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On a New Ostrowski-Type Inequality and Related Results

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Abstract. We provide a new Ostrowski-type inequality involving functions of two independent variables, as well as some related results.

1. Introduction

The well-known Ostrowski inequality (see [3]) states that

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) M, \tag{1}$$

where $f : [a,b] \to \mathbb{R}$ is a differentiable function such that $|f'| \leq M$ for all $x \in [a,b]$.

This article is greatly motivated and inspired also by the following results.

Theorem 1.([1]) *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I, a < b$. $f : I \to \mathbb{R}$ is a differentiable function such that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \leq f'(x) \leq \Gamma, x \in [a,b]$. Then we have*

$$\begin{aligned} &\left| \frac{1}{2} f(x) - \frac{(x-b) f(b) - (x-a) f(a)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ \leq\;& \frac{(x-a)^2 + (b-x)^2}{8(b-a)} (\Gamma - \gamma) \end{aligned} \tag{2}$$

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for all $x \in [a,b]$.

Theorem 2.([10]) *Let* $f:[a,b]\to\mathbb{R}$ *be a twice differentiable mapping on* (a,b) *and suppose that* $\gamma \le f''(t) \le \Gamma$ *for all* $t\in(a,b)$. *Then we have the double inequality*

$$\frac{3S-\Gamma}{24}(b-a)^2 \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \le \frac{3S-\gamma}{24}(b-a)^2 \tag{3}$$

where $S=\frac{f'(b)-f'(a)}{b-a}$.

Theorem 3.([2]) *Under the assumptions of Theorem 2, we have*

$$\begin{aligned} &\frac{\Gamma\left[(x-a)^3+(b-x)^3\right]}{12(b-a)} + \frac{1}{8}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^2(S-\Gamma) \\ \le\ & \frac{1}{2}\left[f(x)+\frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(t)dt \\ \le\ & \frac{\gamma\left[(x-a)^3+(b-x)^3\right]}{12(b-a)} + \frac{1}{8}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]^2(S-\gamma), \end{aligned} \tag{4}$$

for all $x\in[a,b]$, *where* $S=\frac{f'(b)-f'(a)}{b-a}$.

Theorem 4.([7]) *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be an continuous function such that the partial derivative of order 2 exists and supposes that there exist constants* $\gamma,\Gamma\in\mathbb{R}$ *with* $\gamma\le\frac{\partial^2 f(t,s)}{\partial t\partial s}\le\Gamma$ *for all* $(t,s)\in[a,b]\times[c,d]$. *Then, we have*

$$\begin{aligned} &\left|\frac{1}{4}f(x,y)+\frac{1}{4}H(x,y)-\frac{1}{2(b-a)}\int_a^b f(t,y)dt-\frac{1}{2(d-c)}\int_c^d f(x,s)ds\right. \\ &-\frac{1}{2(b-a)(d-c)}\int_a^b \left[(y-c)f(t,c)+(d-y)f(t,d)\right]dt \\ &-\frac{1}{2(b-a)(d-c)}\int_c^d \left[(x-a)f(a,s)+(b-x)f(b,s)\right]ds \\ &\left.+\frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \\ \le\ & \frac{\left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right]}{32(b-a)(d-c)}(\Gamma-\gamma) \end{aligned} \tag{5}$$

for all $(x,y)\in[a,b]\times[c,d]$ *where*

$$
\begin{aligned}
&H(x,y)\\
=\;&\frac{(x-a)\left[(y-c)f(a,c)+(d-y)f(a,d)\right]}{(b-a)(d-c)}\\
&+\frac{(b-x)\left[(y-c)f(b,c)+(d-y)f(b,d)\right]}{(b-a)(d-c)}\\
&+\frac{(x-a)f(a,y)+(b-x)f(b,y)}{b-a}\\
&+\frac{(y-c)f(x,c)+(d-y)f(x,d)}{d-c}
\end{aligned}
$$

Theorem 5.([5]) *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be an absolutely continuous function such that the partial derivative of order* 2 *exists and suppose that there exist constants* $\gamma,\Gamma\in\mathbb{R}$ *with* $\gamma\le\frac{\partial^2 f(t,s)}{\partial t\partial s}\le\Gamma$ *for all* $(t,s)\in[a,b]\times[c,d]$. *Then, we have*

$$
\begin{aligned}
&\Bigg|(1-\lambda)^2 f(x,y)+\frac{\lambda}{2}(1-\lambda)\left[f(a,y)+f(b,y)+f(x,c)+f(x,d)\right]\\
&+\left(\frac{\lambda}{2}\right)^2\left[f(a,c)+f(b,c)+f(a,d)+f(b,d)\right]\\
&-\frac{1}{b-a}\left\{(1-\lambda)\int_a^b f(t,y)dt+\frac{\lambda}{2}\int_a^b\left[f(t,c)+f(t,d)\right]dt\right\}\\
&-\frac{1}{d-c}\left\{(1-\lambda)\int_c^d f(x,s)ds+\frac{\lambda}{2}\int_c^d\left[f(a,s)+f(b,s)\right]ds\right\}\\
&-\frac{\Gamma+\gamma}{2}(1-\lambda)^2\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)\\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\Bigg|\\
&\le\frac{\Gamma-\gamma}{2}\frac{1}{(b-a)(d-c)}\left[(\lambda^2+(1-\lambda)^2)\frac{(b-a)^2}{4}+\left(x-\frac{a+b}{2}\right)^2\right]\\
&\times\left[(\lambda^2+(1-\lambda)^2)\frac{(d-c)^2}{4}+\left(y-\frac{c+d}{2}\right)^2\right]
\end{aligned}
\tag{6}
$$

for all $(x,y) \in \left[a+\lambda\frac{b-a}{2}, b-\lambda\frac{b-a}{2}\right] \times \left[c+\lambda\frac{d-c}{2}, d-\lambda\frac{d-c}{2}\right]$ *and* $\lambda \in [0,1]$.

For the other recent Ostrowski type results, see [4],[6],[8] and [9].

The main purpose of this article is to establish a new inequality similar to the inequalities (2)-(6) for higher-order derivatives of f involving functions of two independent variables.

2. Main Results

Theorem 6. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be an continuous function such that the partial derivative of order 4 exists and supposes that there exist constants* $\gamma, \Gamma \in \mathbb{R}$ *with* $\gamma \leq \frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} \leq \Gamma$ *for all* $(t,s) \in [a,b] \times [c,d]$. *Then, we have*

$$\left| E(x,y) - \frac{\left[(x-a)^3 + (b-x)^3\right]\left[(y-c)^3 + (d-y)^3\right]}{288}(\Gamma+\gamma) \right|$$

$$\leq \quad \frac{\left[(x-a)^3 + (b-x)^3\right]\left[(y-c)^3 + (d-y)^3\right]}{288}(\Gamma-\gamma)$$

for all $(x,y) \in [a,b] \times [c,d]$ *,where*

$$\begin{aligned}
E(x,y) \quad = \quad & \frac{(x-a)(y-c)}{4}\left[f(x,y)+f(x,c)+f(a,y)+f(a,c)\right] \\
& + \frac{(d-y)(x-a)}{4}\left[f(x,y)+f(x,d)+f(a,y)+f(a,d)\right] \\
& + \frac{(y-c)(b-x)}{4}\left[f(x,y)+f(x,c)+f(b,y)+f(b,c)\right] \\
& + \frac{(d-y)(b-x)}{4}\left[f(x,y)+f(x,d)+f(b,y)+f(b,d)\right] \\
& - \frac{y-c}{2}\int_a^b \left[f(t,y)+f(t,c)\right]dt - \frac{d-y}{2}\int_a^b \left[f(t,d)+f(t,y)\right]dt \\
& - \frac{x-a}{2}\int_c^d \left[f(x,s)+f(a,s)\right]ds - \frac{b-x}{2}\int_c^d \left[f(b,s)+f(x,s)\right]ds \\
& + \int_a^b \int_c^d f(t,s)dsdt.
\end{aligned}$$

Proof. We first define the following kernel functions: $p : [a,b] \times [a,b] \to \mathbb{R}$ and

$q : [c,d] \times [c,d] \to \mathbb{R}$ given by

$$p(x,t) = \begin{cases} \frac{(x-t)(t-a)}{2}, & a \leq t \leq x \\ \frac{(x-t)(t-b)}{2}, & x < t \leq b \end{cases}$$

and

$$q(y,s) = \begin{cases} \frac{(y-s)(s-c)}{2}, & c \leq s \leq y \\ \frac{(y-s)(s-d)}{2}, & y < s \leq d \end{cases}.$$

By the definitions of these kernel functions, we can write

$$\begin{aligned} & \int_a^b \int_c^d p(x,t)q(y,s)\frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} dsdt \\ = \; & \frac{1}{4}\int_a^x \int_c^y (x-t)(t-a)(y-s)(s-c)\frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} dsdt \\ & + \frac{1}{4}\int_a^x \int_y^d (x-t)(t-a)(y-s)(s-d)\frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} dsdt \\ & + \frac{1}{4}\int_x^b \int_c^y (x-t)(t-b)(y-s)(s-c)\frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} dsdt \\ & + \frac{1}{4}\int_x^b \int_y^d (x-t)(t-b)(y-s)(s-d)\frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} dsdt. \end{aligned} \tag{1.1}$$

Integrating by parts in the right hand side of (1.1), we have

$$\begin{aligned} & \frac{1}{4}\int_a^x \int_c^y (x-t)(t-a)(y-s)(s-c)\frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} dsdt \\ & = \frac{(x-a)(y-c)}{4}[f(x,y)+f(x,c)+f(a,y)+f(a,c)] \\ & -\frac{y-c}{2}\int_a^x [f(t,y)+f(t,c)]dt - \frac{x-a}{2}\int_c^y [f(x,s)+f(a,s)]ds \\ & \int_a^x \int_c^y f(t,s)dsdt. \end{aligned} \tag{1.2}$$

$$\begin{aligned}
&\frac{1}{4}\int_a^x\int_y^d (x-t)(t-a)(y-s)(s-d)\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}dsdt\\
&=\frac{(d-y)(x-a)}{4}[f(x,y)+f(x,d)+f(a,y)+f(a,d)]\\
&-\frac{d-y}{2}\int_a^x[f(t,d)+f(t,y)]dt-\frac{x-a}{2}\int_y^d[f(x,s)+f(a,s)]ds\\
&+\int_a^x\int_y^d f(t,s)dsdt.
\end{aligned} \tag{1.3}$$

$$\begin{aligned}
&\frac{1}{4}\int_x^b\int_c^y (x-t)(t-b)(y-s)(s-c)\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}dsdt\\
&=\frac{(y-c)(b-x)}{4}[f(x,y)+f(x,c)+f(b,y)+f(b,c)]\\
&-\frac{y-c}{2}\int_x^b[f(t,y)+f(t,c)]dt-\frac{b-x}{2}\int_c^y[f(b,s)+f(x,s)]ds\\
&+\int_x^b\int_c^y f(t,s)dsdt.
\end{aligned} \tag{1.4}$$

$$\begin{aligned}
&\frac{1}{4}\int_x^b\int_y^d (x-t)(t-b)(y-s)(s-d)\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}dsdt\\
&=\frac{(d-y)(b-x)}{4}[f(x,y)+f(x,d)+f(b,y)+f(b,d)]\\
&-\frac{d-y}{2}\int_x^b[f(t,d)+f(t,y)]dt-\frac{b-x}{2}\int_y^d[f(b,s)+f(x,s)]ds\\
&+\int_x^b\int_y^d f(t,s)dsdt.
\end{aligned} \tag{1.5}$$

Adding (1.2)-(1.5), we easily deduce

$$
\begin{aligned}
&\int_a^b \int_c^d p(x,t)\, q(y,s)\, \frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} ds dt = \\
&\frac{(x-a)(y-c)}{4}\left[f(x,y)+f(x,c)+f(a,y)+f(a,c)\right] \\
&+\frac{(d-y)(x-a)}{4}\left[f(x,y)+f(x,d)+f(a,y)+f(a,d)\right] \\
&+\frac{(y-c)(b-x)}{4}\left[f(x,y)+f(x,c)+f(b,y)+f(b,c)\right] \\
&+\frac{(d-y)(b-x)}{4}\left[f(x,y)+f(x,d)+f(b,y)+f(b,d)\right] \\
&-\frac{y-c}{2}\int_a^b \left[f(t,y)+f(t,c)\right] dt-\frac{d-y}{2}\int_a^b \left[f(t,d)+f(t,y)\right] dt \\
&-\frac{x-a}{2}\int_c^d \left[f(x,s)+f(a,s)\right] ds-\frac{b-x}{2}\int_c^d \left[f(b,s)+f(x,s)\right] ds \\
&+\int_a^b \int_c^d f(t,s) ds dt.
\end{aligned} \tag{1.6}
$$

We also have

$$
\int_a^b \int_c^d p(x,t)\, q(y,s)\, ds dt = \frac{\left[(x-a)^3+(b-x)^3\right]\left[(y-c)^3+(d-y)^3\right]}{144} \tag{1.7}
$$

Let $M=\frac{\Gamma+\gamma}{2}$. From (1.6) and (1.7), we can write

$$
\begin{aligned}
&\int_a^b \int_c^d p(x,t)\, q(y,s)\left[\frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2}-M\right] ds dt \\
&= \int_a^b \int_c^d p(x,t)\, q(y,s)\, \frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} ds dt \\
&\quad -\frac{\Gamma+\gamma}{2}\,\frac{\left[(x-a)^3+(b-x)^3\right]\left[(y-c)^3+(d-y)^3\right]}{144}
\end{aligned} \tag{1.8}
$$

On the other hand, we get

$$\left|\int_a^b\int_c^d p(x,t)\,q(y,s)\left[\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}-M\right]dsdt\right| \tag{1.9}$$
$$\le \max_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}-M\right|\int_a^b\int_c^d |p(x,t)\,q(y,s)|\,dsdt$$

Since

$$\max_{(t,s)\in[a,b]\times[c,d]}\left|\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}-M\right|\le\frac{\Gamma-\gamma}{2} \tag{1.10}$$

and

$$\int_a^b\int_c^d |p(x,t)\,q(y,s)|\,dsdt=\frac{\left[(x-a)^3+(b-x)^3\right]\left[(y-c)^3+(d-y)^3\right]}{144}. \tag{1.11}$$

By using (1.10) and (1.11) in (1.9), we get

$$\left|\int_a^b\int_c^d p(x,t)\,q(y,s)\left[\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}-M\right]dsdt\right| \tag{1.12}$$
$$\le \frac{\left[(x-a)^3+(b-x)^3\right]\left[(y-c)^3+(d-y)^3\right]}{288}(\Gamma-\gamma).$$

From (1.8) and (1.12), we see that the required result holds. □

Corollaty 7. *By taking $x=a$ and $y=c$ or $x=b$ and $y=d$ in Theorem 6, we have*

$$\left|\frac{(d-c)(b-a)}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\right.$$
$$-\frac{d-c}{2}\int_a^b\left[f(t,c)+f(t,d)\right]dt-\frac{b-a}{2}\int_c^d\left[f(a,s)+f(b,s)\right]ds$$
$$\left.+\int_a^b\int_c^d f(t,s)dsdt-\frac{(b-a)^3(d-c)^3}{288}(\Gamma+\gamma)\right|$$
$$\le\frac{(b-a)^3(d-c)^3}{288}(\Gamma-\gamma).$$

Corollaty 8. *By taking $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in Theorem 6, we have*

$$\left| \frac{(b-a)(d-c)}{16} \left\{ 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \right.$$
$$+2\left[f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) + f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right) \right]$$
$$\left. +[f(a,c) + f(a,d) + f(b,c) + f(b,d)] \right\}$$
$$-\frac{d-c}{4} \int_a^b \left[f(t,d) + 2f\left(t, \frac{c+d}{2}\right) + f(t,c) \right] dt$$
$$-\frac{b-a}{2} \int_c^d \left[f(a,s) + 2f\left(\frac{a+b}{2}, s\right) + f(b,s) \right] ds$$
$$\left. + \int_a^b \int_c^d f(t,s) ds dt - \frac{(b-a)^3(d-c)^3}{16 \times 288} (\Gamma + \gamma) \right|$$
$$\leq \frac{(b-a)^3(d-c)^3}{16 \times 288} (\Gamma - \gamma).$$

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