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A SMOOTHING NEWTON METHOD FOR NCP BASED ON A NEW CLASS OF SMOOTHING FUNCTIONS†

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Abstract. A new class of smoothing functions is introduced in this paper, which includes some important smoothing complementarity functions as its special cases. Based on this new smoothing function, we proposed a smoothing Newton method. Our algorithm needs only to solve one linear system of equations. Without requiring the nonemptyness and boundedness of the solution set, the proposed algorithm is proved to be globally convergent. Numerical results indicate that the smoothing Newton method based on the new proposed class of smoothing functions with $\theta \in (0,1)$ seems to have better numerical performance than those based on some other important smoothing functions, which also demonstrate that our algorithm is promising.

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1. Introduction

Consider the following nonlinear complementarity problem (NCP): to find a vector $x \in \Re^n$ such that

$$x \geq 0, F(x) \geq 0, x^T F(x) = 0. \quad (1)$$

where $F_i : \Re^n \to \Re (i = 1, \ldots, n)$ is continuously differentiable with $F := (F_1, F_2, \ldots, F_n)^T$. The NCP has been studied extensively due to its many applications in operation research, engineering and economics(see, for example, [1, 2]).

For the NCPs, many solution methods, such as interior point methods [3, 4], smoothing methods [5, 6, 7]. In this paper, we are interested in smoothing Newton methods for solving NCP. This method is to reformulate NCP as a

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system of smoothing equations by using smoothing function, and to solve the equation at each iteration by Newton method. Smoothing function plays an important role in smoothing Newton algorithms. Up to now, many smoothing functions have been proposed: the Kanzow smoothing function [8], Chen-Harker-Kanzow-Smale smoothing function [5], Chen-Mangasarian smoothing function [9], Huang-Han-Chen smoothing function [10], and so on. Generally, the construction of a smoothing function is based on a so-called NCP-function: An NCP-function is a mapping $\phi : \Re^2 \to \Re$ having the property

$$\phi(a,b) = 0 \Leftrightarrow a \geq 0, b \geq 0, ab = 0.$$

Many NCP-functions have been studied. Among them, the Fischer-Burmeister function and the minimum function are the most prominent NCP-functions, which are defined respectively by

$$\phi(a,b) := \sqrt{a^2+b^2} - a - b,\ \forall (a,b) \in \Re^2,$$
$$\phi(a,b) := \min\{a,b\},\ \forall (a,b) \in \Re^2.$$

By smoothing the symmetric perturbed Fischer-Burmeister function, Huang, Han, Xu and Zhang [11] proposed the following smoothing function:

$$\phi(\mu,a,b) := (1+\mu)(a+b) - \sqrt{(a+\mu b)^2 + (b+\mu a)^2 + 2\mu^2},\ \forall (\mu,a,b) \in \Re^3, \quad (2)$$

By smoothing the symmetric perturbed minimum function, Huang et. al. [10] proposed the following smoothing function:

$$\phi(\mu,a,b) := (1+\mu)(a+b) - \sqrt{(1-\mu)^2(a-b)^2 + 2\mu^2},\ \forall (\mu,a,b) \in \Re^3. \quad (3)$$

Recently, by combining the Fischer-Burmeister function and the minimum function, Liu and Wu [12] proposed the following function:

$$\phi_\theta(a,b) := a + b - \sqrt{\theta(a-b)^2 + (1-\theta)(a^2+b^2)},\ \theta \in [0,1], \forall (a,b) \in \Re^2.$$

Motivated by [10, 11, 12], we introduce in this paper the following smoothing function:

$$\begin{aligned} &\phi_\theta(\mu,a,b) \\ =&(1+\mu)(a+b) - \sqrt{\theta(1-\mu)^2(a-b)^2 + (1-\theta)[(a+\mu b)^2 + (b+\mu a)^2] + 2\mu^2}. \end{aligned} \quad (4)$$

where θ is a given constant with $\theta \in [0,1]$. It is easy to see that when $\theta = 1, \phi_\theta$ reduces to the smoothing function defined by (1.3); and when $\theta = 0, \phi_\theta$ reduces to smoothing function defined by (1.2). Thus, the class of smoothing functions defined by (4) contains the smoothing function (1.2) and (1.3) as special cases.

Motivated by the above mentioned work, by using the symmetric perturbed technique and the idea of convex combination, we propose a new class of smoothing functions. We also investigate a smoothing Newton method to solve the NCP based on a new class of smoothing functions. Our algorithm has the following nice properties: (a) Our algorithm needs only to solve one linear system of equations and perform one line search per iteration. (b) Here we give the boundedness of the level set and hence the iteration sequence is bounded and thus there exists at least one accumulation point. We do not need to assume the nonemptyness and boundedness of the solution set of NCP (1.1), although this assumption is widely used in the literature. (c) The function we use is a parametric class of smoothing functions containing some important smoothing complementarity functions as its special cases. We can adjust the two parameter to get better effect in practice. The numerical experiments implicate that the algorithm is efficient and promising.

The organization of this paper is as follows. In section 2, we recall some useful definitions and give some properties of new smoothing function. In section 3, we propose a smoothing Newton algorithm. Convergence results are analyzed in section 4. Some preliminary computational results are reported in section 5. Some words about notation are needed. All vectors are column vectors. $\Re^n_+$ and $\Re^n_{++}$ denote the nonnegative and positive orthants of $\Re^n$, respectively. We define $N = \{1, 2, \ldots, n\}$.

2. Preliminaries

In this section, we recall some useful definitions and give some properties of the new smoothing function defined by (4).

Definition 2.1. A matrix $M \in \Re^{n\times n}$ is said to be a P_0-matrix if all its principal minors are non-negative.

Definition 2.2. A function $F : \Re^n \to \Re^n$ is said to be a P_0-function if for all $x, y \in \Re^n$ with $x \neq y$, there exists an index $i_0 \in N$ such that

$$x_{i_0} \neq y_{i_0},\ (x_{i_0} - y_{i_0})\,[F_{i_0}(x) - F_{i_0}(y)] \geq 0.$$

The following lemma gives some properties of the smoothing function $\phi_\theta(\cdot,\cdot,\cdot)$ defined by (4). Its proof is obviously.

Lemma 2.3. *Let $(\mu, a, b) \in \Re^3$ and $\phi_\theta(\mu, a, b)$ be defined by (4). Then,*
(i) $\phi_\theta(0, a, b) = 0 \Leftrightarrow a \geq 0, b \geq 0, ab = 0$.
(ii) $\phi_\theta(\mu, a, b)$ is continuously differentiable for all points in $\Re^3$ different from $(0, c, c)$ for arbitrary $c \in \Re$. In particular, $\phi_\theta(\mu, a, b)$ is continuously differentiable for arbitrary $(\mu, a, b) \in \Re^3$ with $\mu \neq 0$.
(iii) $\phi_\theta(\mu, a, b)$ is semismooth on $\Re_{++} \times \Re^2$.

Let $z := (\mu, x) \in \Re_{++} \times \Re^n$ and

$$H(z) := \begin{pmatrix} e^\mu - 1 \\ \Phi_\theta(\mu, x) \end{pmatrix}, \tag{5}$$

where

$$\Phi_\theta(\mu, x) := \begin{pmatrix} \phi_\theta(\mu, x_1, F_1(x)) \\ \vdots \\ \phi_\theta(\mu, x_n, F_n(x)) \end{pmatrix}. \tag{6}$$

By (5) and Lemma 2.1, we known that solving NCP (1) is equivalent to solve $H(z) = 0$.

Define merit function $h : \Re_{++} \times \Re^n \to \Re_+$ by

$$h(z) := \|H(z)\|^2. \tag{7}$$

We also know that the NCP (1) is equivalent to the following equation:

$$h(z) = 0. \tag{8}$$

For simplicity, we denote

$$h_\theta(\mu, a, b) = \sqrt{\theta(1-\mu)^2(a-b)^2 + (1-\theta)[(a+\mu b)^2 + (b+\mu a)^2] + 2\mu^2}.$$

Lemma 2.4. *Let $H : \Re^{n+1} \to \Re^{n+1}$ and $\Phi_\theta : \Re^{n+1} \to \Re^n$ be defined by (5) and (6), respectively. Then:*
(i) Φ_θ is continuously differentiable at any $z = (\mu, x) \in \Re^{n+1}$ with $\mu \neq 0$.
(ii) H is continuously differentiable at any $z = (\mu, x) \in \Re_{++} \times \Re^n$ with its Jacobian

$$H'(z) = \begin{pmatrix} e^\mu & 0 \\ v(z) & w(z) \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned} v(z) &:= \operatorname{vec}\{x_i + F_i(x) - (d_\mu)_i,\ i \in N\}, \\ w(z) &:= D_1(z) + D_2(z)F'(x), \\ D_1(z) &:= diag\{1 + \mu - (d_1)_i,\ i \in N\}, \\ D_2(z) &:= diag\{1 + \mu - (d_2)_i,\ i \in N\}, \end{aligned}$$

with

$$\begin{aligned} (d_\mu)_i &= \frac{2x_iF_i(x) - \theta x_i^2 - \theta F_i^2(x) + (x_i^2 + F_i^2(x) - 2\theta x_i F_i(x) + 2)\mu}{h_\theta(\mu, x_i, F_i(x))},\ i \in N, \\ (d_1)_i &= \frac{x_i + \mu F_i(x) - \theta(F_i(x) + \mu x_i) + \mu[F_i(x) + \mu x_i - \theta(x_i + \mu F_i(x))]}{h_\theta(\mu, x_i, F_i(x))},\ i \in N, \\ (d_2)_i &= \frac{F_i(x) + \mu x_i - \theta(x_i + \mu F_i(x)) + \mu[x_i + \mu F_i(x) - \theta(F_i(x) + \mu x_i)]}{h_\theta(\mu, x_i, F_i(x))},\ i \in N. \end{aligned}$$

If F is a P_0-function, then the matrix $H'(z)$ is nonsingular on $\Re_{++} \times \Re^n$.

Proof. It is easy to see that Φ_θ is continuously differentiable at any $z = (\mu, x) \in \Re^{n+1}$ with $\mu \neq 0$.

Next we prove (ii). It follows from (i) and F is continuously differentiable that H is continuously differentiable at any $z = (\mu, x) \in \Re_{++} \times \Re^n$. From the definition of $H(z)$ (5), it follows that (9) holds. For all $i \in N$,

$$\begin{aligned} h_\theta(\mu, x_i, F_i(x)) &= \sqrt{[x_i + \mu F_i(x) - \theta(F_i(x) + \mu x_i)]^2 + (1-\theta^2)(F_i(x) + \mu x_i)^2 + 2\mu^2} \\ &= \sqrt{[F_i(x) + \mu x_i - \theta(x_i + \mu F_i(x))]^2 + (1-\theta^2)(x_i + \mu F_i(x))^2 + 2\mu^2}. \end{aligned}$$

By the above equation, we have

$$\begin{aligned} -1 &< \frac{x_i + \mu F_i(x) - \theta(F_i(x) + \mu x_i)}{h_\theta(\mu, x_i, F_i(x))} < 1 \text{ and} \\ -1 &< \frac{F_i(x) + \mu x_i - \theta(x_i + \mu F_i(x))}{h_\theta(\mu, x_i, F_i(x))} < 1. \end{aligned} \tag{10}$$

Since

$$\begin{aligned} (d_1)_i &= \frac{x_i + \mu F_i(x) - \theta(F_i(x) + \mu x_i)}{h_\theta(\mu, x_i, F_i(x))} + \mu \frac{F_i(x) + \mu x_i - \theta(x_i + \mu F_i(x))}{h_\theta(\mu, x_i, F_i(x))}, \\ (d_2)_i &= \frac{F_i(x) + \mu x_i - \theta(x_i + \mu F_i(x))}{h_\theta(\mu, x_i, F_i(x))} + \mu \frac{x_i + \mu F_i(x) - \theta(F_i(x) + \mu x_i)}{h_\theta(\mu, x_i, F_i(x))}, \end{aligned}$$

which together with (2.6), we have

$$|(d_1)_i| < 1 + \mu \text{ and } |(d_2)_i| < 1 + \mu, \text{ for all } i \in N.$$

Thus,

$$0 < 1 + \mu - (d_1)_i < 2 + 2\mu, \qquad 0 < 1 + \mu - (d_2)_i < 2 + 2\mu,$$

which imply that $D_1(z)$ and $D_2(z)$ are positive diagonal matrices for any $(\mu, x) \in \Re_{++} \times \Re^n$. Since F is a P_0-function, then $F'(x)$ is a P_0-matrix for any $x \in \Re^n$ by Lemma 5.4 in [13]. In view of the fact that $D_2(z)$ is a positive diagonal matrix, by a straightforward calculation we have that all principal minors of the matrix $D_2(z)F'(x)$ are nonnegative. By Definition 2.1, we know that the matrix $D_2(z)F'(x)$ is a P_0-matrix. Hence, by Theorem 3.1 in [14], the matrix $D_1(z) + D_2(z)F'(x)$ is obviously nonsingular, which implies that $H'(z)$ is nonsingular. □

3. Algorithm

In this section we shall present a smoothing Newton method for NCP and prove that the proposed algorithm is well defined.

Algorithm 3.1. (Smoothing Newton algorithm)

S0 Choose $\delta \in (0, 1), \sigma \in (0, \frac{1}{2})$ and $\overline{\mu} > 0$.

Take $\gamma \in (0, 1)$ such that $2\gamma\overline{\mu} < 1$.

Let $\mu_0 = \overline{\mu}, x_0 \in \Re^n$ be an arbitrary vector, $z^0 = (\mu_0, x^0), \overline{z} = (\overline{\mu}, 0), k := 0$.

S1 Termination criterion. If $\|H(z^k)\| = 0$, stop.

S2 Compute $\Delta z^k := (\Delta\mu_k, \Delta x^k) \in \Re^{n+1}$ by

$$H(z^k) + H'(z^k)\Delta z^k = e^{\mu_k}\beta_k \overline{z}, \tag{11}$$

where $\beta_k = \beta(z^k)$ is defined by $\beta(z) := \gamma min\{1, h(z)\}$.

S3 Let m_k is the smallest nonnegative integer such that

$$h(z^k + \delta^{m_k}\Delta z^k) \le [1 - 2\sigma(1 - 2\gamma\overline{\mu})\delta^{m_k}]h(z^k). \tag{12}$$

Let $\lambda_k := \delta^{m_k}$.
S4 Set $z^{k+1} = z^k + \lambda_k \Delta z^k$ and $k := k + 1$. Go to S1.

The following theorem proves that Algorithm 3.1 is well-defined and generates an infinite sequence. Define the set

$$\Omega := \{z = (\mu, x) \in \Re_+ \times \Re^n : \mu \geq \beta(z)\overline{\mu}\}. \tag{13}$$

Theorem 3.1. *Suppose F is a continuously differentiable P_0-function. Then, Algorithm 3.1 is well-defined and generates infinite sequence $\{z^k = (\mu_k, x^k)\}$ with $\mu_k \in \Re_{++}$ and $z^k \in \Omega$ for all $k \geq 0$.*

Proof. If $\mu_k > 0$, since F is a continuously differentiable P_0-function, then it follows from Lemma 2.2 that the matrix $H'(z^k)$ is nonsingular. Hence, step S2 is well-defined at the $k-$th iteration. By (11) we have

$$e^{\mu_k} - 1 + e^{\mu_k}\Delta\mu_k = e^{\mu_k}\beta_k\overline{\mu},$$

which implies

$$\Delta\mu_k = \beta_k\overline{\mu} + \frac{1 - e^{\mu_k}}{e^{\mu_k}} \geq \beta_k\overline{\mu} - \mu_k,$$

where the second inequality follows from $\frac{1-e^{\mu}}{e^{\mu}} \geq -\mu$ for any $\mu > 0$.
Hence, by the first equation of (3.1), we can get

$$\mu_{k+1} = \mu_k + \lambda_k\Delta\mu_k \geq \mu_k + \lambda_k(\beta_k\overline{\mu} - \mu_k) = (1 - \lambda_k)\mu_k + \lambda_k\beta_k\overline{\mu} > 0.$$

From (2.1) and (2.4), we have

$$e^{\mu_k} - 1 \leq \sqrt{h(z^k)}. \tag{14}$$

Let $R^k(\alpha) = h(z^k + \alpha\Delta z^k) - h(z^k) - \alpha h'(z^k)\Delta z^k$. It is easy to see that $R(\alpha) = o(\alpha)$. When $h(z) > 1$, $\beta(z) = \gamma < \gamma\sqrt{h(z)} = \gamma\|H(z)\|$, while $h(z) < 1$, $\beta(z) = \gamma h(z) \leq \gamma\sqrt{h(z)} = \gamma\|H(z)\|$, thus

$$\beta(z) \leq \gamma\|H(z)\|. \tag{15}$$

Then by (3.1), (3.2), (3.4) and (3.5), we have

$$\begin{aligned}
h(z^k + \alpha\Delta z^k) =& R^k(\alpha) + h(z^k) + \alpha h'(z^k)\Delta z^k \\
=& R^k(\alpha) + h(z^k) + 2\alpha H(z^k)^T H'(z^k)\Delta z^k \\
=& R^k(\alpha) + h(z^k) + 2\alpha H(z^k)^T(-H(z^k) + e^{\mu_k}\beta_k\overline{\mu}) \\
=& (1 - 2\alpha)h(z^k) + 2\alpha H(z^k)^T e^{\mu_k}\beta_k\overline{\mu} + o(\alpha) \\
\leq& (1 - 2\alpha)h(z^k) + 2\alpha\|H(z^k)\|(e^{\mu_k} - 1)\beta_k\overline{\mu} + 2\alpha\|H(z^k)\|\beta_k\overline{\mu} + o(\alpha) \\
\leq& (1 - 2\alpha)h(z^k) + 2\alpha\gamma\overline{\mu}h(z^k) + 2\alpha\gamma\overline{\mu}h(z^k) + o(\alpha) \\
=& [1 - 2(1 - 2\gamma\overline{\mu})\alpha]h(z^k) + o(\alpha) \\
=& [1 - 2\sigma(1 - 2\gamma\overline{\mu})\alpha]h(z^k) - 2(1 - \sigma)(1 - 2\gamma\overline{\mu})\alpha h(z^k) + o(\alpha).
\end{aligned}$$

Since $\sigma \in (0, \frac{1}{2})$ and $2\gamma\overline{\mu} < 1$, then$(1 - \sigma)(1 - 2\gamma\overline{\mu})h(z^k) > 0$. For α sufficiently small, we can get $h(z^k + \alpha\Delta z^k) \leq [1 - 2\sigma(1 - 2\gamma\overline{\mu})\alpha]h(z^k)$, this shows that step

S3 is well-defined at the k-th iteration. Therefore, Algorithm 3.1 is well-defined and generates an infinite sequence $\{z^k = (\mu_k, x^k)\}$ with $\mu_k \in \Re_{++}$.

Next, we will prove $z^k \in \Omega$ for $k \geq 0$. This can be obtained by inductive method. Firstly, it is evident from the choice of the starting point $z^0 \in \Omega$. Secondly, suppose that $z^k \in \Omega$, then by (13) we have $\mu_k \geq \beta(z^k)\overline{\mu}$, then

$$\begin{aligned}\mu_{k+1} - \beta(z^{k+1})\overline{\mu} =&\mu_k + \lambda_k \beta(z^k)\overline{\mu} + \lambda_k \frac{1-e^{\mu_k}}{e^{\mu_k}} - \beta(z^{k+1})\overline{\mu}\\ \geq&(1-\lambda_k)\mu_k + \lambda_k \beta(z^k)\overline{\mu} - \beta(z^{k+1})\overline{\mu}\\ \geq&(1-\lambda_k)\beta(z^k)\overline{\mu} + \lambda_k \beta(z^k)\overline{\mu} - \beta(z^{k+1})\overline{\mu}\\ =&\left(\beta(z^k) - \beta(z^{k+1})\right)\overline{\mu}\\ \geq&0.\end{aligned}$$

□

4. Convergence of Algorithm 3.1

In this section, we discuss the global convergence and local superlinear convergence of Algorithm 3.1. We need the following Lemma 4.1 which can be founded in [15].

Lemma 4.1. *Let $\varepsilon > 0$ and the function $\phi : \Re^2 \to \Re$ be defined by*

$$\phi(a, b) := a + b - \sqrt{a^2 + b^2 + \varepsilon}.$$

Let $\{a^k\}, \{b^k\} \subseteq \Re$ be any two sequences such that $a^k, b^k \to +\infty$ or $a^k \to -\infty$ or $b^k \to -\infty$. Then $|\phi(a^k, b^k)| \to +\infty$.

Lemma 4.2. *Let $\widetilde{\phi_\theta}$ be defined by*

$$\widetilde{\phi_\theta}(\mu, a, b) = a + b - \sqrt{\theta(a-b)^2 + (1-\theta)(a^2+b^2) + 2\mu^2},\ \forall (a,b) \in \Re^2,\ \mu > 0.$$

Assume that $\{a^k\}, \{b^k\} \subseteq \Re$ be any two sequences such that $a^k, b^k \to +\infty$ or $a^k \to -\infty$ or $b^k \to -\infty$. Then $|\widetilde{\phi_\theta}(\mu_k, a^k, b^k)| \to +\infty$.

Proof. (i) Suppose that $a^k \to -\infty$. If $\{b^k\}$ is bounded, then the result holds obviously; else if $b^k \to +\infty$, we have $-a^k > 0$ and $b^k > 0$ for all k sufficiently large, and hence,

$$\begin{aligned}&\sqrt{\theta(a^k-b^k)^2 + (1-\theta)((a^k)^2+(b^k)^2) + 2\mu_k^2} - b^k\\ \geq&\sqrt{\theta(b^k)^2 + (1-\theta)(b^k)^2 + 2\mu_k^2} - b^k > 0,\end{aligned}$$

which, together with $-a^k \to +\infty$, implies that $\widetilde{\phi_\theta} \to -\infty$. Thus $|\widetilde{\phi_\theta}| \to +\infty$.

(ii) For the case of $b^k \to -\infty$. By using the symmetry of function $\widetilde{\phi_\theta}$ about a^k, b^k, we know the result holds.

(iii) Suppose that $a^k \to +\infty$ and $b^k \to +\infty$. Thus, for sufficiently large k,

$$\sqrt{\theta(a^k-b^k)^2 + (1-\theta)((a^k)^2+(b^k)^2) + 2\mu_k^2} \leq \sqrt{(a^k)^2+(b^k)^2+2\mu_k^2},$$

hence,

$$\begin{aligned}&a^k + b^k - \sqrt{\theta(a^k - b^k)^2 + (1-\theta)((a^k)^2 + (b^k)^2) + 2\mu_k^2}\\ \geq& a^k + b^k - \sqrt{(a^k)^2 + (b^k)^2 + 2\mu_k^2}.\end{aligned}$$

By Lemma 4.1, we know that

$$|\widetilde{\phi_\theta}(\mu_k, a^k, b^k)| = a^k + b^k - \sqrt{\theta(a^k - b^k)^2 + (1-\theta)((a^k)^2 + (b^k)^2) + 2\mu_k^2} \to +\infty.$$

□

Lemma 4.3. *Let F be a continuous P_0-function and $\Phi_\theta(\mu, x)$ be defined by (6). For any $\mu > 0$ and $c > 0$, define the level set*

$$L_\mu(c) := \{x \in \Re^n : \|\Phi_\theta(\mu, x)\| \leq c\}. \tag{16}$$

Then, for any $0 < \mu_1 \leq \mu_2$ and $c > 0$, the set $L(c) := \cup_{\mu_1 \leq \mu \leq \mu_2} L_\mu(c)$ is bounded.

Proof. Suppose, to the contrary, that $L_\mu(c)$ is unbounded. Then for some fixed $c > 0$, we can find a sequence $\{(\mu_k, x^k)\}$ such that $\mu_1 \leq \mu_k \leq \mu_2$ and $\|\Phi_\theta(\mu_k, x^k)\| \leq c$, $\|x^k\| \to \infty$.

Since the sequence $\{x^k\}$ is unbounded, then the index set $J := \{i \in N : \{x_i^k\}$ is unbounded $\}$ is nonempty. Without loss of generality, we can assume that $\{|x_i^k| \to \infty\}$ for all $i \in J$. Let the sequence $\{\widetilde{x}^k\}$ be defined by

$$\widetilde{x}^k = \begin{cases} 0 & \text{if } i \in J \\ x_i^k & \text{if } i \notin J. \end{cases} \tag{17}$$

Then, $\{\widetilde{x}^k\}$ is bounded. Note that F is a P_0-function, by Definition 2.2, we have

$$\begin{aligned} 0 \leq& \max_{i \in N} (x_i^k - \widetilde{x}_i^k) \left[F_i(x^k) - F_i(\widetilde{x}^k)\right] \\ =& \max_{i \in J} x_i^k \left[F_i(x^k) - F_i(\widetilde{x}^k)\right] \\ =& x_j^k \left[F_j(x^k) - F_j(\widetilde{x}^k)\right], \end{aligned} \tag{18}$$

where j is one of the indices for which the max is attained, and j is assumed, without loss of generality, to be independent of k, we obtained $|x_j^k| \to \infty$.
We consider the following two cases:
case 1: $x_j^k \to +\infty$. In this case, since $\{F_j(\widetilde{x}^k)\}$ is bounded by the continuity of F_j, we deduce from Equation (4.3) $F_j(x^k) \nrightarrow -\infty$. Since $\mu_1 \leq \mu_k \leq \mu_2$, we have

$$\mu_k x_j^k + F_j(x^k) \to +\infty, \; x_j^k + \mu_k F_j(x^k) \to +\infty.$$

By Lemma 4.2, we know that

$$|\Phi_{\theta,j}(\mu_k, x^k)| \to \infty.$$

case 2: $x_j^k \to -\infty$. In this case, since $\{F_j(\widetilde{x}^k)\}$ is bounded by the continuity of F_j, we deduce from Equation (4.3) $F_j(x^k) \le F_j(\widetilde{x}^k)$ for any k. Since $\mu_1 \le \mu_k \le \mu_2$, we have

$$\mu_k x_j^k + F_j(x^k) \to -\infty, \ x_j^k + \mu_k F_j(x^k) \to -\infty,$$

which, together with Lemma 4.2, gives

$$|\Phi_{\theta,j}(\mu_k, x^k)| \to \infty.$$

In either case, we obtained $\|\Phi_\theta(\mu_k, x^k)\| \to +\infty$, which contradicts with $\|\Phi_\theta(\mu_k, x^k)\| \le c$. This completes the proof. □

Corollary 4.3 Suppose that F is a P_0-function and $\mu > 0$. Then the function $\|\Phi_\theta(\mu, x)\|$ is coercive, i.e., $\lim_{\|x\|\to\infty} \|\Phi_\theta(\mu, x)\| = +\infty$.

Theorem 4.4. *Suppose F is a continuously differentiable P_0-function, and the sequence $\{z^k = (\mu_k, x^k)\}$ is generated by Algorithm 3.1. Then the sequence $\{z^k\}$ is bounded and any accumulation point $z^* = (\mu_*, x^*)$ of the sequence $\{z^k\}$ is a solution of $H(z^k) = 0$.*

Proof. Since $h(z^k)$ is monotonically decreasing and bounded from below by zero, it then follows that the sequence $\|\Phi_\theta(z^k)\|$ is bounded. By Corollary 4.3, we immediately obtain $\{x^k\}$ is bounded. Note that the boundedness of $\{h(z^k)\}$ implies the boundedness of μ_k. So $\{z^k\}$ is bounded. Without loss of generality, suppose $z^k \to z^*$. Then $h(z^k) \to h^*, \beta(z^k) \to \beta^*$. If $h(z^k) = 0$, we obtain the desired result. Now, we prove $h^* = 0$ by contradiction. In fact, if $h^* \ne 0$, then $h^* > 0$, then $\beta^* = \gamma \min\{1, h^*\} > 0$, and $\mu^* \ge \beta^* \overline{\mu}$. It follows from Lemma 2.2 that $H'(z^*)$ is nonsingular. By the continuity of $H'(z)$, there exists a closed neighborhood $N(z^*)$ of z^* such that for any $z \in N(z^*)$, we have $\mu \in \Re_{++}$ and $H'(z)$ is invertible. So, for all sufficiently large k, $z^k \in N(z^*)$ and $H'(z^k)$ is invertible. Let $\Delta z^k = (\Delta\mu_k, \Delta x^k) \in \Re \times \Re^n$ be the unique solution of the following system:

$$H(z^k) + H'(z^k)\Delta z^k = e^{\mu_k}\beta_k \overline{z},$$

It follows from the continuity of H and the definition of $\beta(.)$ that $\{\mu_k\}$ and $\{\beta_k\}$ converge to μ_* and β^*, respectively. That together with (3.2), implies that

$$\lim_{k\to\infty} \lambda_k = 0.$$

Thus, for sufficiently large k, the stepsize $\widehat{\lambda}_k := \frac{\lambda_k}{\delta}$ does not satisfy (3.2), then

$$h(z^k + \widehat{\lambda}_k \Delta z^k) > \left[1 - 2\sigma(1 - 2\gamma\overline{\mu})\widehat{\lambda}_k\right] h(z^k), \tag{19}$$

which implies that

$$\frac{h(z^k + \widehat{\lambda}_k \Delta z^k) - h(z^k)}{\widehat{\lambda}_k} > -2\sigma(1 - 2\gamma\overline{\mu})h(z^k). \tag{20}$$

$$\begin{aligned} H(z^*)^T H'(z^*)\Delta z^* &= \lim_{k\to\infty} H(z^k)^T H'(z^k)\Delta z^k \\ &= \lim_{k\to\infty} H(z^k)^T(-H(z^k) + e^{\mu_k}\beta_k\overline{\mu}) \end{aligned}$$

$$\begin{aligned}&=\lim_{k\to\infty}\left(-h(z^k)+H(z^k)^T(e^{\mu_k}-1)\beta_k\overline{\mu}+H(z^k)^T\beta_k\overline{\mu}\right) \quad (21)\\&\le\lim_{k\to\infty}\left(-h(z^k)+2\gamma\overline{\mu}\|H(z^k)\|^2\right)\\&=(2\gamma\overline{\mu}-1)h(z^*).\end{aligned}$$

Taking limits on both sides of the inequalities (4.5), from (4.6) we have

$$\begin{aligned}-2\sigma(1-2\gamma\overline{\mu})h(z^*) &\le 2H(z^*)^TH'(z^*)\Delta z^*\\&\le 2(2\gamma\overline{\mu}-1)h(z^*).\end{aligned}$$

This indicates that $-\sigma(1-2\gamma\overline{\mu})\le 2\gamma\overline{\mu}-1$, since $2\gamma\overline{\mu}<1$, we have $\sigma\ge 1$, which contradicts $\sigma<\frac{1}{2}$. Thus, $h(z^*)=0$ and $\mu_*=0$. Hence $z^*=(\mu_*,x^*)$ is a solution of $H(\mu,x)=0$. □

Theorem 4.5. *Suppose that F is a continuously differentiable P_0-function. Let z^* be an accumulation point of the iteration sequence $\{z^k\}$ generated by Algorithm 3.1. If all $V\in\partial H(z^*)$ are nonsingular, then:*
(1) $\lambda_k\equiv 1$, for all z^k sufficiently close to z^*;
(2) the whole sequence $\{z^k\}$ converges to z^*;
(3) $\|z^{k+1}-z^*\|=o(\|z^k-z^*\|)$(or $\|z^{k+1}-z^*\|=O(\|z^k-z^*\|^2)$ if F' is Lipschitz continuous on $\Re^n$).

Proof. The proof is similar to the one given in [16], Theorem 3.2. □

5. Numerical experiments

In this section, we report some numerical results of Algorithm 3.1. All experiments are done using a PC with CPU of 1.6 GHz and RAM of 512 MB, and all codes are finished in MATLAB 7.5. Throughout our computational experiments, the parameters used in the algorithm are chosen as

$$\delta=0.5,\ \sigma=0.06,\ \gamma=0.001,\ \overline{\mu}=1.0.$$

In our implementation, we use $\|H(z^k)\|\le 10^{-6}$ as the stopping rule.

Example 5.1. Kojima-Shindo Problem. This test problem was used by Pang and Gabriel [17], Mangasarian and Solodov [18], Kanzow [19], and Jiang and Qi [20] with four variables. Let

$$\begin{aligned}F_1(x)&=3x_1^2+2x_1x_2+2x_2^2+x_3+3x_4-6,\\F_2(x)&=2x_1^2+x_1+x_2^2+10x_3+2x_4-2,\\F_3(x)&=3x_1^2+x_1x_2+2x_2^2+2x_3+9x_4-9,\\F_4(x)&=x_1^2+3x_2^2+2x_3+3x_4-3.\end{aligned}$$

Table 1 gives the results for this example with starting points $a_1=(0,0,0,1)^T$, $a_2=(1,-2,1,-2)^T$, $a_3=(1,2,6,8)^T$.

TABLE 1. Numerical results for Examples 5.1 to 5.4

		a_1			a_2			a_3		
EX	θ	IT	NF	CPU	IT	NF	CPU	IT	NF	CPU
5.1	0	9	14	0.060319	10	16	0.063772	10	18	0.070342
	0.25	8	13	0.053006	10	15	0.055583	11	19	0.080942
	0.5	8	13	0.053622	10	15	0.056014	7	8	0.050825
	0.75	8	13	0.071610	9	12	0.055434	7	8	0.058709
	1	-	-	-	11	18	0.076599	8	10	0.051401
5.2	0	10	23	0.063856	16	81	0.112669	14	33	0.084473
	0.25	12	32	0.069427	13	36	0.080319	12	30	0.067422
	0.5	13	35	0.066243	11	22	0.061275	12	29	0.070216
	0.75	12	33	0.065865	11	19	0.062824	11	23	0.071821
	1	14	38	0.089583	-	-	-	-	-	-
5.3	0	21	45	0.072896	24	51	0.063188	15	24	0.065481
	0.25	8	20	0.055192	15	27	0.051503	7	11	0.043257
	0.5	7	12	0.050989	17	31	0.062504	6	7	0.040183
	0.75	6	8	0.040433	18	33	0.063182	18	33	0.055909
	1	23	45	0.082194	23	56	0.075995	24	60	0.074259
5.4	0	13	26	0.051879	15	39	0.075145	24	98	0.083144
	0.25	10	20	0.048800	12	25	0.070647	21	96	0.080437
	0.5	11	24	0.052045	9	15	0.055815	23	86	0.081092
	0.75	10	22	0.048697	12	31	0.053310	14	29	0.053238
	1	15	42	0.057394	14	37	0.075407	20	70	0.072042

Example 5.2. Josephy Problem. This test problem was used by Dirkse and Ferris [22] with four variables. Let

$$\begin{aligned} F_1(x) &= 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6, \\ F_2(x) &= 2x_1^2 + x_1 + x_2^2 + 3x_3 + 2x_4 - 2, \\ F_3(x) &= 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 3x_4 - 1, \\ F_4(x) &= x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3. \end{aligned}$$

Table 1 gives the results for this example with starting points $a_1 = (2, -2, -2, -2)^T$, $a_2 = (2, 3, 4, 6)^T$, $a_3 = (0, 2, 0, 6)^T$.

Example 5.3. Mathiesen Problem. This test problem was used by Pang and Gabriel [17] with four variables, which was also tested by Kanzow [19] . Let

$$\begin{aligned} F_1(x) &= -x_2 + x_3 + x_4, \\ F_2(x) &= x_1 - \alpha(b_2x_3 + b_3x_4)/x_2, \\ F_3(x) &= b_2 - x_1 - (1 - \alpha)(b_2x_3 + b_3x_4)/x_3, \\ F_4(x) &= b_3 - x_1, \end{aligned}$$

where $\alpha = 0.75$, $b_2 = 1$, $b_3 = 2$. Table 1 gives the results for this example with starting points $a_1 = (0.5, 0.5, 0.5, 2)^T$, $a_2 = (2, -2, -2, -2)^T$, $a_3 = (0, -2, -2, 0)^T$.

Example 5.4. HS 34 Problem. This test problem was from the book of Hock and Schittkowski [21]: Their Karush-Kuhn-Tucker (KKT) optimality conditions lead to complementarity problems of dimensions 8. Let

$$\begin{aligned}
F_1(x) &= -1 + x_4 e^{x_1} + x_6,\\
F_2(x) &= -x_4 + x_5 e^{x_2} + x_7,\\
F_3(x) &= -x_5 + x_8,\\
F_4(x) &= x_2 - e^{x_1},\\
F_5(x) &= x_3 - e^{x_2},\\
F_6(x) &= 100 - x_1,\\
F_7(x) &= 100 - x_2,\\
F_8(x) &= 10 - x_3.
\end{aligned}$$

Table 1 gives the results with starting points $a_1 = (-1, -1, -1, 1, 1, 1, 1, 1)^T$, $a_2 = (0, 0, 0, 1, 1, 1, 1, 1)^T$, $a_3 = (1, 1, 1, -10, -10, -10, -10, -10)^T$.

In Table 1, IT denotes the numbers of iteration; NF denotes the numbers of function value's evaluation; CPU denotes the CPU time for solving the underlying problem in second; and $-$ denotes the algorithm fails to find the optimizer in the sense that the iteration numbers are larger than 1000.

Table 1 shows that not all the best numerical results occur in the case of $\theta = 0$(in this case, the smoothing function is proposed by Huang et. al. in [11]) or $\theta = 1$ (in this case, the smoothing function is proposed by Huang et. al. in [10]). These demonstrate that the new smoothing function introduced in this paper is worth investigating. The Figures 1 and 2 below plot the corresponding convergence of merit function $h(z^k)$ versus the iteration number. From the two figures, when $\theta = 0.5$ and $\theta = 0.75$, $h(z^k)$ has a faster decrease than $\theta = 0$ and $\theta = 1$. These also demonstrate that the new smoothing function introduced in this paper is worth investigating. Numerical experiments also demonstrate the feasibility and efficiency of the new algorithm.This new proposed class of complementarity functions have great advantage because we can adjust the parameter θ to obtain an optimal solution to NCP.

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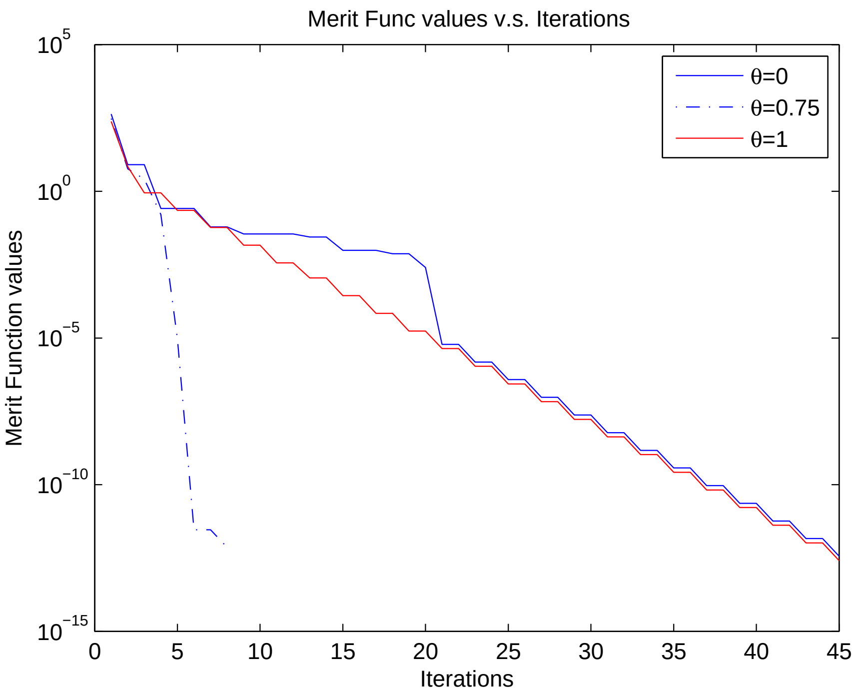

FIGURE 1. Convergence behavior of Example 5.3 with the initial point a_1

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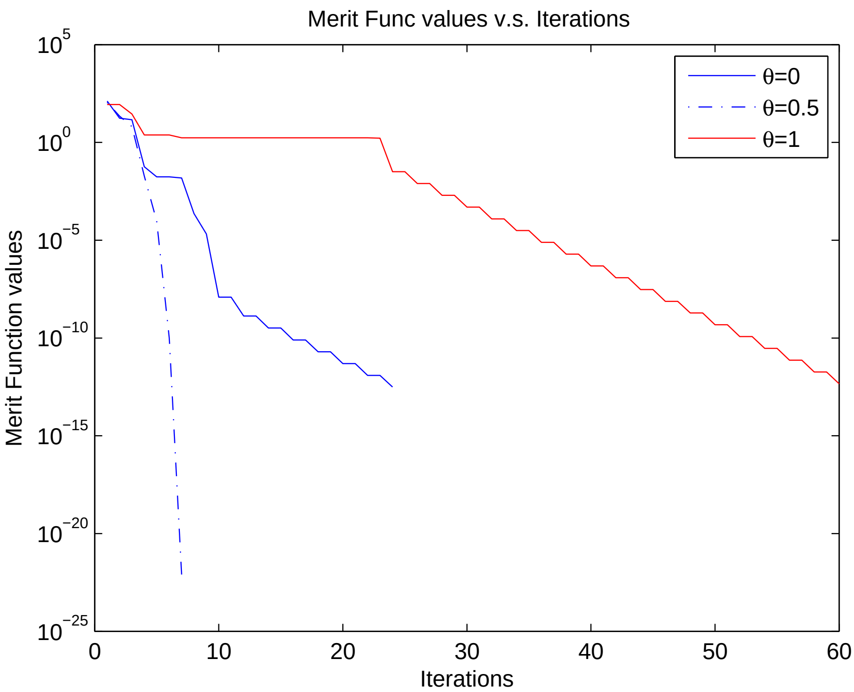

FIGURE 2. Convergence behavior of Example 5.3 with the initial point a_3

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