KYUNGPOOK Math. J. 54(2014), 521-529
http://dx.doi.org/10.5666/KMJ.2014.54.4.521

Some Optimal Convex Combination Bounds for Arithmetic Mean

Gao Hongya* and Xue Ruihong
College of Mathematics and Computer Science, Hebei University, Baoding, 071002, P. R. China
e-mail : ghy@hbu.cn *and* xueruihong100@163.com

Abstract. In this paper we derive some optimal convex combination bounds related to arithmetic mean. We find the greatest values α_1 and α_2 and the least values β_1 and β_2 such that the double inequalities

$$\alpha_1 T(a,b) + (1-\alpha_1)H(a,b) < A(a,b) < \beta_1 T(a,b) + (1-\beta_1)H(a,b)$$

and

$$\alpha_2 T(a,b) + (1-\alpha_2)G(a,b) < A(a,b) < \beta_2 T(a,b) + (1-\beta_2)G(a,b)$$

holds for all $a,b>0$ with $a \neq b$. Here $T(a,b)$, $H(a,b)$, $A(a,b)$ and $G(a,b)$ denote the second Seiffert, harmonic, arithmetic and geometric means of two positive numbers a and b, respectively.

1. Introduction

For $a,b>0$ with $a \neq b$, the first and second Seiffert means $P(a,b)$ and $T(a,b)$ was introduced by Seiffert [1,2] as follows:

$$(1.1) \qquad P(a,b) = \frac{a-b}{4\arctan(\sqrt{a/b})-\pi} = \frac{a-b}{2\arcsin\frac{a-b}{a+b}}, \quad T(a,b) = \frac{a-b}{2\arctan\frac{a-b}{a+b}}.$$

Recently, both means P and T have been the subject of intensive research. In particular, many remarkable inequalities for P and T can be found in the literature [2-6].

Let $A(a,b) = (a+b)/2$, $G(a,b) = \sqrt{ab}$ and $H(a,b) = 2ab/(a+b)$ be the arithmetic, geometric and harmonic means of two positive real numbers a and b with $a \neq b$. Then

$$(1.2) \qquad \min\{a,b\} < H(a,b) < G(a,b) < P(a,b) < A(a,b) < T(a,b) < \max\{a,b\}.$$

* Corresponding Author.
Received June 9, 2011; accepted April 29, 2013.
2010 Mathematics Subject Classification: 26D15.
Key words and phrases: Optimal convex combination bound, arithmetic mean, harmonic mean, geometric mean, the second Seiffert mean.
This work was supported by NSF of Hebei Province (Grant No. A2011201011).

In [7], Seiffert proved

$$P(a,b) > \frac{3A(a,b)G(a,b)}{A(a,b)+2G(a,b)} \quad \text{and} \quad P(a,b) > \frac{2}{\pi}A(a,b),$$

for all $a, b > 0$ with $a \neq b$.

In [8], the authors found the greatest value α and the least value β such that the double inequality

$$\alpha A(a,b) + (1-\alpha)H(a,b) < P(a,b) < \beta A(a,b) + (1-\beta)H(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

For other useful inequalities, see [9-20].

The purpose of the present paper is to find the greatest values α_1 and α_2 and the least values β_1 and β_2 such that the double inequalities

$$\alpha_1 T(a,b) + (1-\alpha_1)H(a,b) < A(a,b) < \beta_1 T(a,b) + (1-\beta_1)H(a,b)$$

and

$$\alpha_2 T(a,b) + (1-\alpha_2)G(a,b) < A(a,b) < \beta_2 T(a,b) + (1-\beta_2)G(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

2. Main Results

The first result in this paper is an optimal convex combination bounds of the second Seiffert and harmonic means for arithmetic mean.

Theorem 2.1. *The double inequality* $\alpha_1 T(a,b) + (1-\alpha_1)H(a,b) < A(a,b) < \beta_1 T(a,b) + (1-\beta_1)H(a,b)$ *holds for all* $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_1 \leqslant \frac{3}{4}$ *and* $\beta_1 \geqslant \frac{\pi}{4}$.

Proof. Firstly, we prove that

$$A(a,b) < \frac{\pi}{4}T(a,b) + \left(1 - \frac{\pi}{4}\right)H(a,b), \tag{2.1}$$

$$A(a,b) > \frac{3}{4}T(a,b) + \frac{1}{4}H(a,b), \tag{2.2}$$

for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume $a > b$. Let $t = a/b > 1$ and $p \in \{\frac{3}{4}, \frac{\pi}{4}\}$. Then (1.1) leads to

$$\begin{aligned} &\frac{1}{b}\{A(a,b) - [pT(a,b) + (1-p)H(a,b)]\} \\ = \; & A(t,1) - [pT(t,1) + (1-p)H(t,1)] \\ = \; & \frac{t^2 + 2(2p-1)t + 1}{2(t+1)\arctan\frac{t-1}{t+1}} f(t), \end{aligned} \tag{2.3}$$

where

$$f(t) = \arctan \frac{t-1}{t+1} - \frac{p(t^2-1)}{t^2+2(2p-1)t+1}. \tag{2.4}$$

Simple computations lead to

$$\lim_{t\to 1^+} f(t) = 0, \quad \lim_{t\to +\infty} f(t) = \frac{\pi}{4} - p. \tag{2.5}$$

$$\begin{aligned} f'(t) &= \frac{(-4p^2+2p+1)t^4+4(p-1)t^3+2(4p^2-6p+3)t^2+4(p-1)t+(-4p^2+2p+1)}{(1+t^2)[t^2+2(2p-1)t+1]^2} \\ &= \frac{(t-1)^2 g(t)}{(1+t^2)[t^2+2(2p-1)t+1]^2}, \end{aligned} \tag{2.6}$$

where

$$g(t) = (-4p^2+2p+1)t^2 - 2(4p^2-4p+1)t + (-4p^2+2p+1). \tag{2.7}$$

Now we distinguish between two cases:
case 1 $p = \frac{3}{4}$. In this case,

$$g(t) = \frac{1}{4}(t^2-2t+1) = \frac{1}{4}(t-1)^2 > 0, \text{ for } t > 1. \tag{2.8}$$

Therefore, inequality (2.2) follows from (2.3)-(2.7). Notice that in this case, the second equality in (2.5) becomes

$$\lim_{t\to +\infty} f(t) = \frac{\pi}{4} - \frac{3}{4} > 0.$$

case 2 $p = \frac{\pi}{4}$. From (2.7) we have

$$\lim_{t\to 1^+} g(t) = 4p(3-4p) = \pi(3-\pi) < 0, \quad \lim_{t\to +\infty} g(t) = +\infty, \tag{2.9}$$

$$g'(t) = 2(-4p^2+2p+1)t - 2(4p^2-4p+1), \tag{2.10}$$

$$\lim_{t\to 1^+} g'(t) = \pi(3-\pi) < 0, \quad \lim_{t\to +\infty} g'(t) = +\infty, \tag{2.11}$$

$$g''(t) = 2(-4p^2+2p+1) = \frac{1}{2}(-\pi^2+2\pi+4) > 0, \tag{2.12}$$

From (2.12) we clearly see that $g'(t)$ is increasing for $t > 1$, which together with (2.11) implies that there exists $\lambda_1 > 1$ such that $g'(t) < 0$ for $t \in (1, \lambda_1)$ and $g'(t) > 0$ for $t \in (\lambda_1, +\infty)$. Hence $g(t)$ is strictly decreasing for $t \in (1, \lambda_1)$ and strictly increasing for $t \in (\lambda_1, +\infty)$. (2.9) implies that there exists $\lambda_2 > 1$ such that $g(t) < 0$ for $t \in (1, \lambda_2)$ and $g(t) > 0$ for $t \in (\lambda_2, +\infty)$. This result together with (2.6) implies that $f(t)$ is strictly decreasing for $t \in (1, \lambda_2)$ and strictly increasing for $t \in (\lambda_2, +\infty)$. Notice that if $p = \pi/4$, then the second equality in (2.5) becomes

$$\lim_{t \to +\infty} f(t) = 0.$$

Thus $f(t) < 0$ for all $t > 1$. Inequality (2.1) follows.

Secondly, we prove that $\frac{3}{4}T(a, b) + \frac{1}{4}H(a, b)$ is the best possible lower convex combination bound of the second Seiffert and harmonic means for arithmetic mean.

If $\alpha_1 > \frac{3}{4}$, then (2.7) (with α_1 in place of p) leads to

$$\lim_{t \to 1^+} g(t) = 4\alpha_1(3 - 4\alpha_1) < 0.$$

From this result and the continuity of $g(t)$ we clearly see that there exists $\delta = \delta(\alpha_1) > 0$ such that $g(t) < 0$ for $t \in (1, 1 + \delta)$. Then (2.6) implies $f'(t) < 0$ for $t \in (1, 1 + \delta)$. Thus $f(t)$ is decreasing for $t \in (1, 1 + \delta)$. Since (2.5), then $f(t) < 0$ for $t \in (1, 1 + \delta)$, which is equivalent to, by (2.3), that

$$A(t, 1) < \alpha_1 T(t, 1) + (1 - \alpha_1)H(t, 1),$$

for $t \in (1, 1 + \delta)$.

Finally, we prove that $\frac{\pi}{4}T(a, b)+(1-\frac{\pi}{4})H(a, b)$ is the best possible upper convex combination bound of the second Seiffert and harmonic means for arithmetic mean.

If $\beta_1 < \frac{\pi}{4}$, then from (1.1) one has

$$\begin{aligned} &\lim_{t \to +\infty} \frac{\beta_1 T(t, 1) + (1 - \beta_1)H(t, 1)}{A(t, 1)} \\ = &\lim_{t \to +\infty} \frac{\beta_1(t^2 - 1) + 4(1 - \beta_1)t \arctan \frac{t-1}{t+1}}{(t + 1)^2 \arctan \frac{t-1}{t+1}} = \frac{4\beta_1}{\pi} < 1. \end{aligned} \tag{2.13}$$

Inequality (2.13) implies that for any $\beta_1 < \frac{\pi}{4}$ there exists $X = X(\beta_1) > 1$ such that

$$\beta_1 T(t, 1) + (1 - \beta_1)H(t, 1) < A(t, 1)$$

for $t \in (X, +\infty)$. □

The second result in this paper is an optimal convex combination bounds of the second Seiffert and geometric means for arithmetic Mean.

Theorem 2.2. *The double inequality $\alpha_2 T(a, b) + (1 - \alpha_2)G(a, b) < A(a, b) < \beta_2 T(a, b) + (1 - \beta_2)G(a, b)$ holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leqslant \frac{3}{5}$ and $\beta_2 \geqslant \frac{\pi}{4}$.*

Proof. Firstly, we prove that

$$A(a,b) < \frac{\pi}{4}T(a,b) + \left(1 - \frac{\pi}{4}\right) G(a,b) \tag{2.14}$$

$$A(a,b) > \frac{3}{5}T(a,b) + \frac{2}{5}G(a,b) \tag{2.15}$$

for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume $a > b$. Let $t = \sqrt{a/b} > 1$ and $p \in \{\frac{3}{5}, \frac{\pi}{4}\}$. Then (1.1) leads to

$$\begin{aligned} & \frac{1}{b}\left\{A(a,b) - [pT(a,b) + (1-p)G(a,b)]\right\} \\ = \; & A(t^2,1) - [pT(t^2,1) + (1-p)G(t^2,1)] \\ = \; & \frac{t^2 + 2(p-1)t + 1}{2\arctan\frac{t^2-1}{t^2+1}} f(t), \end{aligned} \tag{2.16}$$

where

$$f(t) = \arctan\frac{t^2-1}{t^2+1} - \frac{p(t^2-1)}{t^2 + 2(p-1)t + 1}. \tag{2.17}$$

Simple computations lead to

$$\lim_{t\to 1^+} f(t) = 0, \quad \lim_{t\to+\infty} f(t) = \frac{\pi}{4} - p, \tag{2.18}$$

$$f'(t) = \frac{h(t)}{(1+t^4)[t^2 + 2(p-1)t + 1]^2} = \frac{(t-1)^2 g(t)}{(1+t^4)[t^2 + 2(p-1)t + 1]^2}, \tag{2.19}$$

where

$$\begin{aligned} h(t) = \; & 2p(-p+1)t^6 + 2(-2p+1)t^5 + 2(-p^2+5p-4)t^4 + 4(2p^2-4p+3)t^3 \\ & + 2(-p^2+5p-4)t^2 + 2(-2p+1)t + 2p(-p+1) \end{aligned}$$

and

$$g(t) = 2p(-p+1)t^4 + 2(-2p^2+1)t^3 + 4(-2p^2+2p-1)t^2 + 2(-2p^2+1)t + 2p(-p+1). \tag{2.20}$$

It is easy to see that

$$\lim_{t\to 1^+} g(t) = 4p(3-5p), \lim_{t\to+\infty} g(t) = +\infty, \tag{2.21}$$

$$g'(t) = 8p(1-p)t^3 + 6(-2p^2+1)t^2 + 8(-2p^2+2p-1)t + 2(-2p^2+1), \tag{2.22}$$

$$\lim_{t\to 1^+} g'(t) = 8p(3-5p), \lim_{t\to+\infty} g(t) = +\infty, \tag{2.23}$$

$$g''(t) = 24p(1-p)t^2 + 12(-2p^2+1)t + 8(-2p^2+2p-1), \tag{2.24}$$

$$\lim_{t\to 1^+} g''(t) = 4(-16p^2+10p+1), \lim_{t\to+\infty} g''(t) = +\infty, \tag{2.25}$$

$$g'''(t) = 48p(1-p)t + 12(-2p^2+1). \tag{2.26}$$

Now we distinguish between two cases.
case 1 $p = \frac{3}{5}$. It follows from (2.21), (2.23), (2.25) and (2.26) that

$$\lim_{t\to 1^+} g(t) = 0, \quad \lim_{t\to+\infty} g(t) = +\infty, \tag{2.27}$$

$$\lim_{t\to 1^+} g'(t) = 0, \quad \lim_{t\to+\infty} g'(t) = +\infty, \tag{2.28}$$

$$\lim_{t\to 1^+} g''(t) = \frac{124}{25} > 0, \quad \lim_{t\to+\infty} g''(t) = +\infty, \tag{2.29}$$

$$g'''(t) = \frac{12}{25}(24t+7) > 0, \tag{2.30}$$

From (2.30) we clearly see that $g''(t)$ is strictly increasing for $t > 1$, which together with (2.29) implies that $g''(t) > 0$ for all $t > 1$. Thus $g'(t)$ is strictly increasing for $t > 1$. From (2.28) we get $g'(t) > 0$ for all $t > 1$. Therefore $g(t)$ is strictly increasing for $t > 1$. (2.27) implies that $g(t) > 0$ for all $t > 1$. Thus from (2.19) we clearly see that $f'(t) > 0$ for $t > 1$, from which one has $f(t)$ is strictly increasing for $t > 1$. Notice that the second equality in (2.18) becomes

$$\lim_{t\to+\infty} f(t) = \frac{\pi}{4} - \frac{3}{5} > 0.$$

Hence $f(t) > 0$ and (2.15) follows from (2.18) and (2.16).
case 2 $p = \frac{\pi}{4}$. From (2.21), (2.23), (2.25) and (2.26) we have

$$\lim_{t\to 1^+} g(t) = \pi(3-5\pi/4) < 0, \quad \lim_{t\to+\infty} g(t) = +\infty, \tag{2.31}$$

$$\lim_{t\to 1^+} g'(t) = 2\pi(3 - 5\pi/4) < 0, \quad \lim_{t\to +\infty} g'(t) = +\infty, \tag{2.32}$$

$$\lim_{t\to 1^+} g''(t) = 4(-\pi^2 + \frac{5\pi}{2} + 1) < 0, \quad \lim_{t\to +\infty} g''(t) = +\infty, \tag{2.33}$$

$$\lim_{t\to 1^+} g'''(t) = -\frac{9\pi^2}{8} + \frac{7\pi}{4} + 3 > 0, \quad \lim_{t\to +\infty} g'''(t) = +\infty, \tag{2.34}$$

Since

$$g^{(4)}(t) = 3\pi(4 - \pi) > 0, \tag{2.35}$$

then we clearly see that $g'''(t)$ is strictly increasing for $t > 1$, which together with (2.34) implies that $g'''(t) > 0$ for $t > 1$. Thus $g''(t)$ is strictly increasing for $t > 1$. From (2.33), we derive that there exists $\lambda_3 > 1$ such that $g''(t) < 0$ for $t \in (1, \lambda_3)$ and $g''(t) > 0$ for $t \in (\lambda_3, +\infty)$. Hence $g'(t)$ is strictly decreasing for $t \in (1, \lambda_3)$ and strictly increasing for $t \in (\lambda_3, +\infty)$. From (2.32), there exists $\lambda_4 > 1$ such that $g'(t) < 0$ for $t \in (1, \lambda_4)$ and $g'(t) > 0$ for $t \in (\lambda_4, +\infty)$. Thus $g(t)$ is strictly decreasing for $t \in (1, \lambda_4)$ and strictly increasing for $t \in (\lambda_4, +\infty)$. (2.31) implies that there exists $\lambda_5 > 1$ such that $g(t) < 0$ for $t \in (1, \lambda_5)$ and $g(t) > 0$ for $t \in (\lambda_5, +\infty)$. (2.19) implies that $f(t)$ is strictly decreasing for $t \in (1, \lambda_5)$ and strictly increasing for $t \in (\lambda_5, +\infty)$. Notice that in this case, the second equality in (2.18) becomes

$$\lim_{t\to +\infty} f(t) = 0.$$

Thus $f(t) < 0$ for all $t > 1$, and (2.14) follows.

Secondly, we prove that $\frac{3}{5}T(a, b) + \frac{2}{5}G(a, b)$ is the best possible lower convex combination bound of the second Seiffert and geometric means for arithmetic mean.

If $\alpha_2 > \frac{3}{5}$, then (2.21) (with α_2 in place of p) leads to

$$\lim_{t\to 1^+} g(t) = 4\alpha_2(3 - 5\alpha_2) < 0. \tag{2.36}$$

From (2.36) and the continuity of $g(t)$ we see that there exists $\delta = \delta(\alpha_2) > 0$ such that

$$g(t) < 0 \tag{2.37}$$

for $t \in (1, 1 + \delta)$. Then (2.19) and the first equality of (2.18) imply that

$$f(t) < 0 \tag{2.38}$$

for $t \in (1, 1 + \delta)$. Therefore, by (2.16), $A(t^2, 1) < \alpha_2 T(t^2, 1) + (1 - \alpha_2)G(t^2, 1)$ for $t \in (1, 1 + \delta)$.

Finally, we prove that $\frac{\pi}{4}T(a,b)+(1-\frac{\pi}{4})G(a,b)$ is the best possible upper convex combination bound of the second Seiffert and geometric means for arithmetic mean.

If $\beta_2 < \frac{\pi}{4}$, then from (1.1) one has

$$\begin{aligned} & \lim_{t\to+\infty} \frac{\beta_2 T(t,1)+(1-\beta_2)G(t,1)}{A(t,1)} \\ = & \lim_{t\to+\infty} \frac{\beta_2(t-1)+2(1-\beta_2)\sqrt{t}\arctan\frac{t-1}{t+1}}{(t+1)\arctan\frac{t-1}{t+1}} = \frac{4\beta_2}{\pi} < 1. \end{aligned} \tag{2.39}$$

Inequality (2.39) implies that for any $\beta_2 < \frac{\pi}{4}$ there exists $X = X(\beta_2) > 1$ such that

$$\beta_2 T(t,1)+(1-\beta_2)G(t,1) < A(t,1)$$

for $t \in (X, +\infty)$. □

References

[1] H. J. Seiffert, Problem 887, *Nieuw Archief voor Wiskunde*, **11**(1993), 176-176.

[2] H. J. Seiffert, *Aufgabe* β16, Die Wurzel, **29**(1995), 221-222.

[3] P. A. Hästö, *Optimal inequalities between Seiffert's mean and power mean*, Mathematical Inequalities and Applications, **7**(2004), 47-53.

[4] E. Neuman and J. Sändor, *On certain means of two arguments and their extensions*, International Journal of Mathematics and Mathematical Sciences, **16**(2003), 981-993.

[5] E. Neuman and J. Sändor, *On the Schwab-Borchardt mean*, Mathematica Pannonica, **14**(2003), 253-266.

[6] P. A. Hästö, *A monotonicity property of ratios of symmetric homogeneous means*, Journal of Inequalities in Pure and Applied Mathematics, **3**(2002), 1-54.

[7] H. J. Seiffert, *Ungleichungen für einen bestimmten mittelwert*, Nieuw Archief voor Wiskunde, **13**(1995), 195-198.

[8] Y. M. Chu, Y. F. Qiu, M. K. Wang and G. D. Wang, *The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert's mean*, Journal of Inequalities and Applications, Article ID 436457, dio: 10.1155/436457, 7 pages, 2010.

[9] M. K. Wang, Y. M. Chu and Y. F. Qiu, *Some comparison inequalities for generalized Muirhead and identric means*, Journal of Inequalities and Applications, **2010**, Article ID 295620,10 pages, 2010.

[10] B. Y. Long and Y. M. Chu, *Optimal inequalities for generalized logarithmic, arithmetic and geometric means*, Journal of Inequalities and Applications, **2010**, Article ID 806825, 10 pages, 2010.

[11] B. Y. Long and Y. M. Chu, *Optimal power mean bounds for the weighted geometric mean of classical means*, Journal of Inequalities and Applications, **2010**, Article ID 905679, 6 pages, 2010.

[12] W. F. Xia, Y. M. Chu and G. D. Wang, *The optimal upper and lower power mean bounds for a convex combination of the arithmetic and logarithmic means*, Abstract and Applied Analysis, **2010**, Article ID604804, 9 pages, 2010.

[13] Y. M. Chu and B. Y. Long, *Best possible inequalities between generalized logarithmic mean and classical means*, Abstract and Applied Analysis, **2010**, Article ID 303286, 13 pages, 2010.

[14] M. Y. Shi, Y. M. Chu and Y. P. Jiang, *Optimal inequalities among various means of two arguments*, Abstract and Applied Analysis, **2009**, Article ID 694394, 10 pages, 2009.

[15] Y. M. Chu and W. F. Xia, *Two sharp inequalities for power mean, geometric mean and harmonic mean*, Journal of Inequalities and Applications, **2009**, Article ID 741923, 6 pages, 2009.

[16] Y. M. Chu and W. F. Xia, *Inequalities for generalized logarithmic means*, Journal of Inequalities and Applications, **2009**, Article ID 763252, 7 pages, 2009.

[17] J. J. Wen and W. L. Wang, *The optimization for the inequalities of power means*, Journal of Inequalities and Applications, **2006**, Article ID 46782, 25 pages, 2006.

[18] T. Hara, M. Uchiyama and S. E. Takahasi, *A refinement of various mean inequalities*, Journal of Inequalities and Applications, **2**(1998), 387-395.

[19] E. Neuman and J. Sändor, *On the Schwab-Borchardt mean*, Mathematica Pannonica, **17**(2006), 49-59.

[20] A. A. Jagers, *Solution of problem 887*, Nieuw Archief voor Wiskunde, **12**(1994), 230-231.