,$$

where $k_1(t)$ as defined in (2.7) The required inequality in (2.6) follows from (2.8) and (2.9). The proof is complete. \square

Remark 2.2.

1. It is interesting to note that the special case when $n(t) = n_0$ (any constant), the inequality given in Theorem 2.2, reduces to the inequality given in Theorem 1.2.

2. When $p = 1$ the inequality given in Theorem 2.2, reduces to the inequality given in Theorem 7 in [7].
3. If we put $n(t) = n_0$, $h(t) = 0$, and $p = 1$, in Theorem 2.2, we get the well known Gronwall-Bellman inequality (see Theorem 1.1).
4. If we put $n(t) = n_0$, and $p = 1$, The inequality given in Theorem 2.2, reduces to the Willett and Wong inequality [12].
5. When we put $p = 2$, $q = 1$, $n(t) = n_0$, and $f(t) = 0$, the inequality given in Theorem 2.2, reduces to the well known Ou-lmag inequality [10].
6. When $f(t) = 0$ the inequality given in Theorem 2.2, reduces to the inequality given in Theorem 2.1.

Theorem 2.3. *Let $x(t)$ be a real-valued positive continuous function and $f(t)$, $g(t)$ are real-valued non-negative continuous functions defined on $I = [0, \infty)$ and satisfy the inequality*

$$(2.10) \quad x^p(t) \leq x_0 + \int_0^t f(s) \left[x^q(s) + \int_0^s g(\lambda)x(\lambda)d\lambda \right] ds,$$

where x_0 is non-negative constant and $p > q \geq 0$. Then

$$(2.11) \quad x(t) \leq \left[x_0 + \int_0^t f(s)k_2(s) \exp \left(\int_0^s g(\lambda)d\lambda \right) ds \right]^{\frac{1}{p}}, \quad \forall t \in I,$$

where

$$(2.12) \quad k_2(t) = \left[x_0^{\frac{q(p-q)}{p}} + \left[\frac{q[p-q]}{p} \right] \int_0^t f(s) \exp \left(-[p-q] \int_0^s g(\lambda)d\lambda \right) ds \right]^{\left[\frac{1}{p-q} \right]},$$

for all $t \in I$.

Proof. Let $J^p(t)$ equal the right hand side in (2.10), we have $J(0) = x_0^{\frac{1}{p}}$ and

$$(2.13) \quad x(t) \leq J(t), \quad \forall t \in I.$$

Differentiating $J^p(t)$, leads to

$$(2.14) \quad pJ^{[p-1]}(t) \frac{dJ(t)}{dt} \leq f(t)Y(t), \quad \forall t \in I,$$

where $Y(t) = J^q(t) + \int_0^t g(s)J(s)ds$, thus we have $Y(0) = J^q(0) = x_0^{\frac{q}{p}}$, and

$$(2.15) \quad J(t) \leq Y(t), \quad \forall t \in I.$$

Differentiating $Y(t)$ and using (2.14) and (2.15), produces

$$\frac{dY(t)}{dt} \leq \frac{q}{p} f(t) Y^{[q-p+1]}(t) + g(t) Y(t), \quad \forall t \in I,$$

but $Y(t) > 0$, thus we have

$$(2.16) \quad Y^{[p-q-1]}(t) \frac{dY(t)}{dt} - g(t) Y^{[p-q]}(t) \leq \frac{q}{p} f(t), \quad \forall t \in I.$$

Now if we put

$$(2.17) \quad Z_1(t) = Y^{[p-q]}(t), \quad \forall t \in I,$$

then we have $Z_1(0) = Y^{[p-q]}(0) = x_0^{\frac{q[p-q]}{p}}$ and $Y^{[p-q-1]}(t) \frac{dY(t)}{dt} = \left[\frac{1}{p-q} \right] \frac{dZ_1}{dt}$, thus from (2.16) we obtain

$$\frac{dZ_1(t)}{dt} - [p-q]g(t)Z_1(t) \leq \left[\frac{q[p-q]}{p} \right] f(t), \quad \forall t \in I.$$

The above inequality implies the following estimation for $Z_1(t)$

$$(2.18) \quad Z_1(t) \leq \exp \left([p-q] \int_0^t g(s) ds \right) \times \left[x_0^{\frac{q[p-q]}{p}} + \left[\frac{q[p-q]}{p} \right] \int_0^t f(s) \exp \left(-[p-q] \int_0^s g(\lambda) d\lambda \right) ds \right],$$

for all $t \in I$, then from (2.18) in (2.17), we obtain

$$(2.19) \quad Y(t) \leq k_2(t) \exp \left(\int_0^t g(s) ds \right), \quad \forall t \in I,$$

where $k_2(t)$ is as defined in (2.12), thus from (2.14) we have

$$pJ^{[p-1]}(t) \frac{dJ(t)}{dt} \leq f(t) k_2(t) \exp \left(\int_0^t g(s) ds \right), \quad \forall t \in I.$$

Taking $t = s$ in the above inequality and Integrating from 0 to t , gives

$$(2.20) \quad J(t) \leq \left[x_0 + \int_0^t f(s) k_2(s) \exp \left(\int_0^s g(\lambda) d\lambda \right) ds \right]^{\frac{1}{p}}, \quad \forall t \in I.$$

Using (2.20) in (2.13), leads to the required inequality in (2.11). The proof is complete. \square

Remark 2.3. It is interesting to note that the special case when $p = 1$, Theorem 2.3 reduces to Theorem 3 in [7].

If we put $p = 2$ and, $q = 0$ in Theorem 2.3, we can easily drive the following corollary:

Corollary 2.1. Let $x(t)$ be a real-valued positive continuous function and $f(t)$, $g(t)$ are real-valued non-negative continuous functions defined on $I = [0, \infty)$ and satisfy the inequality

$$x^2(t) \leq x_0 + \int_0^t f(s) \left[1 + \int_0^s g(\lambda)x(\lambda)d\lambda \right] ds, \quad \forall t \in I,$$

where x_0 is non-negative constant. Then

$$x(t) \leq \sqrt{x_0 + \int_0^t f(s) \exp \left(\int_0^s g(\lambda)d\lambda \right) ds}, \quad \forall t \in I.$$

Theorem 2.4. Let $x(t)$ be a real-valued positive continuous function and $f(t)$, $g(t)$ are real-valued non-negative continuous functions defined on $I = [0, \infty)$ and satisfy the inequality

$$(2.21) \quad x^p(t) \leq x_0 + \int_0^t f(s) \left[x^p(s) + \int_0^s g(\lambda)x(\lambda)d\lambda \right] ds, \quad \forall t \in I,$$

where x_0 and p are positive constants. Then

$$(2.22) \quad x(t) \leq x_0^{\frac{1}{p}} \left[1 + \int_0^t f(s) \exp \left(\int_0^s [f(\lambda) + g(\lambda)]d\lambda \right) ds \right]^{\frac{1}{p}}, \quad \forall t \in I.$$

Proof. Let $J_1^p(t)$ equal the right hand side in (2.21), we have $J_1(0) = x_0^{\frac{1}{p}}$ and

$$(2.23) \quad x(t) \leq J_1(t), \quad \forall t \in I.$$

Differentiating $J_1^p(t)$, leads to

$$(2.24) \quad pJ_1^{[p-1]}(t) \frac{dJ_1(t)}{dt} \leq f(t)Y_1(t), \quad \forall t \in I,$$

where $Y_1(t) = J_1^p(t) + \int_0^t g(s)J_1(s)ds$, thus we have $Y_1(0) = J_1^p(0) = x_0$ and

$$(2.25) \quad J_1(t) \leq Y_1(t), \quad \forall t \in I.$$

Differentiating $Y_1(t)$ and using (2.24) and (2.25), produces

$$\frac{dY_1(t)}{dt} \leq [f(t) + g(t)]Y_1(t), \quad \forall t \in I.$$

Taking $t = s$ in the above inequality and Integrating from 0 to t , we get

$$(2.26) \quad Y_1(t) \leq x_0 \exp \left(\int_0^t [f(s) + g(s)]ds \right), \quad \forall t \in I.$$

Substituting (2.26) in (2.24) we have

$$J_1^{[p-1]}(t) \frac{dJ_1(t)}{dt} \leq \frac{1}{p} x_0 f(t) \exp \left(\int_0^t [f(s) + g(s)] ds \right), \quad \forall t \in I.$$

The above inequality implies an estimation for $J_1(t)$ as follows

$$(2.27) \quad J_1(t) \leq x_0^{\frac{1}{p}} \left[1 + \int_0^t f(s) \exp \left(\int_0^s [f(\lambda) + g(\lambda)] d\lambda \right) ds \right]^{\frac{1}{p}}, \quad \forall t \in I.$$

Using (2.27) in (2.23), we get the required inequality in (2.22). The proof is complete. \square

Theorem 2.5. Let $x(t)$, $f(t)$ and $g(t)$ be real-valued positive continuous functions defined on $I = [0, \infty)$ and satisfy the inequality

$$(2.28) \quad x(t) \leq x_0 + \int_0^t f(s) \left[x^{[2-p]}(s) + \int_0^s g(\lambda) x^q(\lambda) d\lambda \right]^p ds, \quad \forall t \in I,$$

where $x_0 > 0$, and $0 < p \leq 2$, $0 \leq q < 1$, are constants. Then

$$(2.29) \quad x(t) \leq x_0 + \int_0^t f(s) k_3(s) ds, \quad \forall t \in I,$$

where

$$(2.30) \quad K_3(t) = \exp \left(p(2-p) \int_0^t f(s) ds \right) \times \left[x_0^{[1-q][2-p]} + (1-q) \int_0^t g(s) \exp \left((1-q)(2-p) \int_0^s f(\lambda) d\lambda \right) ds \right]^{\left[\frac{p}{1-q} \right]},$$

for all $t \in I$.

Proof. Let $J_2(t)$ equal the right hand side in (2.28), we have $J_2(0) = x_0$ and

$$(2.31) \quad x(t) \leq J_2(t), \quad \forall t \in I.$$

Differentiating $J_2(t)$, produces

$$(2.32) \quad \frac{dJ_2(t)}{dt} \leq f(t) Y_2^p(t), \quad \forall t \in I,$$

where $Y_2(t) = J_2^{[2-p]}(t) + \int_0^t g(s) J_2^q(s) ds$, thus we have $Y_2(0) = J_2^{[2-p]}(0) = x_0^{[2-p]}$ and

$$(2.33) \quad J_2(t) \leq Y_2(t), \quad \forall t \in I.$$

Differentiating $Y_2(t)$ and using (2.32) and (2.33), leads to

$$(2.34) \quad \frac{dY_2(t)}{dt} \leq (2-p)f(t)Y_2(t) + g(t)Y_2^q(t), \quad \forall t \in I,$$

but $Y_2(t) > 0$ then we can write the inequality (2.34) in the following form

$$(2.35) \quad Y_2^{-q}(t) \frac{dY_2(t)}{dt} - (2-p)f(t)Y_2^{[1-q]}(t) \leq g(t), \quad \forall t \in I.$$

If we let, $Y_2^{[1-q]}(t) = Z_2(t)$, we have $Z_2(0) = Y_2^{[1-q]}(0) = x_0^{[1-q][2-p]}$, and $Y_2^{-q}(t) \frac{dY_2(t)}{dt} = \frac{1}{(1-q)} \frac{dZ_2}{dt}$, then we can write the inequality (2.35) as follows

$$(2.36) \quad \frac{dZ_2}{dt} - (1-q)(2-p)f(t)Z_2 \leq (1-q)g(t), \quad \forall t \in I.$$

The inequality (2.36) implies an estimation for $Z_2(t)$ as in the following

$$(2.37) \quad Z_2(t) \leq \exp \left((1-q)(2-p) \int_0^t f(s)ds \right) \times \left[x_0^{[1-q][2-p]} + (1-q) \int_0^t g(s) \exp \left(-(1-q)(2-p) \int_0^s f(\lambda)d\lambda \right) ds \right],$$

for all $t \in I$.

But $Y_2^{[1-q]}(t) = Z_2(t)$, then from (2.37), we have

$$(2.38) \quad Y_2^p(t) \leq k_3(t), \quad \forall t \in I,$$

where $k_3(t)$ as defined in (2.30), and from (2.38) in (2.32), we obtain

$$\frac{dJ_2(t)}{dt} \leq f(t)k_3(t), \quad \forall t \in I.$$

The above inequality implies an estimation for $J_2(t)$ as in the following

$$(2.39) \quad J_2(t) \leq x_0 + \int_0^t f(s)k_3(s)ds, \quad \forall t \in I.$$

Using (2.39) in (2.31), we get the required inequality in (2.29). The proof is complete. □

Remark 2.4. When $g(t) = 0$ and $p = 1$, the inequality given in Theorem 2.5, reduces to the Gronwall-Bellman inequality (see Theorem 1.1).

In the special case when $p = 2$, and $q = \frac{1}{2}$, Theorem 2.5 takes the following form:

Corollary 2.2. Let $x(t)$, $f(t)$ and $g(t)$ be real-valued positive continuous functions defined on $I = [0, \infty)$ and satisfy the inequality

$$x(t) \leq x_0 + \int_0^t f(s) \left[1 + \int_0^s g(\lambda) \sqrt{x(\lambda)} d\lambda \right]^2 ds, \quad \forall t \in I,$$

where $x_0 > 0$. Then

$$x(t) \leq x_0 + \int_0^t f(s) \left[1 + \frac{1}{2} \int_0^s g(\lambda) \right]^4 ds, \quad \forall t \in I.$$

Theorem 2.6. Let $x(t)$ be a real-valued positive continuous function and $f(t)$, $g(t)$ are non-negative real-valued continuous functions defined on $I = [0, \infty)$, and satisfy the inequality

$$(2.40) \quad x^p(t) \leq x_0 + \int_0^t f(s)x(s) \left[x^q(s) + \int_0^s g(\lambda)x(\lambda)d\lambda \right]^r ds,$$

for all $t \in I$, where $x_0 > 0$, $r > 0$, $q > 0$, and $r + q > p$, $p > 1$ are constants. Then

$$(2.41) \quad x(t) \leq \left[x_0^{\left[\frac{p-1}{p}\right]} + \left(\frac{p-1}{p}\right) \int_0^t f(s)k_4(s)ds \right]^{\left[\frac{1}{p-1}\right]}, \quad \forall t \in I,$$

where

$$(2.42) \quad k_4(t) = \frac{x_0^{\frac{qr}{p}} \exp\left(r \int_0^t g(s)ds\right)}{\left[1 - \frac{q[r+q-p]}{p} x_0^{\frac{q[r+q-p]}{p}} \int_0^t f(s) \exp([r+q-p] \int_0^s g(\lambda)d\lambda) ds \right]^{\left[\frac{r}{r+q-p}\right]}},$$

for all $t \in I$, such that

$$\frac{q[r+q-p]}{p} x_0^{\frac{q[r+q-p]}{p}} \int_0^t f(s) \exp([r+q-p] \int_0^s g(\lambda)d\lambda) ds < 1, \text{ for all } t \in I.$$

Proof. Let $J_3^p(t)$ equal the right hand side in (2.40), we have $J_3(0) = x_0^{\frac{1}{p}}$ and

$$(2.43) \quad x(t) \leq J_3(t), \quad \forall t \in I.$$

Differentiating $J_3^p(t)$, gives

$$(2.44) \quad pJ_3^{p-1} \frac{dJ_3(t)}{dt} \leq f(t)J_3(t)Y_3^r(t), \quad \forall t \in I,$$

where $Y_3(t) = J_3^q(t) + \int_0^t g(s)J_3(s)ds$, hence $Y_3(0) = J_3^q(0) = x_0^{\frac{q}{p}}$, and

$$(2.45) \quad J_3(t) \leq Y_3(t) \quad \forall t \in I.$$

Differentiating $Y_3(t)$ and using (2.44) and (2.45), leads to

$$(2.46) \quad \frac{dY_3(t)}{dt} \leq \frac{q}{p} f(t) Y_3^{[r+q-p+1]}(t) + g(t) Y_3(t), \quad \forall t \in I,$$

but $Y_3(t) > 0$ then we can write the inequality (2.46) in the following form

$$(2.47) \quad Y_3^{-[r+q-p+1]}(t) \frac{dY_3(t)}{dt} - g(t) Y_3^{-[r+q-p]}(t) \leq \frac{q}{p} f(t), \quad \forall t \in I.$$

If we let

$$(2.48) \quad Y_3^{-[r+q-p]}(t) = Z_3(t), \quad \forall t \in I,$$

we have

$Z_3(0) = Y_3^{-[r+q-p]}(0) = x_0^{-\frac{q[r+q-p]}{p}}$, and $Y_3^{-[r+q-p+1]}(t) \frac{dY_3(t)}{dt} = \frac{-1}{[r+q-p]} \frac{dZ_3}{dt}$, then we can write the inequality (2.47) as follows

$$\frac{dZ_3}{dt} + (r + q - p)g(t)Z_3(t) \geq \frac{-q[r + q - p]}{p} f(t), \quad \forall t \in I.$$

The above inequality implies an estimation for $Z_3(t)$ as in the following inequality

$$(2.49) \quad Z_3(t) \geq \frac{\left[1 - \frac{q[r+q-p]}{p} x_0^{\frac{q[r+q-p]}{p}} \int_0^t f(s) \exp\left([r + q - p] \int_0^s g(\lambda) d\lambda\right) ds \right]}{x_0^{\frac{q[r+q-p]}{p}} \exp\left((r + q - p) \int_0^t g(s) ds\right)},$$

for all $t \in I$, then from (2.48) and (2.49), we have

$$(2.50) \quad Y_3^r(t) \leq k_4(t), \quad \forall t \in I,$$

where $k_4(t)$ as defined in (2.42), thus from (2.50) in (2.44), we have

$$J_3^{[p-2]}(t) \frac{dJ_3(t)}{dt} \leq \frac{1}{p} f(t) k_4(t), \quad \forall t \in I.$$

Taking $t = s$ in the above inequality and integrating from 0 to t , produces

$$(2.51) \quad J_3(t) \leq \left[x_0^{\left[\frac{p-1}{p}\right]} + \left(\frac{p-1}{p}\right) \int_0^t f(s) k_4(s) ds \right]^{\left[\frac{1}{p-1}\right]}, \quad \forall t \in I.$$

Using (2.51) in (2.43), we get the required inequality in (2.41). The proof is complete. \square

Theorem 2.7. Let $x(t)$ be a real-valued positive continuous function and $f(t)$, $g(t)$ are non-negative real-valued continuous functions defined on $I = [0, \infty)$, and satisfy the inequality

$$(2.52) \quad x(t) \leq x_0 + \int_0^t f(s)x(s) \left[x^q(s) + \int_0^s g(\lambda)x(\lambda)d\lambda \right]^p ds,$$

for all $t \in I$, where $x_0 > 0$, $p > 0$, $q > 0$, and $p + q > 1$, are constants. Then

$$(2.53) \quad x(t) \leq x_0 \exp\left(\int_0^t f(s)k_5(s)ds\right), \quad t \in I,$$

where

$$(2.54) \quad k_5(t) = \frac{x_0^{pq} \exp\left(p \int_0^t g(s)ds\right)}{\left[1 - q[p + q - 1]x_0^{q[p+q-1]} \int_0^t f(s) \exp([p + q - 1] \int_0^s g(\lambda)d\lambda)ds\right]^{\frac{p}{p+q-1}}},$$

for all $t \in I$, such that

$$q[p + q - 1]x_0^{q[p+q-1]} \int_0^t f(s) \exp([p + q - 1] \int_0^s g(\lambda)d\lambda)ds < 1, \text{ for all } t \in I.$$

Proof. Let $J_4(t)$ equal the right hand side in (2.52), we have $J_4(0) = x_0$ and

$$(2.55) \quad x(t) \leq J_4(t), \quad \forall t \in I.$$

Differentiating $J_4(t)$, gives

$$(2.56) \quad \frac{dJ_4(t)}{dt} \leq f(t)J_4(t)Y_4^p(t), \quad \forall t \in I,$$

where $Y_4(t) = J_4^q(t) + \int_0^t g(s)J_4(s)ds$, hence $Y_4(0) = J_4^q(0) = x_0^q$, and

$$(2.57) \quad J_4(t) \leq Y_4(t) \quad \forall t \in I.$$

Differentiating $Y_4(t)$ and using (2.56) and (2.57), leads to

$$\frac{dY_4(t)}{dt} \leq qf(t)Y_4^{[p+q]}(t) + g(t)Y_4(t), \quad \forall t \in I,$$

but $Y_4(t) > 0$, then we have

$$(2.58) \quad Y_4^{-[p+q]}(t) \frac{dY_4(t)}{dt} - g(t)Y_4^{[1-(p+q)]}(t) \leq qf(t), \quad \forall t \in I.$$

If we let

$$(2.59) \quad Y_4^{[1-(p+q)]}(t) = Z_4(t), \quad \forall t \in I,$$

we have

$Z_4(0) = Y_4^{-[p+q-1]}(0) = x_0^{-q[p+q-1]}$, and $Y_4^{-[p+q]}(t) \frac{dY_4(t)}{dt} = \frac{-1}{[p+q-1]} \frac{dZ_4}{dt}$, then we can write the inequality (2.58) as follows

$$(2.60) \quad \frac{dZ_4}{dt} + (p + q - 1)g(t)Z_4(t) \geq -q(p + q - 1)f(t), \quad \forall t \in I.$$

The above inequality implies an estimation for $Z_4(t)$ as in the following inequality

$$(2.61) \quad Z_4(t) \geq \frac{\left[1 - q(p + q - 1)x_0^{q[p+q-1]} \int_0^t f(s) \exp([p + q - 1] \int_0^s g(\lambda)d\lambda) ds \right]}{x_0^{q[p+q-1]} \exp\left((p + q - 1) \int_0^t g(s) ds \right)},$$

for all $t \in I$, then from (2.59) and (2.61), we have

$$(2.62) \quad Y_4^p(t) \leq k_5(t), \quad \forall t \in I,$$

where $k_5(t)$ as defined in (2.54), thus from (2.62) in (2.56), we have

$$(2.63) \quad \frac{dJ_4(t)}{dt} \leq f(t)k_5(t), \quad \forall t \in I.$$

Taking $t = s$ in the above inequality and integrating from 0 to t , produces

$$(2.64) \quad J_4(t) \leq x_0 \exp\left(\int_0^t f(s)k_5(s) ds \right), \quad t \in I.$$

Using (2.64) in (2.55), we get the required inequality in (2.53).

The proof is complete. □

Remark 2.5. If we put $q = 1$, the inequality given in Theorem 2.7, reduces to Theorem 3.2 in [1].

Theorem 2.8. Let $x(t)$, $f(t)$ and $g(t)$ be real-valued positive continuous functions defined on $I = [0, \infty)$ and satisfy the inequality

$$(2.65) \quad x(t) \leq x_0 + \int_0^t f(s) \left[x^p(s) + \int_0^s g(\lambda)x^{[2p-1]}(\lambda)d\lambda \right]^p ds,$$

for all $t \in I$, where $x_0 > 0$, and $p \in (0, 1)$, are constants. Then

$$(2.66) \quad x(t) \leq x_0 + \int_0^t f(s)k_6(s), \quad \forall t \in I,$$

where

$$(2.67) \quad k_6(t) = \left[x_0^{2p[1-p]} + 2(1-p) \int_0^s [pf(\lambda) + g(\lambda)]d\lambda \right]^{\left[\frac{p}{2(1-p)} \right]} ds, \quad \forall t \in I.$$

Proof. Let $J_5(t)$ equal the right hand side in (2.65), we have $J_5(0) = x_0$ and

$$(2.68) \quad x(t) \leq J_5(t), \quad \forall t \in I.$$

Differentiating $J_5(t)$, gives

$$(2.69) \quad \frac{dJ_5(t)}{dt} \leq f(t)Y_5^p(t), \quad \forall t \in I,$$

where $Y_5(t) = J_5^p(t) + \int_0^t g(s)J_5^{[2p-1]}(s)ds$, thus we have $Y_5(0) = J_5^p(0) = x_0^p$ and

$$(2.70) \quad J_5(t) \leq Y_5(t), \quad \forall t \in I.$$

Differentiating $Y_5(t)$ and using (2.69), and (2.70), leads to

$$\frac{dY_5(t)}{dt} \leq [pf(t) + g(t)]Y_5^{[2p-1]}(t), \quad \forall t \in I,$$

then

$$Y_5^{[1-2p]}(t) \frac{dY_5(t)}{dt} \leq [pf(t) + g(t)], \quad \forall t \in I.$$

Taking $t = s$ in the last inequality and integrating both sides from 0 to t and using $Y_5^{2[1-p]}(0) = x_0^{2p[1-p]}$, gives

$$(2.71) \quad Y_5^p(t) \leq k_6(t), \quad \forall t \in I,$$

where $k_6(t)$ as defined in (2.67), thus from (2.71) in (2.69) we have

$$\frac{dJ_5(t)}{dt} \leq f(t)k_6(t), \quad \forall t \in I,$$

which implies the estimation for $J_5(t)$ as

$$(2.72) \quad J_5(t) \leq x_0 + \int_0^t f(s)k_6(s), \quad \forall t \in I.$$

Using (2.72) in (2.68), we get the required inequality in (2.66). The proof is complete. \square

Theorem 2.9. Let $x(t)$ be a real-valued positive continuous function and $f(t)$, $g(t)$ are non-negative real-valued continuous functions defined on $I = [0, \infty)$, and $n(t)$ be a positive monotonic non-decreasing continuous function on $I = [0, \infty)$ and satisfy the inequality

$$(2.73) \quad x(t) \leq n(t) + \int_0^t f(s) \left[x(s) + \int_0^s g(\lambda)x(\lambda)d\lambda \right]^p ds,$$

for all $t \in I$, where $p \in (0, 1)$. Then

$$(2.74) \quad x(t) \leq n(t) \left[1 + \int_0^t f(s)k_7(s)n^{[p-1]}(s) \right], \quad \forall t \in I,$$

where

$$(2.75) \quad k_7(t) \leq \exp\left(p(1-p) \int_0^t g(s)ds\right) \left[1 + (1-p) \int_0^t f(s)n^{[p-1]}(s) \right. \\ \left. \times \exp\left(- (1-p) \int_0^s g(\lambda)d\lambda\right) ds \right]^{\lceil \frac{p}{1-p} \rceil}, \quad \forall t \in I.$$

Proof. Since $n(t)$ is a positive monotonic nondecreasing function, we observe from the inequality (2.73)

$$\left[\frac{x(t)}{n(t)} \right] \leq 1 + \int_0^t f(s)n^{[p-1]}(s) \\ \times \left[\left(\frac{x(s)}{n(s)} \right) + \int_0^s g(\lambda) \left(\frac{x(\lambda)}{n(\lambda)} \right) d\lambda \right]^p ds, \quad \forall t \in I.$$

Let

$$(2.76) \quad m(t) = \frac{x(t)}{n(t)}, \quad m(0) \leq 1, \quad \forall t \in I.$$

Hence

$$m(t) \leq 1 + \int_0^t f(s)n^{[p-1]}(s) \left[m(s) + \int_0^s g(\lambda)m(\lambda)d(\lambda) \right]^p, \quad \forall t \in I.$$

Let

$$J_6(t) = 1 + \int_0^t f(s)n^{[p-1]}(s) \left[m(s) + \int_0^s g(\lambda)m(\lambda)d(\lambda) \right]^p, \quad \forall t \in I,$$

we can easily obtain $J_6(0) = 1$, and

$$(2.77) \quad m(t) \leq J_6(t), \quad \forall t \in I.$$

Differentiating $J_6(t)$, gives

$$(2.78) \quad \frac{dJ_6(t)}{dt} \leq f(t)n^{[p-1]}(t)Y_6^p(t), \quad \forall t \in I,$$

where $Y_6(t) = J_6(t) + \int_0^t g(s)J_6(s)d(s)$, $J_6(0) = 1$, and

$$(2.79) \quad J_6(t) \leq Y_6(t), \quad \forall t \in I.$$

Differentiating $Y_6(t)$, and using (2.78) and (2.79), leads to

$$\frac{dY_6(t)}{dt} = \frac{dJ_6(t)}{dt} + g(t)J_6(t) \leq f(t)n^{[p-1]}(t)Y_6^p(t) + g(t)Y_6(t), \quad \forall t \in I,$$

which implies the estimation for $Y_6(t)$, such that

$$(2.80) \quad Y_6^p(t) \leq k_7(t), \quad \forall t \in I,$$

where $k_7(t)$ is as given in (2.75), thus from (2.80) in (2.78), we have

$$\frac{dJ_6(t)}{dt} \leq f(t)n^{[p-1]}(t)k_7(t), \quad \forall t \in I,$$

the above inequality implies the following estimation for $J_6(t)$

$$(2.81) \quad J_6(t) \leq 1 + \int_0^t f(s)k_7(s)n^{[p-1]}ds, \quad \forall t \in I.$$

Using (2.81) in (2.77), we get

$$(2.82) \quad m(t) \leq 1 + \int_0^t f(s)k_7(s)n^{[p-1]}ds, \quad \forall t \in I.$$

We get the desired bound in (2.74) from (2.76) and (2.82). the proof is complete. \square

If we put $p = \frac{1}{2}$ in Theorem 2.9, we can easily drive the following corollary:

Corollary 2.3. Let $x(t)$ be a real-valued positive continuous function and $f(t)$, $g(t)$ are non-negative real-valued continuous functions defined on $I = [0, \infty)$, and $n(t)$ be a positive monotonic non-decreasing continuous function on $I = [0, \infty)$ and satisfy the inequality

$$x(t) \leq n(t) + \int_0^t f(s) \sqrt{x(s) + \int_0^s g(\lambda)x(\lambda)d\lambda} ds,$$

for all $t \in I$. Then

$$x(t) \leq n(t) \left[1 + \int_0^t \frac{f(s)k_8(s)}{\sqrt{n(s)}} ds \right], \quad \forall t \in I,$$

where

$$k_8(t) \leq \exp\left(\frac{1}{4} \int_0^t g(s)ds\right) \times \left[1 + \frac{1}{2} \int \frac{f(s)}{\sqrt{n(s)}} \exp\left(-\frac{1}{2} \int_0^s g(\lambda)d\lambda\right) ds \right], \quad \forall t \in I.$$

Theorem 2.10. *Let $x(t)$ be a real-valued positive continuous function and $f(t)$, $g(t)$ are non-negative real-valued continuous functions defined on $I = [0, \infty)$, and $n(t)$ be a positive monotonic non-decreasing continuous function on $I = [0, \infty)$ and satisfy the inequality*

$$(2.83) \quad x(t) \leq n(t) + \int_0^t f(s) \left[x(s) + \int_0^s g(\lambda) x^p(\lambda) d\lambda \right]^p ds,$$

for all $t \in I$, where $p > 1$. Then

$$(2.84) \quad x(t) \leq n(t) \left[1 + \int_0^t f(s) k_9(s) n^{[p-1]}(s) \right], \quad \forall t \in I,$$

where

$$(2.85) \quad k_9(t) \leq \left[1 + (1 - p) \int_0^t [f(s) n^{[p-1]}(s) + g(s) n(s)] ds \right]^{\left[\frac{p}{1-p} \right]}, \quad \forall t \in I.$$

Proof. Since $n(t)$ is a positive monotonic nondecreasing function, we observe from (2.83)

$$\left[\frac{x(t)}{n(t)} \right] \leq 1 + \int_0^t f(s) n^{[p-1]}(s) \left[\left(\frac{x(s)}{n(s)} \right) + \int_0^s g(\lambda) n^{[p-1]}(\lambda) \left(\frac{x(\lambda)}{n(\lambda)} \right)^p d\lambda \right]^p ds,$$

for all $t \in I$.

Let

$$(2.86) \quad m(t) = \frac{x(t)}{n(t)}, \quad m(0) \leq 1, \quad \forall t \in I.$$

Hence

$$m(t) \leq 1 + \int_0^t f(s) n^{[p-1]}(s) \left[m(t) + \int_0^s g(\lambda) n^{[p-1]}(\lambda) m^p(\lambda) \right]^p ds, \quad \forall t \in I.$$

Let

$$J_7(t) = 1 + \int_0^t f(s) n^{[p-1]}(s) \left[m(t) + \int_0^s g(\lambda) n^{[p-1]}(\lambda) m^p(\lambda) \right]^p ds, \quad \forall t \in I,$$

we can easily obtain $J_7(0) = 1$, and

$$(2.87) \quad m(t) \leq J_7(t), \quad \forall t \in I.$$

Differentiating $J_7(t)$, gives

$$(2.88) \quad \frac{dJ_7(t)}{dt} \leq f(t) n^{[p-1]}(t) Y_7^p(t), \quad \forall t \in I,$$

where $Y_7(t) = J_7(t) + \int_0^t g(s)n^{[p-1]}(s)Y_7^p(s)ds$, $Y_7(0) = J_7(0) = 1$, and

$$(2.89) \quad J_7(t) \leq Y_7(t), \quad \forall t \in I.$$

Differentiating $Y_7(t)$, and using (2.88) and (2.89), leads to

$$\frac{dY_7(t)}{dt} \leq [f(t)n^{[p-1]}(t) + g(t)n(t)]Y_7^p(t), \quad \forall t \in I,$$

which implies the estimation for $Y_7(t)$, such that

$$(2.90) \quad Y_7^p(t) \leq k_9(t), \quad \forall t \in I,$$

where $k_9(t)$ is as given in (2.85), thus from (2.90) in (2.88), we get

$$\frac{dJ_7(t)}{dt} \leq f(t)n^{[p-1]}(t)k_9(t), \quad \forall t \in I,$$

which implies the estimation for $J_7(t)$

$$(2.91) \quad J_7(t) \leq 1 + \int_0^t f(s)k_9(s)n^{[p-1]}(s)ds, \quad \forall t \in I.$$

Using (2.91) in (2.87) we get

$$(2.92) \quad m(t) \leq 1 + \int_0^t f(s)k_9(s)n^{[p-1]}(s)ds, \quad \forall t \in I.$$

The desired bound in (2.84) follows from (2.86) and (2.92).

The proof is complete. \square

3. Some Applications

In this section, we present some applications of the above results in order to illustrate the usefulness of this work. For instance, let us introduce some applications of the inequalities obtained in Theorems 2.2 and Theorem 2.3 in studying the boundedness and asymptotic behaviour of the solutions of nonlinear integral and integrodifferential equations.

We present the following example, as an application of the inequality obtained in Theorem 2.2:

Example 3.1. We discuss the boundedness and asymptotic behaviour of the solution of nonlinear integral equation of the form

$$(3.1) \quad x^p(t) = g(t) + \int_0^t K(t, s)H(s, x(s))ds, \quad \forall t \in I = [0, \infty),$$

where p as defined in Theorem 2.2, $g(t)$ is a positive real-valued continuous function defined on I and K, H are non-decreasing real-valued continuous functions defined

on $I \times I$. Here we assume that every solution $x(t)$ of (3.1) under discussion exists on I .

We list the following hypotheses on the functions g, K and H involved in (3.1):

$$(3.2) \quad |g(t)| \leq n^p(t); |K(t, s)| \leq 1; |H(t, x(t))| \leq f(t)|x^p(t)| + h(t)|x^q(t)|,$$

$$(3.3) \quad |g(t)| \leq n^p(t)e^{-pt}; |K(t, s)| \leq e^{-pt}; |H(t, x(t))| \leq f(t)|x^p(t)| + h(t)|x^q(t)|,$$

$$(3.4) \quad y_1(t)n(t) \times \left[1 + p_1 \int_0^t h(s)n^{-[p-q]}(s) \exp \left(-p_1 \int_0^s f(\lambda)d\lambda \right) ds \right]^{\left[\frac{1}{p-q} \right]} < \infty,$$

$$(3.5) \quad y_2(t)n(t) \times \left[1 + p_1 \int_0^t e^{-qs}h(s)n^{-[p-q]}(s) \exp \left(-p_1 \int_0^s e^{-p\lambda}f(\lambda)d\lambda \right) ds \right]^{\left[\frac{1}{p-q} \right]} < \infty,$$

for all $t \in I$, where $y_1(t) = \exp \left(\frac{1}{p} \int_0^t f(s)ds \right)$, $y_2(t) = \exp \left(\frac{1}{p} \int_0^t e^{ps}f(s)ds \right)$, $p_1 = \frac{p-q}{p}$ and $x_0, q, f(t), h(t)$ as defined in Theorem 2.2.

Suppose that the hypotheses (3.2) and (3.4) are satisfied, and let $x(t), t \in I$ be a solution of (3.1). Then from (3.1) and (3.2) we have

$$|x^p(t)| \leq n^p(t) + \int_0^t f(s)|x^p(s)|ds + \int_0^t h(s)|x^q(s)|ds, \quad \forall t \in I.$$

Now, a suitable application of the inequality given in Theorem 2.2 to yields

$$(3.6) \quad |x^p(t)| \leq y_1(t)n(t) \times \left[1 + p_1 \int_0^t h(s)n^{-[p-q]}(s) \exp \left(-p_1 \int_0^s f(\lambda)d\lambda \right) ds \right]^{\left[\frac{1}{p-q} \right]} < \infty,$$

for all $t \in I$ thus, from the hypotheses (3.4) and the estimation in (3.6) implies the boundedness of solution $x(t)$ of (3.1).

We now consider the (3.1) and the hypotheses (3.3) and (3.5), and let $x(t), t \in I$ be a solution of (3.1). Then from (3.1) and using (3.3) it is easily to observe that

$$(3.7) \quad [e^t|x(t)|]^p \leq n^p(t) + \int_0^t e^{-sp}f(s)[e^s|x^p(s)|]^p ds + \int_0^t e^{-sq}h(s)[e^s|x^q(s)|]^q ds,$$

for all $t \in I$. Now, a suitable application of the inequality given in Theorem 2.2 to (3.7) yields:

$$(3.8) \quad e^t|x(t)| \leq y_2(t)n(t) \times \left[1 + p_1 \int_0^t e^{-qs}h(s)n^{-[p-q]}(s) \exp \left(-p_1 \int_0^s e^{-p\lambda}f(\lambda)d\lambda \right) ds \right]^{\left[\frac{1}{p-q} \right]},$$

for all $t \in I$ thus, from (3.5) and (3.8) we observe that

$$(3.9) \quad |x(t)| \leq \alpha e^{-t}, \quad \forall t \in I,$$

where $\alpha \geq 0$ is constant. From (3.9) we see that the solution $x(t)$ approaches zero as $t \rightarrow \infty$.

Remark 3.1. In [1], Abdeldaim and Yakout taking $|g(t)| \leq x_0$, where x_0 is a constant. But in the above example we taking $|g(t)| \leq n^p(t)$ where $n(t)$ is a function which is a generalization.

As an application of the inequality given in Theorem 2.3 we present the following example :

Example 3.2. We discuss the boundedness of the solution of a nonlinear integro-differential equation of the form:

$$(3.10) \quad px^{[p-1]}(t) \frac{dx(t)}{dt} = F(t, x(t), y(t), \int_0^t K(s, x(s)) ds), x(0) = x_0,$$

for all $t \in I$, where p, x_0 as defined in Theorem 2.3, and $y(t), F, K$ be a nondecreasing real-valued continuous functions defined on $I \times I$. Here we assume that every solution $x(t)$ of (3.10) under discussion exists on I .

We list the following hypotheses on the functions $y(t), H, K$ involved in (3.10):

$$(3.11) \quad |y(t)| \leq f(t); |K(t, x(t))| \leq g(t)x(t),$$

$$(3.12) \quad |F(t, x(t), y(t), \int_0^t K(s, x(s)) ds)| \leq |y(t)|[|x^q(t)| + \int_0^t |K(s, x(s))| ds],$$

$$(3.13) \quad \left[x_0 + \int_0^t f(s)k_{10}(s) \exp\left(\int_0^s g(\lambda) d\lambda\right) ds \right]^{\frac{1}{p}} < \infty,$$

where

$$(3.14) \quad k_{10}(t) = \left[x_0^{q[p-q]} + \left[\frac{q[p-q]}{p} \right] \int_0^t f(s) \exp\left(-[p-q] \int_0^s g(\lambda) d\lambda\right) ds \right]^{\left[\frac{1}{p-q}\right]},$$

for all $t \in I$, where $f(t)$ as defined in Theorem 2.3.

By taking $t = s$ in (3.10) and integrating from 0 to t we have

$$(3.15) \quad x^p(t) = x_0^p + \int_0^t [F(s, x(s), y(s), \int_0^s K(\lambda, x(\lambda)) d\lambda)] ds.$$

Suppose that the hypotheses (3.11) and (3.12) are satisfied and from (3.15), it is easily to see that the solution $x(t)$ satisfies the equivalent integral equation

$$(3.16) \quad |x^p(t)| \leq x_0^p + \int_0^t f(s) \left[|x^q(s)| + \int_0^s g(\lambda) |x(\lambda)| d\lambda \right] ds.$$

Now, a suitable application of the inequality given in Theorem 2.3 to (3.16) yields

$$(3.17) \quad x(t) \leq \left[x_0 + \int_0^t f(s) k_{10}(s) \exp \left(\int_0^s g(\lambda) d\lambda \right) ds \right]^{\frac{1}{p}}, \quad \forall t \in I,$$

where $k_{10}(t)$ as defined in (3.14) thus, from the hypotheses (3.13) and the estimation in (3.17) implies the boundedness of the solution $x(t)$ of (3.10).

Acknowledgments. The authors would like to thank Dr. Ahmed Abdeldaim, for his valuable comments on this manuscript.

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