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LEGENDRE MULTIWAVELET GALERKIN METHODS FOR DIFFERENTIAL EQUATIONS†

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Abstract. The multiresolution analysis for Legendre multiwavelets are given, anti-derivatives of Legendre multiwavelets are used for the numerical solution of differential equations, a special form of multilevel augmentation method algorithm is proposed to solve the disrete linear system efficiently, convergence rate of the Galerkin methods is given and numerical examples are presented.

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1. Introduction

The idea of applying wavelet bases to discretize differential equations in wavelets-Galerkin methods has been explored by many authors [6, 10, 11, 14]. Among them, Xu and Shann [14] presented a thorough study of one dimensional problems. Instead of using wavelets directly, they took the anti-derivatives of wavelets as trial functions. In this way, singularity of wavelets is smoothed, the boundary condition can be treated easily. Since the introduction of multiwavelets into the numerical solution of the integral equation [2], multiwavelets bases have also been applied to discretize the differential equations because some difficulties of using wavelets for the representation of differential operators may be eliminated by using multiwavelets that may possess all the following properties: orthogonality on a finite interval, symmetry, compact support without overlap and high order of vanishing moments. In recent years, Legendre wavelets and multiwavelets, which are not differentiable on [0,1], has drawn a lot of attention in this direction [3, 9, 12, 13]. Recent developments include a special multilevel augmentation method (MAM), which was proposed by Chen, Wu and Xu [5] to

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solve some linear system of equations arising from discretizing differential equations that requires the use of special multiscale bases. Under certain conditions it leads to an efficient, stable and accurate solver for the discrete linear system. The most important point is that this MAM is not an iterative method.

In this paper we shall study Legendre multiwavelets-Galerkin methods based on variational principles. We shall take anti-derivatives of multiwavelets as trial functions as in [14], and then we show that the problem of evaluating the integrals of multiwavelets (cf. [7, 11]) can be resolved by using the properties of Legendre polynomials. The convergence rate of the method is given, and MAM algorithm [5] is applied for the two-point boundary value problem. The paper is organized as follows. Section 2 reviews the constructions of Legendre multiscaling functions and multiwavelets. Section 3 defines the anti-derivatives of Legendre multiscaling functions and multiwavelets, and a general operational matrix of integration is derived by using a derivative formula of Legendre polynomials. Section 4 discusses how to calculate the product matrix of basis functions and resolve the problem of evaluating the integrals of multiwavelets. In section 5 we use the anti-derivatives of Legendre multiwavelets as trial functions in the Galerkin methods for the two-point boundary value problems of ODEs, and propose a special form of MAM algorithm to solve the discrete linear system of equations. In section 6 we use the tensor product of Legendre multiwavelet basis to solve a Dirichlet problem for the elliptic equation on a rectangle. Finally, section 7 presents some numerical examples and conclusion.

2. Legendre multiwavelets

Let $L^2[0,1]$ be the Hilbert space equipped with the inner product

$$< u, v >:= \int_{[0,1]} u(x)v(x)dx, u, v \in L^2[0,1]$$

and the norm

$$||v|| := \sqrt{< v, v >}, v \in L^2[0,1].$$

We will define a multiresolution approximation of $L^2[0,1]$ of multiplicity r generated by Legendre multiwavelets (cf. [2, 3, 9]).

2.1. Legendre multiscaling functions

The Legendre polynomial is given by the following Rodrigues formula:

$$P_0(x) := 1, P_m(x) := \frac{1}{2^m m!}\frac{d^m}{dx^m}[(x^2-1)^m], x \in [-1,1], m = 1, 2, ... \qquad (2.1)$$

Clearly $P_m(x)$ is a polynomial of degree m. It has the following properties on the interval [-1,1] (cf. [1]):

(i) orthogonality:

$$< P_m, P_n >= \int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0, & m \neq n, \\ \frac{2}{2n+1}, & m = n \end{cases}$$

(ii) recurrence relation:

$$(m+1)P_{m+1}(x) = (2m+1)xP_m(x) - mP_{m-1}(x), m = 1, 2, \cdots$$

(iii) derivative formula:

$$(2m+1)P_m(x) = P'_{m+1}(x) - P'_{m-1}(x), m = 1, 2, \cdots ;$$

(iv) symmetry(or asymmetry) about $x = 0$: $P_m(-x) = (-1)^m P_m(x)$;
(v) function values at endpoints: $P_m(1) = 1, P_m(-1) = (-1)^m$.

Legendre multiscaling functions are defined as follows:

$$\varphi^m(x) = \begin{cases} \sqrt{2m-1}P_{m-1}(2x-1), & x \in [0,1], \\ 0, & otherwise \end{cases}, m = 1, 2, \cdots . \tag{2.2}$$

Then they have the following properties on the interval [0,1]:

(i) orthonormality:

$$< \varphi^m, \varphi^n >= \int_0^1 \varphi^m(x)\varphi^n(x)dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n \end{cases} \tag{2.3}$$

(ii) recurrence relation:

$$\frac{m}{\sqrt{4m^2-1}}\varphi^{m+1}(x) = (2x-1)\varphi^m(x) - \frac{m-1}{\sqrt{4(m-1)^2-1}}\varphi^{m-1}(x), m = 2, 3, \cdots ; \tag{2.4}$$

(iii) derivative formula:

$$\varphi^m(x) = \frac{1}{2\sqrt{4m^2-1}}\frac{d\varphi^{m+1}(x)}{dx} - \frac{1}{2\sqrt{4(m-1)^2-1}}\frac{d\varphi^{m-1}(x)}{dx}, m = 2, 3, \cdots ; \tag{2.5}$$

(iv) symmetry(or asymmetry) about $x = \frac{1}{2}$:

$$\varphi^m(1-x) = (-1)^{m-1}\varphi^m(x), \ m = 1, 2, \cdots ; \tag{2.6}$$

(v) function values at endpoints:

$$\varphi^m(1) = \sqrt{2m-1}, \varphi^m(0) = (-1)^{m-1}\sqrt{2m-1}, m = 1, 2, \cdots . \tag{2.7}$$

Now let $r > 1$ be an integer, and take the first r multiscaling functions $\varphi^1, \varphi^2, \cdots, \varphi^r$, which form an orthonormal bases of the function space

$$V_0 = \overline{span\{\varphi^1, \varphi^2, \cdots, \varphi^r\}}.$$

Let $\vec{\varphi}$ be the vector of the multiscaling functions:

$$\vec{\varphi} = (\varphi^1, \varphi^2, \cdots, \varphi^r)^T,$$

and denote for integers $j > 0$ and $k = 0, 1, \cdots, 2^j - 1$

$$\vec{\varphi}_{jk}(x) = (\varphi^1_{jk}(x), \varphi^2_{jk}(x), \cdots, \varphi^r_{jk}(x))^T$$

the dilates and translates of the multiscaling functions, where

$$\varphi^m_{jk}(x) = 2^{j/2}\varphi^m(2^jx - k)(m = 1, 2, \cdots, r)$$

is supported on the interval $[2^{-j}k, 2^{-j}(k+1)] \subset [0,1]$. For a fixed $j > 0$, all elements $\{\varphi_{jk}^m(x)\}$ of the vector

$$\Phi_j(x) = (\vec{\varphi}_{j0}^T(x), \vec{\varphi}_{j1}^T(x), \cdots, \vec{\varphi}_{j\ell}^T(x))^T \ (\ell = 2^j - 1) \tag{2.8}$$

form an orthonormal bases of the $2^j r$-dimensional space

$$V_j = \overline{span\{\vec{\varphi}_{j0}^T(x), \vec{\varphi}_{j1}^T(x), \cdots, \vec{\varphi}_{j\ell}^T(x)\}}. \tag{2.9}$$

then one has (i) $V_j \subset V_{j+1}, j \geq 0$; (ii) $\overline{\cup_{j\geq 0} V_j} = L^2[0,1]$; (iii) $f(\cdot) \in V_j \Leftrightarrow f(2\cdot) \in V_{j+1}$; and (iv) r functions $\varphi^1, \varphi^2, \cdots, \varphi^r$ form an orthonormal bases of the space V_0. Therefore, $\{V_j\}_{j\geq 0}$ is an orthonormal multiresolution approximation of $L^2[0,1]$ of multiplicity r.

2.2. Legendre multiwavelet functions

Since $V_0 \subset V_1$, we denote W_0 the orthonormal complement of V_0 in V_1, $V_1 = V_0 \oplus W_0$, then an orthonormal basis $\{\psi^1, \cdots, \psi^r\}$ of the space W_0 can be obtained by the Gram-Schmidt process: for $m = 1, 2, \cdots, r$ we define inductively

$$y^m = \varphi_{10}^m - \sum_{i=1}^{r} < \varphi_{10}^m, \varphi^i > \varphi^i - \sum_{i=1}^{m-1} < \varphi_{10}^m, \psi^i > \psi^i, \ \psi^m = \frac{y^m}{\|y^m\|} \tag{2.10}$$

For instance, when $r = 3$, we take Legendre polynomials

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1),$$

By formula (2.2), we have the multiscaling functions on their support $[0,1]$

$$\varphi_1(x) = 1, \varphi_2(x) = \sqrt{3}(2x-1), \varphi_3(x) = \sqrt{5}(6x^2 - 6x + 1) \tag{2.11}$$

and through formula (2.10) the multiwavalet functions can be constructed as following

$$\psi^1(x) = \begin{cases} 6x - 1, & 0 \leq x \leq \frac{1}{2}, \\ 6x - 5 & \frac{1}{2} \leq x \leq 1, \end{cases}$$

$$\psi^2(x) = \begin{cases} \sqrt{3}(30x^2 - 14x + 1), & 0 \leq x \leq \frac{1}{2}, \\ \sqrt{3}(30x^2 - 46x + 17) & \frac{1}{2} \leq x \leq 1, \end{cases}$$

$$\psi^3(x) = \begin{cases} \sqrt{5}(24x^2 - 12x + 1), & 0 \leq x \leq \frac{1}{2}, \\ -\sqrt{5}(24x^2 - 36x + 13) & \frac{1}{2} \leq x \leq 1. \end{cases}$$

2.3. Legendre multiwavelet basis

Let $\vec{\psi}$ be the vector of the multiwavelet functions:

$$\vec{\psi} = (\psi^1, \psi^2, \cdots, \psi^r)^T,$$

and denote for integers $j \geq 0$ and $k = 0, 1, \cdots, 2^j - 1$

$$\vec{\psi}_{jk}(x) = (\psi_{jk}^1(x), \psi_{jk}^2(x), \cdots, \psi_{jk}^r(x))^T$$

the dilates and translates of the multiwavelet functions, where

$$\psi_{jk}^m(x) = 2^{j/2}\psi^m(2^j x - k)(m = 1, 2, \cdots, r)$$

is supported on the interval $[2^{-j}k, 2^{-j}(k+1)] \subset [0,1]$. Let $\ell = 2^j - 1$, and

$$\begin{aligned} \vec{\psi}_j^T(x) &= \{\vec{\psi}_{j0}^T(x), \vec{\psi}_{j1}^T(x), \cdots, \vec{\psi}_{j\ell}^T(x)\}, \\ W_j &= \overline{span\{\vec{\psi}_{j0}^T(x), \vec{\psi}_{j1}^T(x), \cdots, \vec{\psi}_{j\ell}^T(x)\}}, j = 0,1,2,\cdots, \end{aligned} \tag{2.12}$$

then W_j is the orthonormal complement of V_j in V_{j+1}: $V_{j+1} = V_j \oplus W_j$, so we inductively obtain the decomposition

$$V_n = V_{n-1} \oplus W_{n-1} = V_{n-2} \oplus W_{n-2} \oplus W_{n-1} = \cdots = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{n-1} \tag{2.13}$$

and

$$L^2[0,1] = \lim_{n\to\infty} V_n = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{n-1} \oplus \cdots. \tag{2.14}$$

Hence $\{\varphi^m, \psi_{jk}^m, m = 1,2,\cdots,r, j = 0,1,2,\cdots, k = 0,1,\cdots,2^j - 1\}$ is an orthonormal multiwavelet basis of $L^2[0,1]$, and all elements of the vector

$$\Psi_j(x) = (\vec{\varphi}^T, \vec{\psi}^T, \vec{\psi}_1^T(x), \cdots, \vec{\psi}_{j-1}^T(x))^T, \tag{2.15}$$

form another orthonormal bases of the space V_j (see (2.8)-(2.9)).

For any $f(x) \in L^2[0,1]$, we have

$$f(x) = \sum_{m=1}^{r} c_m \varphi^m + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} \sum_{m=1}^{r} c_{jk}^m \psi_{jk}^m, \tag{2.16}$$

the projection of $f(x)$ in V_n is

$$f_n(x) = \sum_{m=1}^{r} c_m \varphi^m + \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-1} \sum_{m=1}^{r} c_{jk}^m \psi_{jk}^m = \vec{C}^T \Psi_n(x), \tag{2.17}$$

which can also be expanded by the orthonormal basis $\Phi_n(x)$ defined on (2.8):

$$f_n(x) = \sum_{k=0}^{2^n-1} \sum_{m=1}^{r} d_{nk}^m \varphi_{nk}^m = \vec{D}^T \Phi_n(x), \tag{2.18}$$

where $(\ell = 2^{n-1} - 1, \jmath = 2^n - 1)$

$$\begin{aligned} \vec{C}^T &= (c_1, \cdots, c_r, c_{00}^1, \cdots, c_{00}^r, c_{10}^1, \cdots, c_{10}^r, \cdots, c_{n-1,\ell}^1, \cdots, c_{n-1,\ell}^r), \\ \vec{D}^T &= (d_{n0}^1, \cdots, d_{n0}^r, \cdots, d_{n\jmath}^1, \cdots, d_{n\jmath}^r). \end{aligned}$$

Since $\psi^m (1 \le m \le r)$ is orthogonal to $\{\varphi^1, \varphi^2, \cdots, \varphi^r\}$,which is equivalent to the basis $\{1, x, x^2, \cdots, x^{r-1}\}$, the first r moments of ψ^m vanish: $\int_0^1 \psi^m(x) x^i dx = 0, i = 0,1,\cdots,r-1$. Thus, if $f(x) \in C^r[0,1]$, then [2]

$$\|f(x) - f_n(x)\| \le \frac{C}{2^{rn}} \sup_{x\in[0,1]} |f^{(r)}(x)| \tag{2.19}$$

2.4. The transformation matrix between two bases

Now we give out the transformation matrix T_j between the two orthonormal basis $\Phi_j(x)$ and $\Psi_j(x)$ of the space V_j defined on (2.8) and (2.15) respectively, such that

$$\Psi_j(x) = T_j\Phi_j(x),\ \Phi_j(x) = T_j^T\Psi_j(x), j \geq 1. \tag{2.20}$$

Rewrite formula (2.10) as (where $< \varphi_{10}^m, \psi^m >= \|y^m\|$))

$$\varphi_{10}^m = \sum_{i=1}^{r} < \varphi_{10}^m, \varphi^i > \varphi^i + \sum_{i=1}^{m} < \varphi_{10}^m, \psi^i > \psi^i, m = 1, 2, \cdots, r$$

From the formula (2.6) and $V_0 \perp W_0$ one has $\psi^m(1-x) = (-1)^m\psi^m(x),\ m = 1, 2, \cdots$, then for $m = 1, 2, \cdots, r$

$$\varphi_{11}^m = \sum_{i=1}^{r}(-1)^{m+i} < \varphi_{10}^m, \varphi^i > \varphi^i + \sum_{i=1}^{m}(-1)^{m+i+1} < \varphi_{10}^m, \psi^i > \psi^i.$$

Define matrices

$$H = (< \varphi_{10}^m, \varphi^i >)_{r\times r}, \tilde{H} = ((-1)^{m+i} < \varphi_{10}^m, \varphi^i >)_{r\times r}$$

and lower triangular matrices (for $k < i, let\ G(k,i) = \tilde{G}(k,i) = 0$)

$$G = (< \varphi_{10}^m, \psi^i >)_{r\times r}, \tilde{G} = ((-1)^{m+i+1} < \varphi_{10}^m, \psi^i >)_{r\times r},$$

then

$$\Phi_1(x) = \begin{bmatrix} \vec{\varphi}_{10} \\ \vec{\varphi}_{11} \end{bmatrix} = \begin{bmatrix} H & G \\ \tilde{H} & \tilde{G} \end{bmatrix} \begin{bmatrix} \vec{\varphi} \\ \vec{\psi} \end{bmatrix} = T_1^T\Psi_1,\ \begin{bmatrix} \vec{\varphi} \\ \vec{\psi} \end{bmatrix} = \begin{bmatrix} H^T & \tilde{H}^T \\ G^T & \tilde{G}^T \end{bmatrix} \begin{bmatrix} \vec{\varphi}_{10} \\ \vec{\varphi}_{11} \end{bmatrix}. \tag{2.21}$$

The second equation in (2.21) is usually called dilation equation that leads to

$$\begin{bmatrix} \vec{\varphi}_{j-1,k} \\ \vec{\psi}_{j-1,k} \end{bmatrix} = \begin{bmatrix} H^T & \tilde{H}^T \\ G^T & \tilde{G}^T \end{bmatrix} \begin{bmatrix} \vec{\varphi}_{j,2k} \\ \vec{\varphi}_{j,2k+1} \end{bmatrix},\ k = 0, 1, \cdots, 2^{j-1} - 1.$$

or in its matrix form

$$\begin{bmatrix} \Phi_{j-1}(x) \\ \vec{\psi}_{j-1}(x) \end{bmatrix} = L_j\Phi_j(x), L_j = \begin{bmatrix} P_j \\ Q_j \end{bmatrix}, \tag{2.22}$$

where

$$P_j = \begin{bmatrix} H^T & \tilde{H}^T & & & & & \\ & & H^T & \tilde{H}^T & & & \\ & & & & \ddots & & \\ & & & & & H^T & \tilde{H}^T \end{bmatrix},$$

$$Q_j = \begin{bmatrix} G^T & \tilde{G}^T & & & & & \\ & & G^T & \tilde{G}^T & & & \\ & & & & \ddots & & \\ & & & & & G^T & \tilde{G}^T \end{bmatrix}$$

Finally, from (2.15),(2.21), and (2.22) we obtain (by using induction)

$$T_1 = L_1 = \begin{bmatrix} H^T & \tilde{H}^T \\ G^T & \tilde{G}^T \end{bmatrix}, \ T_j = \begin{bmatrix} T_{j-1} & O \\ O & I \end{bmatrix} L_j, \ j = 2, 3, \cdots \tag{2.23}$$

3. Anti-derivatives of Legendre multiwavelets

Let $q^m(x) := \int_0^x \varphi^m(t)dt, \ m = 1, 2, \cdots, r$. From (2.11) we have

$$q^1(x) = \begin{cases} \frac{1}{2}\varphi^1(x) + \frac{1}{2\sqrt{3}}\varphi^2(x), & x \in (-\infty, 1), \\ 1, & x \in [1, +\infty). \end{cases} \tag{3.1}$$

From the derivative formula (2.5) and the formula (2.7) it is readily seen that

$$q^m(x) = \frac{1}{2\sqrt{4m^2-1}}\varphi^{m+1}(x) - \frac{1}{2\sqrt{4(m-1)^2-1}}\varphi^{m-1}(x), m = 2, 3, \cdots, r \tag{3.2}$$

Define the coefficients

$$a_0 = \frac{1}{2}, \ a_m = \frac{1}{2\sqrt{4m^2-1}}, \ m = 1, 2, \cdots, \tag{3.3}$$

then

$$\vec{q}(x) := (q^1(x), q^2(x), \cdots, q^r(x))^T = L\vec{\varphi}(x) + \vec{\Gamma}(x), \tag{3.4}$$

where L is a $r \times r$ matrix and $\vec{\Gamma}(x)$ is a $r \times 1$ vector,

$$L = \begin{bmatrix} a_0 & a_1 & & & \\ -a_1 & 0 & a_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_{r-2} & 0 & a_{r-1} \\ & & & -a_{r-1} & 0 \end{bmatrix}, \ \vec{\Gamma}(x) = \begin{bmatrix} \chi_{[1,+\infty)} \\ 0 \\ \vdots \\ 0 \\ a_r\varphi^{r+1}(x) \end{bmatrix}. \tag{3.5}$$

Dilating and translating $\vec{q}(x)$ we get

$$q^1_{nk}(x) = \begin{cases} a_0\varphi^1_{nk}(x) + a_1\varphi^2_{nk}(x), & x \in [0, \frac{1+k}{2^n}], \\ 1, & x \in [\frac{1+k}{2^n}, 1]. \end{cases} \tag{3.6}$$

$$q^m_{nk}(x) = a_m\varphi^{m+1}_{nk}(x) - a_{m-1}\varphi^{m-1}_{nk}(x), m = 2, 3, \cdots \tag{3.7}$$

Let $Q_n(x) := 2^{-n}(\vec{q}^T_{n0}(x), \vec{q}^T_{n1}(x), \cdots, \vec{q}^T_{n\ell}(x))^T$ ($\ell = 2^n - 1$), then

$$Q_n(x) = \int_0^x \Phi_n(t)dt = P\Phi_n(x) + \vec{S}(x), \tag{3.8}$$

where P is a $(\ell+1) \times (\ell+1)$ block matrix and $\vec{S}(x)$ a $(\ell+1) \times 1$ block matrix,

$$P = 2^{-n}\begin{bmatrix} L & F & F & \dots & F \\ \bigcirc & L & F & \dots & F \\ \bigcirc & \bigcirc & L & \dots & F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bigcirc & \bigcirc & \bigcirc & \dots & L \end{bmatrix}, \ \vec{S}(x) = 2^{-n}a_r\begin{bmatrix} \vec{s}_{n0}(x) \\ \vec{s}_{n1}(x) \\ \vdots \\ \vec{s}_{n\ell}(x) \end{bmatrix}. \tag{3.9}$$

$$F = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{r\times r}, \quad \vec{s}_{nk}(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \varphi_{nk}^{r+1}(x) \end{bmatrix}_{r\times 1}, \quad k = 0, 1, \cdots, \ell. \tag{3.10}$$

Now we define $\vec{R}_n(x)$ as the anti-derivative of $\Psi_n(x)$:

$$\vec{R}_n(x) := \int_0^x \Psi_n(t)dt, \tag{3.11}$$

then by (2.20) and (3.8)

$$\vec{R}_n(x) = T_n Q_n(x) = T_n P \Phi_n(x) + T_n \vec{S}(x). \tag{3.12}$$

Remark 3.1. The matrix P in (3.9) is called operational matrix of integration for an approximate formula $\int_0^x \Phi_n(t)dt \approx P\Phi_n(x)$ in [13]. Therefore, (3.8) gives the exact definition of the operational matrix of integration.

4. Evaluation of the connection coefficients

The integrals of the products of several basis functions are called the connection coefficients [7, 11]. By orthonormality of the multiscaling functions and multiwavelet functions one has $\int_0^1 \Phi_n(x)\Phi_n^T(x)dx = \int_0^1 \Psi_n(x)\Psi_n^T(x)dx = I$, and by (2.20) one has $\int_0^1 \Phi_n(x)\Psi_n^T(x)dx = T_n^T$, so the integrals of the products of two basis functions are obtained. Next we want to evaluate the integrals of the products of three basis functions

$$\int_0^1 \varphi_{nk}^m(x)\varphi_{nk}^i(x)\varphi_{nk}^j(x)dx = 2^{n/2}\int_0^1 \varphi^m(x)\varphi^i(x)\varphi^j(x)dx \tag{4.1}$$

For this purpose, we show that the product $\varphi^m(x)\varphi^i(x)$ of two multiscaling functions can be expressed as a linear combination of finite number of functions among $\{\varphi^m(x), m \geq 1\}$. Note that $\varphi_1(x) = 1$, one has

$$\varphi^m(x)\varphi^1(x) = \varphi^m(x), m \geq 1 \tag{4.2}$$

By defining the coefficients

$$b_m = \frac{\sqrt{3}m}{\sqrt{4m^2-1}}, m = 1, 2, \cdots, \ell. \tag{4.3}$$

the recurrence relation (2.4) can be rewritten as (note that $\varphi^2(x) = \sqrt{3}(2x-1)$)

$$\varphi^m(x)\varphi^2(x) = b_{m-1}\varphi^{m-1}(x) + b_m\varphi^{m+1}(x), m \geq 2 \tag{4.4}$$

Multiplying this equation with $\varphi^i(x)$ we obtain

$$\varphi^m(x)\varphi^i(x)\varphi^2(x) = b_{m-1}\varphi^{m-1}(x)\varphi^i(x) + b_m\varphi^{m+1}(x)\varphi^i(x)$$

Exchanging the index m and i we obtain

$$\varphi^m(x)\varphi^i(x)\varphi^2(x) = b_{i-1}\varphi^m(x)\varphi^{i-1}(x) + b_i\varphi^m(x)\varphi^{i+1}(x)$$

Then equalling right hand sides of above two equations, we get

$$\varphi^m(x)\varphi^{i+1}(x) = \frac{1}{b_i}[b_{m-1}\varphi^{m-1}(x)\varphi^i(x) + b_m\varphi^{m+1}(x)\varphi^i(x) - b_{i-1}\varphi^m(x)\varphi^{i-1}(x)]$$

This means if all products $\{\varphi^m(x)\varphi^{i-1}(x), m \geq i-1\}$ and $\{\varphi^m(x)\varphi^i(x), m \geq i\}$ are linear combinations of finite number of functions among $\{\varphi^m(x), m \geq 1\}$, then by induction all products $\{\varphi^m(x)\varphi^{i+1}(x), m \geq i+1\}$ are also linear combinations of finite number of functions among $\{\varphi^m(x), m \geq 1\}$. Starting from (4.2) and (4.4), we inductively deduce the following equation

$$\varphi^m(x)\varphi^i(x) = \sum_{k=0}^{i-1} C_k^{m,i}\varphi^{m-i+1+2k}(x), \tag{4.5}$$

where the coefficients for $i = 1, 2$ are from (4.2) and (4.4)

$$C_0^{m,1} = 1,\ C_0^{m,2} = b_{m-1}, C_1^{m,2} = b_m,$$

and the coefficients for $i \geq 2$ are

$$\begin{cases} C_0^{m,i+1} = \dfrac{b_{m-1}}{b_i} C_0^{m-1,i}, \\ C_k^{m,i+1} = \dfrac{1}{b_i}(b_{m-1}C_k^{m-1,i} + b_m C_{k-1}^{m+1,i} - b_{i-1}C_{k-1}^{m,i-1}),\ k = 1, 2, \cdots, i-1, \\ C_i^{m,i+1} = \dfrac{b_m}{b_i} C_{i-1}^{m+1,i} \end{cases}$$

Inserting (4.5) into (4.1), the integrals of the products of three basis functions are obtained:

$$\int_0^1 \varphi^m(x)\varphi^i(x)\varphi^j(x)dx = \sum_{k=0}^{i-1} C_k^{m,i}\delta_{m-i+1+2k,j} = C_k^{m,i}, \tag{4.6}$$

$$k = 0, 1, \cdots, i-1, m-i+1+2k = j\ (m \geq i).$$

Other integrals of the products of three basis functions can be obtained by combining above formula with one of the formula (2.20),(3.8) and (3.12).

5. Applications to two-point boundary value problems

In this section, we apply Legendre multiwavelets to numerical solutions of a two-point boundary value problem of ODE

$$Tu := -(p(x)u')' + q(x)u = f(x), for\ x \in (0, 1), \tag{5.1}$$

with either one of the boundary conditions:

$$\begin{cases} u(0) = u(1) = 0, & (Dirichlet\ condition) \\ u(0) = u'(1) = 0, & (mixed\ condition) \\ u'(0) = u'(1) = 0, & (Neumann\ condition) \end{cases}$$

We assume that $f \in L^2[0,1]$, the coefficients $p(x)$ and $q(x)$ are continuously differentiable in $I=[0,1]$ with

$$0 < p_1 \leq p(x) \leq p_2 \text{ and } 0 < q_1 \leq q(x) \leq q_2. \tag{5.2}$$

Let $H^s(I)$ denote the standard Sobolev space with the norm $\| \cdot \|_s$ and semi-norm $| \cdot |_s$ given by

$$\| v \|_s^2 = \sum_{i=0}^{s} \int_0^1 |v^{(i)}(x)|^2 dx \text{ and } | v |_s^2 = \int_0^1 |v^{(s)}(x)|^2 dx.$$

We define

$$H_0^1(I) = \{v \in H^1(I) \mid v(0) = v(1) = 0\}, \text{ and } H_\star^1(I) = \{v \in H^1(I) \mid v(0) = 0\}.$$

It is well-known that the semi-norm $| \cdot |_1$ is a norm in these two spaces and is equivalent to the norm $\| \cdot \|_1$.

5.1. Error estimates

The variational form of (5.1) is

$$a(u,v) = < f, v >, \ for \ all \ v \in H_0^1(I), \tag{5.3}$$

where $a(\cdot,\cdot)$ is a bilinear form defined by

$$a(u,v) = \int_0^1 (pu'v' + quv) dx. \tag{5.4}$$

Clearly $a(\cdot,\cdot)$ is continuous and coercive on $H_0^1(I)$. It is well-known that, by Lax-Milgram lemma, (5.3) admits a unique weak solution $u \in H^1(I)$.

Now we apply Legendre multiwavelets to Galerkin methods for solving (5.3). According to the boundary condition, we construct a finite dimensional space S_n, then we solve numerically the Galerkin projection of the solution u on S_n defined by

$$a(u_n, v) = < f, v >, \ for \ all \ v \in S_n. \tag{5.5}$$

It is also clear that (5.5) admits a unique solution $u_n \in S_n$ such that [14]

$$\| u - u_n \|_1 \leq C \ inf\{\| u - v \|_1, v \in S_n\}. \tag{5.6}$$

By (3.11) the elements of the vector $\vec{R}_n(x)$ are $q^1(x), q^2(x), \cdots, q^r(x)$ and a part of the functions of the set

$$\Xi = \{R_{jk}^m(x), j = 0,1,\cdots; k = 0,1,\cdots,2^j - 1; m = 1,2,\cdots,r\}$$

where $R_{jk}^m(x) = \int_0^x \psi_{jk}^m(x) dx$. We have

Lemma 5.1. *The three sets* $\{q^2(x),\cdots,q^r(x)\} \cup \Xi$, $\{q^1(x),\cdots,q^r(x)\} \cup \Xi$, *and* $\{q^1(x),\cdots,q^r(x)\} \cup \Xi \cup \{1\}$ *form a basis for the spaces* $H_0^1(I)$, $H_\star^1(I)$, *and* $H^1(I)$, *respectively.*

Proof. We only prove for space $H_0^1(I)$, the proofs for other two spaces are similar. First, it is easy to see that the functions of $\{q^2(x), \cdots, q^r(x)\} \cup \Xi$ are linearly independent. Then, for any $v \in H_0^1(I)$, let $w = v'$. Note that $\int_0^1 w(x)dx = 0$. Since $w \in L^2(I)$, by (2.16)-(2.17), there are numbers $\{c_1, \cdots, c_r, c_{jk}^m, j = 0, 1, \cdots, k = 0, 1, \cdots, 2^j - 1, m = 1, 2, \cdots, r\}$ such that

$$w_n(x) = \sum_{m=1}^{r} c_m \varphi^m + \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-1} \sum_{m=1}^{r} c_{jk}^m \psi_{jk}^m, \ \lim_{n\to\infty} \int_0^1 (w - w_n)^2 dx = 0.$$

Since $\overline{w}_n = \int_0^1 w_n(x)dx = c_1 \int_0^1 \varphi^1(x)dx = c_1$, and

$$|\overline{w}_n| = |\int_0^1 (w_n - w)dx| \le \int_0^1 (w_n - w)^2 dx \to 0 \ (n \to \infty),$$

we have $c_1 = 0$. Hence, if we take

$$v_n(x) = \sum_{m=2}^{r} c_m q^m(x) + \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-1} \sum_{m=1}^{r} c_{jk}^m R_{jk}^m(x),$$

then $v_n'(x) = w_n(x)$, and

$$|v - v_n|_1 = \|v' - v_n'\| = \|w - w_n\| \to 0 \ (n \to \infty).$$

As $|\cdot|_1$ is a norm on $H_0^1(I)$, the proof is completed. □

Let the set

$$\Xi_n = \{R_{jk}^m(x), j = 0, 1, \cdots, n-1; k = 0, 1, \cdots, 2^j - 1; m = 1, 2, \cdots, r\},$$

then S_n can be defined by its basis as

$$S_n = \begin{cases} \overline{span\{\{q^2(x), \cdots, q^r(x)\} \cup \Xi_n\}}, & (Dirichlet\ condition) \\ \overline{span\{\{q^1(x), \cdots, q^r(x)\} \cup \Xi_n\}}, & (mixed\ condition) \\ \overline{span\{\{q^1(x), \cdots, q^r(x)\} \cup \Xi_n \cup \{1\}\}}, & (Neumann\ condition) \end{cases}$$

By (3.11)-(3.12) the basis of S_n consists of the elements of the vectors

$$\tilde{R}_n(x) = \tilde{T}_n Q_n(x), \ R_n(x), \ or \ \begin{bmatrix} 1 \\ R_n(x) \end{bmatrix} \tag{5.7}$$

respectively according to boundary conditions, where $\tilde{T}_n$ is T_n with its first line deleted. So we also use these vectors to represent the basis of S_n.

From the proof of lemma 5.1 we know that for any $v \in H_0^1(I) \cap C^{r+1}(I)$, by (2.19)

$$|v - v_n|_1 = \|v' - v_n'\| = \|w - w_n\| \le \frac{C}{2^{rn}} \sup_{x\in[0,1]} |v^{(r+1)}(x)|.$$

This means for any $u \in H_0^1(I) \cap C^{r+1}(I)$,

$$inf\{|u - v|_1, v \in S_n\} \le \frac{C}{2^{rn}} \sup_{x\in[0,1]} |u^{(r+1)}(x)|. \tag{5.8}$$

Combining (5.6) with (5.8), we have proved

Theorem 5.2. *Let u and u_n be the solutions of (5.3) and (5.5) respectively. If $u \in H_0^1(I) \cap C^{r+1}(I)$, then we have*

$$\| u - u_n \|_1 \leq \frac{C}{2^{rn}} \sup_{x\in[0,1]} |u^{(r+1)}(x)|. \tag{5.9}$$

5.2. The solution of the linear system of equations

In this subsection we suppose that Dirichlet condition is imposed on the boundary. Let $u_n(x) = \tilde{R}_n^T(x)U$, and v go through all elements of $\tilde{R}_n(x)$ in (5.5), then we have

$$[\int_0^1 p(x)\tilde{\Psi}_n(x)\tilde{\Psi}_n^T(x)dx + \int_0^1 q(x)\tilde{R}_n(x)\tilde{R}_n^T(x)dx]\ U = \int_0^1 f(x)\tilde{R}_n(x)dx,$$

If we denote the matrices

$$A_n = \int_0^1 p(x)\tilde{\Psi}_n(x)\tilde{\Psi}_n^T(x)dx + \int_0^1 q(x)\tilde{R}_n(x)\tilde{R}_n^T(x)dx, f_n = \int_0^1 f(x)\tilde{R}_n(x)dx,$$

then a linear system of equations is obtained:

$$A_nU = f_n \tag{5.10}$$

By assumption (5.2), the coefficient matrix A_n is a symmetric and positive definite matrix. Since the basis $\tilde{R}_n^T(x)$ is an orthonormal basis of $S_n \subset H_0^1(I)$ in the sense of the inner product $< u, v >_1 := < u', v' >$, it is easy to prove that the condition number of A_n is bounded:

Proposition 5.3. *For any $n \geq 0$,*

$$cond(A_n) \leq c := \frac{p_2}{p_1} + \frac{4q_2}{p_1\pi^2}.$$

Proof. see [4], page 166. □

Now we discuss how to solve the linear system (5.10). In practise,we expand the functions $f(x), p(x), q(x)$ in V_n through the basis $\Phi_n(x)$:

$$f(x) = \Phi_n^T(x)F,\ p(x) = \sum_{k=0}^{2^n-1}\sum_{m=1}^{r} p_{nk}^m\varphi_{nk}^m(x),\ q(x) = \sum_{k=0}^{2^n-1}\sum_{m=1}^{r} q_{nk}^m\varphi_{nk}^m(x),$$

By virtue of (2.19), the above approximation of $p(x)$ and $q(x)$ still keep their property (5.2) for sufficiently large n, so A_n is still a symmetric and positive definite matrix. We have (P is defined in (3.9))

$$\begin{aligned}
A_n &= B_n + K_n,\\
B_n &= \int_0^1 p(x)\tilde{\Psi}_n(x)\tilde{\Psi}_n^T(x)dx = \tilde{T}_n \int_0^1 p(x)\Phi_n(x)\Phi_n^T(x)dx\tilde{T}_n^T,\\
K_n &= \int_0^1 q(x)\tilde{R}_n(x)\tilde{R}_n^T(x)dx = \tilde{T}_n \int_0^1 q(x)Q_n(x)Q_n^T(x)dx\tilde{T}_n^T,
\end{aligned}$$

$$f_n = \int_0^1 \tilde{R}_n(x)\Phi_n^T(x)dx\ F = \tilde{T}_n PF.$$

Let

$$D_n := \int_0^1 p(x)\Phi_n(x)\Phi_n^T(x)dx = \sum_{k=0}^{2^n-1}\sum_{m=1}^{r} p_{nk}^m \int_0^1 \varphi_{nk}^m(x)\Phi_n(x)\Phi_n^T(x)dx$$

$$\begin{aligned} E_n &:= \int_0^1 q(x)Q_n(x)Q_n^T(x)dx \\ &= \sum_{k=0}^{2^n-1}\sum_{m=1}^{r} q_{nk}^m \Big[P\int_0^1 \varphi_{nk}^m(x)\Phi_n(x)\Phi_n^T(x)dxP^T + P\int_0^1 \varphi_{nk}^m(x)\Phi_n(x)\vec{S}(x)dx \\ &\qquad + \int_0^1 \varphi_{nk}^m(x)\vec{S}(x)\Phi_n^T(x)dxP^T + \int_0^1 \varphi_{nk}^m(x)\vec{S}(x)\vec{S}^T(x)dx\Big]. \end{aligned} \tag{5.11}$$

Then

$$B_n = \tilde{T}_n D_n \tilde{T}_n^T, K_n = \tilde{T}_n E_n \tilde{T}_n^T.$$

D_n is a block diagonal matrix by virtue of (4.1) and (4.6), thus the inverses of D_n and B_n can be computed very efficiently.

In recent years a special multilevel augmentation method (MAM) has been developed [5] to solve some linear system of equations arising from discretizing differential equations that requires the use of special multiscale bases and leads to an efficient, stable and accurate solver for the discrete linear system. Our basis $\tilde{R}_n(x)$ is just a basis of this type that meets all requirements for this MAM algorithm, but it is different from those bases in [5]. Besides, The MAM algorithm in an example in [5] can be applied to (5.10) only when $p(x) = q(x) = 1$ is valid. Here we propose a special form of the MAM algorithm for (5.10) as follows:

MAM algorithm for (5.10) Let $m_0 > 0$ be a fixed integer.

Step 1 Solve $u_{m_0} \in R^{N_1}(N_1 = 2^{m_0}r)$ from equation $A_{m_0}u_{m_0} = f_{m_0}$.

Step 2 Set $u_{m_0,0} := u_{m_0}$ and split the matrix $K_{m_0} = K_{m_0,0}^L + K_{m_0,0}^H$ where $K_{m_0,0}^L = K_{m_0} = \tilde{T}_{m_0}E_{m_0}\tilde{T}_{m_0}^T, K_{m_0,0}^H = 0$.

Step 3 For integer $m \geq 1$, suppose that $u_{m_0,m-1} \in R^{N_2}(N_2 = 2^{m_0+m-1}r)$ has been obtained and do the following:

(i): Augment the matrices $K_{m_0,m-1}^L$ and $K_{m_0,m-1}^H$ to form

$$K_{m_0,m}^L = \tilde{T}_n \begin{bmatrix} E_{0,0}^{m_0} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tilde{T}_n^T,$$

where $E_{0,0}^{m_0} = E_{m_0}, n = m_0 + m$, and

$$K_{m_0,m}^H = \tilde{T}_n \begin{bmatrix} 0 & E_{0,1}^{m_0} & \cdots & E_{0,m}^{m_0} \\ E_{1,1}^{m_0} & E_{1,1}^{m_0} & \cdots & E_{1,m}^{m_0} \\ \vdots & \vdots & \cdots & \vdots \\ E_{m,1}^{m_0} & E_{m,1}^{m_0} & \cdots & E_{m,m}^{m_0} \end{bmatrix} \tilde{T}_n^T$$

respectively;

(ii): Augment $u_{m_0,m-1}$ by setting $\overline{u}_{m_0,m} = \begin{bmatrix} u_{m_0,m-1} \\ 0 \end{bmatrix}$;

(iii): Solve $u_{m_0,m} \in R^{N_3}(N_3 = 2^{m_0+m}r)$ from the algebraic equations

$$(B_{m_0,m} + K_{m_0,m}^L)u_{m_0,m} = f_{m_0,m} - K_{m_0,m}^H \overline{u}_{m_0,m}$$

Noting that $D_{m_0,m}$ is a block diagonal matrix, $B_{m_0,m} = \tilde{T}_n D_{m_0,m} \tilde{T}_n^T$, and T_n is an orthogonal matrix, the computation of the inverse of $B_{m_0,m} + K_{m_0,m}^L$ can be reduced to the computation of the inverse of one $2^{m_0}r \times 2^{m_0}r$ matrix by using inverse operation for block matrices in basic linear algebra. Therefore this algorithm is very efficient. As for its accuracy, the following proposition ensures that the approximate solution $u_{m_0,m}$ generated by the MAM has the same order of approximation as that of the subspaces S_n.

Proposition 5.4. *Let u and $u_n(x) = \tilde{R}_n^T(x)u_{m_0,m}$ $(n = m_0+m)$ be the solutions of (5.3) and (5.5) respectively, where $u_{m_0,m}$ is the solution of (5.10). If $u \in H_0^1(I) \cap C^{r+1}(I)$, then there exists $m_0 \geq 1$, such that*

$$\| u - u_{m_0,m} \|_1 \leq \frac{C}{2^{rn}} \sup_{x\in[0,1]} |u^{(r+1)}(x)|. \tag{5.12}$$

Proof. From the assumption (5.2) and theorem 5.2, it is obvious that the operator T of (5.1) satisfies the conditions $(H_1) - (H_5)$ in [5]. Then the conclusion comes from theorem 2.3 of [5]. $\square$

6. Application to Dirichlet problem for the elliptic equation on a rectangle

In this section we discuss the application of Legendre multiwavelets to numerical solution of the Dirichlet problem for the elliptic equation on the rectangle $\Omega = [0,1]^2$

$$\begin{cases} T_1 u := -\triangle u + u = f(x,y), \ (x,y) \in (0,1) \times (0,1), \\ u(0,y) = u(1,y) = u(x,0) = u(x,1) = 0. \end{cases} \tag{6.1}$$

Let $L^2(\Omega)$ be the Hilbert space equipped with the inner product

$$< u, v >:= \int_0^1 \int_0^1 u(x,y)v(x,y)dxdy, u, v \in L^2(\Omega)$$

and the norm

$$||v|| := \sqrt{< v, v >}, v \in L^2(\Omega).$$

We assume that $f \in L^2(\Omega)$. Let $H^1(\Omega)$ denotes the standard Sobolev space with the norm $\| \cdot \|_1$ and semi-norm $| \cdot |_1$ given by

$$| v |_1^2 = \int_0^1 \int_0^1 (|\frac{\partial v}{\partial x}(x, y)|^2 + |\frac{\partial v}{\partial y}(x, y)|^2) dxdy, \ \| v \|_1^2 = ||v||^2 + | v |_1^2 .$$

We define

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \mid v|_{\partial\Omega} = 0\}.$$

It is well-known that $| \cdot |_1$ is equivalent to $\| \cdot \|_1$ in $H_0^1(\Omega)$.

The variational form of (6.1) is

$$a(u, v) =< f, v >, \ for \ all \ v \in H_0^1(\Omega), \tag{6.2}$$

where $a(\cdot, \cdot)$ is a bilinear form defined by

$$a(u, v) = \int_0^1 \int_0^1 (u_x v_x + u_y v_y + uv) dxdy. \tag{6.3}$$

Clearly $a(\cdot, \cdot)$ is continuous and coercive on $H_0^1(\Omega)$, and (6.2) admits a unique weak solution $u \in H_0^1(\Omega)$ by Lax-Milgram lemma.

Now we apply Legendre multiwavelets to Galerkin methods to solve (6.2). Some results in section 2 can be extended to $L^2(\Omega)$. Firstly, the tensor products $\{V_n[0, 1] \otimes V_n[0, 1]\}$ form a multiresolution analysis (MRA) of the space $L^2(\Omega) = L^2[0, 1] \otimes L^2[0, 1]$, the elements of $\Phi_n(x) \otimes \Phi_n(y)$ and the elements of $\Psi_n(x) \otimes \Psi_n(y)$ are two equivalent bases of $\{V_n[0, 1] \otimes V_n[0, 1]\}$. (Here and afterward we use tensor product, or Kronecker product, of two matrices. For its definition and properties we refer readers to [8]). Secondly, each function $f \in L^2(\Omega)$ can be approximated by

$$f_n(x, y) := \Psi_n^T(x) \otimes \Psi_n^T(y)\vec{C} = \Phi_n^T(x) \otimes \Phi_n^T(y)\vec{D},$$

and the error of the approximation for $f \in C^{r+1,r+1}(\Omega)$ is

$$\|f(x, y) - f_n(x, y)\| \leq \frac{C}{2^{rn}}, \tag{6.4}$$

i.e., the rate of convergence is of order r/2 (see [2]).

Following the line of section 5, we want to seek the approximate solution on a finite dimensional space $S_n = H_0^1(\Omega) \cap (V_n[0, 1] \otimes V_n[0, 1])$, i.e., we will solve numerically the Galerkin projection u_n of the solution u on S_n defined by

$$a(u_n, v) =< f, v >, \ for \ all \ v \in S_n. \tag{6.5}$$

It is easy to see that $\tilde{R}_n(x) \otimes \tilde{R}_n(y)$ is a basis of S_n. We suppose that $u_n(x, y) = \tilde{R}_n^T(x) \otimes \tilde{R}_n^T(y)U$, and let $v(x, y) = \tilde{R}_n(x) \otimes \tilde{R}_n(y)$ in (6.5), then we have a linear system of equations

$$A_n U = f_n, \tag{6.6}$$

where (P is defined in (3.9))

$$\begin{aligned} A_n :=& I \otimes K_n + K_n \otimes I + K_n \otimes K_n, \\ f_n =& \int_0^1 \int_0^1 f(x,y) \tilde{R}_n(x) \otimes \tilde{R}_n(y) dx dy, \\ K_n =& \int_0^1 \tilde{R}_n(x) \tilde{R}_n^T(x) dx = \tilde{T}_n (PP^T + \int_0^1 \vec{S}(x) \vec{S}^T(x) dx) \tilde{T}_n^T. \end{aligned}$$

This linear system can be solved by using the method of separation of variables. Since K_n is a symmetric matrix with positive eigenvalues, there exists an orthonormal matrix G_n and an invertible diagonal matrix Λ such that $K_n = G_n^T \Lambda G_n$, then the solution of (6.6) is given by

$$U = (G_n^T \otimes G_n^T) H^{-1} (G_n \otimes G_n) f_n.$$

where $H = I \otimes \Lambda + \Lambda \otimes I + \Lambda \otimes \Lambda$ is an invertible diagonal matrix. Thus the problem of solving (6.6) is reduced to an one-dimensional eigenvalue problem for a symmetric matrix K_n, which size is $(2^n r - 1) \times (2^n r - 1)$.

If $u \in H_0^1(\Omega) \cap C^{r+1,r+1}(\Omega)$, then we have the error estimate

$$|u - u_n|_2 := \|u_{xy} - (u_n)_{xy}\| \leq \frac{C}{2^{rn}}. \tag{6.7}$$

Remark 6.1. The operator T_1 of (6.1) satisfies the conditions $(H_1) - (H_5)$ in [5] so the MAM algorithm can be applied to solve (6.6). The elements of $\tilde{R}_n(x) \otimes \tilde{R}_n(y)$ must be reordered according to the standard construction of the bases of $L^2[0,1] \otimes L^2[0,1]$ (see [2] for an outline of the construction). However, further investigations are needed to see how to split A_n of (6.6) such that the inverse of the matrix $B_{m_0,m} + K_{m_0,m}^L$ in Step 3 of MAM algorithm in section 5 can be computed rapidly.

7. Numerical Examples and Conclusions

We present three numerical examples.

Example 7.1. Two-point boundary value problem

$$-((1+x^2)u')' + |2x-1|u = f(x), \ x \in (0,1), u(0) = u(1) = 0. \tag{7.1}$$

The exact solution is $u(x) = |2x-1|^3 - 1$. We take $r = 2$, this means the Legendre multiwavelet basis is a linear basis. And we take $m_0 = 2$ to apply the special form of MAM algorithm of section 5. The numerical results are summarized in Table 1.

Example 7.2. Two-point boundary value problem

$$-(e^x u')' + sin(x^2+1)u = f(x), \ x \in (0,1), u(0) = u(1) = 0. \tag{7.2}$$

The exact solution is $u(x) = sin(\pi x)$. We take $r = 3$, this means the Legendre multiwavelet basis is a quadratic basis. And we take $m_0 = 1$ to apply the special form of MAM algorithm of section 5. The numerical results are also summarized in Table 1.

TABLE 1. Numerical results for (7.1) and (7.2)

(7.1)	m	$\|u-u_{2,m}\|_\infty$	(7.2)	m	$\|u-u_{1,m}\|_\infty$
n=2	0	7.6257e-004	n=1	0	0.0021
n=3	1	9.6956e-005	n=2	1	1.7910e-004
n=4	2	6.1902e-006	n=3	2	1.2359e-005
n=5	3	3.9670e-007	n=4	3	7.2453e-007
n=6	4	4.0248e-008	n=5	4	5.4021e-008
n=7	5	1.5570e-009	n=6	5	2.5154e-008
n=8	6	8.5763e-011	n=7	6	1.9644e-008
n=9	7	7.9934e-012	n=8	7	5.6938e-010

TABLE 2. Numerical results for (7.3)

r	n	$\|u-u_n\|_\infty$	r	n	$\|u-u_n\|_\infty$
2	1	0.0252	4	1	1.7296e-004
2	2	0.0092	4	2	1.7933e-005
2	3	0.0026	4	3	1.2641e-006
2	4	6.9462e-004	4	4	8.1654e-008
2	5	1.7790e-004	4	5	5.1546e-009
2	6	4.4973e-005	4	6	3.2325e-010
2	7	1.1303e-005	4	7	2.0230e-011

Example 7.3. Dirichlet problem for the elliptic equation

$$\begin{cases} -\triangle u + u = f(x,y), \ (x,y) \in (0,1) \times (0,1), \\ u(0,y) = u(1,y) = u(x,0) = u(x,1) = 0. \end{cases} \tag{7.3}$$

The exact solution is $u(x,y) = sin(\pi x)y(y-1)$. We make use of linear basis ($r = 2$) which means each basis function is a product of two linear functions $g_1(x)$ and $g_2(y)$, and cubic basis ($r = 4$) respectively. The numerical results are summarized in Table 2.

Conclusions. Legendre multiwavelets together with its anti-derivatives are suitable for constructing orthonormal bases to solve some boundary value problems of ODEs and PDEs using Galerkin methods. In one dimensional case, applying multilevel augmentation method leads to an efficient, stable, and accurate algorithm, the rate of convergence in Galerkin methods is of order r. In two dimensional case, it is convenient to use tensor product of Legendre multiwavelet bases to form a basis for boundary value problems on a rectangle, the rate of convergence in Galerkin methods is of order r/2, and further investigations are needed to see how to apply the MAM algorithm to solve the discrete linear system more efficiently.

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