J. Appl. Math. & Informatics Vol. **32***(2014), No. 1 - 2, pp. 153 - 159*
http://dx.doi.org/10.14317/jami.2014.153

WOLFE TYPE HIGHER ORDER SYMMETRIC DUALITY UNDER INVEXITY†

KHUSHBOO VERMA* AND T. R. GULATI

ABSTRACT. In this paper, we introduce a pair of higher-order symmetric dual models/problems. Weak, strong and converse duality theorems for this pair are established under the assumption of higher-order invexity. Moreover, self duality theorem is also discussed.

AMS Mathematics Subject Classification : 90C46, 49N15, 90C30.
Key words and phrases : Higher-order dual models, symmetric duality, Duality theorems, Higher-order invexity.

1. Introduction

The concept of symmetric duality was first introduced by Dorn [8] for quadratic programming. Later, in non linear programming this concept was significantly developed by Dantzig et al. [7], Mond [15] and Bazarra and Goode [5]. Mangasarian [14] introduced the concept of second and higher-order duality for nonlinear programming problems, which motivated several authors to work on second order dualty [3, 4, 9, 10, 11]. Subsequently, higher-order symmetric duality for nonlinear problems has been studied in [1, 2, 12, 18]. The study of second and higher-order duality is significant due to the computational advantage over the first-order duality as it provides tighter bounds for the value of the objective function when approximations are used. Mond and Zhang [16] discussed the duality results for various higher-order dual problems under invexity assumptions. For a pair of nondifferentiable programs, Chen [6] also discussed the duality theorems under higher-order generalized $F-$convexity. Yang et al. [19] obtained the duality results for multiobjective higher-order symmetric duality under invexity assumptions.

Received March 6, 2013. Revised July 12, 2013. Accepted July 15, 2013. *Coresponding Author. †The first author is thankful to the MHRD, Government of India for providing financial support.

Recently, Ahmad [1] discussed higher-order duality in nondifferentiable Multiobjective Programming. In this paper, we introduce a pair of higher-order symmetric dual models/problems. Weak, strong and converse duality theorems for this pair are established under the assumption of higher-order generalized invexity. Moreover, self duality theorem is also discussed.

2. Preliminaries

Definition 2.1. A function $\phi : R^n \mapsto R$ is said to be higher-order invexity at $u \in R^n$ with respect to $\eta : R^n \times R^n \mapsto R^n$ and $h : R^n \times R^n \mapsto R$, if for all $(x, p) \in R^n \times R^n$,

$$\phi(x) - \phi(u) - h(u, p) + p^T \nabla_p h(u, p) \geqq \eta^T(x, u)[\nabla_x \phi(u) + \nabla_p h(u, p)].$$

Let $\nabla_{xy} f$ denote the $n \times m$ matrix and $\nabla_{yx} f$ denote the $m \times n$ matrix of second order derivative. Also $\nabla_{xx} f$ and $\nabla_{yy} f$ denote the $n \times n$ and $m \times m$ symmetric Hessian matrices with respect to x and y, respectively.

3. Higher order symmetric duality

We consider the following pair of higher order symmetric duals and establish weak, strong and converse duality theorems.

Primal Problem (WHP):

Minimize

$L(x, y, p) = f(x, y) + h(x, y, p) - p^T \nabla_p h(x, y, p) - y^T [\nabla_y f(x, y) + \nabla_p h(x, y, p)]$

subject to

$$\nabla_y f(x, y) + \nabla_p h(x, y, p) \leqq 0, \tag{1}$$

$$p^T [\nabla_y f(x, y) + \nabla_p h(x, y, p)] \geqq 0, \tag{2}$$

$$x, \ y \geqq 0, \tag{3}$$

Dual Problem (WHD):

Maximize

$M(u, v, r) = f(u, v) + g(u, v, r) - r^T \nabla_r g(u, v, r) - u^T [\nabla_u f(u, v) + \nabla_r g(u, v, r)]$

subject to

$$\nabla_u f(u, v) + \nabla_r g(u, v, r) \geqq 0, \tag{4}$$

$$r^T [\nabla_u f(u, v) + \nabla_r g(u, v, r)] \leqq 0, \tag{5}$$

$$u, \ v \geqq 0, \tag{6}$$

where $f : R^n \times R^m \mapsto R, \ g : R^n \times R^m \times R^n \mapsto R$ and $h : R^n \times R^m \times R^m \mapsto R$ are twice differentiable functions.

Theorem 3.1 (Weak Duality)**.** *Let (x, y, p) and (u, v, r) be feasible solutions for primal and dual problem, respectively. Suppose that*
(i) $f(., v)$ is higher-order invexity at u with respect to η_1 and $g(u, v, r)$,
(ii) $-[f(x, .)]$ is higher-order invexity at y with respect to η_2 and $-h(x, y, p)$,
(iii) $\eta_1(x, u) + u + r \geqq 0$,
(iv) $\eta_2(v, y) + y + p \geqq 0$.
Then

$$L(x, y, p) \geqq M(u, v, r). \tag{7}$$

Proof. It is given (x, y, p) is feasible for (WHP) and (u, v, r) is feasible for (WHD), therefore by the hypothesis (iii) and the dual constraints (4), we get

$$(\eta_1(x, u)^T + u + r)[\nabla_x f(u, v) + \nabla_r g(u, v, r)] \geqq 0,$$

or

$$(\eta_1(x, u)^T + u)[\nabla_x f(u, v) + \nabla_r g(u, v, r)] \geqq -r^T[\nabla_x f(u, v) + \nabla_r g(u, v, r)],$$

which on using the dual constraint (5) implies that

$$(\eta_1(x, u) + u)^T[\nabla_x f(u, v) + \nabla_r g(u, v, r)] \geqq 0. \tag{8}$$

Now by the higher order invexity of $f(., v)$ at v with respect to η_1 and $g(u, v, r)$, we get

$$f(x, v) - f(u, v) - g(u, v, r) + r^T\nabla_r g(u, v, r) + u^T[\nabla_x f(u, v) + \nabla_r g(u, v, r)] \geqq 0. \tag{9}$$

Similarly, hypothesis (iv) along with primal constraints (1) and (3) yields

$$(\eta_2(v, y) + y)^T[\nabla_y f(x, y) + \nabla_p h(x, y, p)] \leqq 0. \tag{10}$$

Therefore, by higher-order invexity of $-[f(x, .)]$ at y with respect to η_2 and $-h(x, y, p)$, we obtain

$$f(x, y) - f(x, v) + h(x, y, p) - p^T\nabla_p h(x, y, p) - y^T[\nabla_y f(x, y) + \nabla_p h(x, y, p)] \geqq 0. \tag{11}$$

Adding inequalities (9) and (11), we get

$$f(x, y) + h(x, y, p) - p^T\nabla_p h(x, y, p) - y^T[\nabla_y f(x, y) + \nabla_p h(x, y, p)]$$
$$\geqq f(u, v) + g(u, v, r) - r^T\nabla_r g(u, v, r) - u^T[\nabla_x f(u, v) + \nabla_r g(u, v, r)].$$

or

$$L(x, y, z, p) \geqq M(u, v, w, r).$$

Thus the result holds. □

Theorem 3.2 (Strong Duality)**.** *Let $(\bar{x}, \bar{y}, \bar{p})$ be a local optimal solution of (WHP). Assume that*
(i) $\nabla_{pp}h(\bar{x}, \bar{y}, \bar{p})$ is negative definite,
(ii)$\nabla_y f(\bar{x}, \bar{y}) + \nabla_p h(\bar{x}, \bar{y}, \bar{p}) \neq 0$,
(iii) $\bar{y}^T[\nabla_y h(\bar{x}, \bar{y}, \bar{p}) - \nabla_p h(\bar{x}, \bar{y}, \bar{p}) + \nabla_{yy} f(\bar{x}, \bar{y})\bar{p}] = 0 \Rightarrow \bar{p} = 0$,
(iv) $h(\bar{x}, \bar{y}, 0) = g(\bar{x}, \bar{y}, 0)$, $\nabla_x h(\bar{x}, \bar{y}, 0) = \nabla_r g(\bar{x}, \bar{y}, 0)$, $\nabla_y h(\bar{x}, \bar{y}, 0) = \nabla_p h(\bar{x}, \bar{y}, 0)$.
Then (I) $(\bar{x}, \bar{y}, \bar{r} = 0)$ is feasible for (WHD) and
(II) $L(\bar{x}, \bar{y}, \bar{p}) = M(\bar{x}, \bar{y}, \bar{r})$. Also, if the hypotheses of Theorem (3.1) hold for all

feasible solutions of (WHP) and (WHD), then $(\bar{x}, \bar{y}, \bar{p} = 0)$ and $(\bar{x}, \bar{y}, \bar{r} = 0)$ are global optimal solutions of (WHP) and (WHD), respectively.

Proof. Since $(\bar{x}, \bar{y}, \bar{p})$ is a local optimal solution of (WHP), there exist $\alpha,\ \delta \in R,\ \beta,\ \xi \in R^m$ and $\mu, \in R^n$ such that the following Fritz-John conditions [13, 17] are satisfied at $(\bar{x}, \bar{y}, \bar{p})$:

$$\alpha[\nabla_x f(\bar{x}, \bar{y}) + \nabla_x h(\bar{x}, \bar{y}, \bar{p}) - \nabla_{px} h(\bar{x}, \bar{y}, \bar{p})\bar{p}] + [\nabla_{yx} f(\bar{x}, \bar{y}) + \nabla_{px} h(\bar{x}, \bar{y}, \bar{p})](\beta - \alpha\bar{y} - \delta\bar{p}) - \mu = 0, \tag{12}$$

$$\alpha[\nabla_y h(\bar{x}, \bar{y}, \bar{p}) - \nabla_{py} h(\bar{x}, \bar{y}, \bar{p})\bar{p} - \nabla_p h(\bar{x}, \bar{y}, \bar{p})] + [\nabla_{yy} f(\bar{x}, \bar{y}) + \nabla_{py} h(\bar{x}, \bar{y}, \bar{p})](\beta - \alpha\bar{y} - \delta\bar{p}) - \xi = 0, \tag{13}$$

$$\nabla_{pp} h(\bar{x}, \bar{y}, \bar{p})(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p}) - \delta[\nabla_y f(\bar{x}, \bar{y}) + \nabla_p h(\bar{x}, \bar{y}, \bar{p})] = 0, \tag{14}$$

$$\beta^T[\nabla_y f(\bar{x}, \bar{y}) + \nabla_p h(\bar{x}, \bar{y}, \bar{p})] = 0, \tag{15}$$

$$\delta\bar{p}^T[\nabla_y f(\bar{x}, \bar{y}) + \nabla_p h(\bar{x}, \bar{y}, \bar{p})] = 0, \tag{16}$$

$$\bar{x}^T\mu = 0, \tag{17}$$

$$\bar{y}^T\xi = 0, \tag{18}$$

$$(\alpha, \beta, \delta, \mu, \xi) \geqq 0,\ (\alpha, \beta, \delta, \mu, \xi) \neq 0. \tag{19}$$

Premultiplying equation (14) by $(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p})$, we get

$$(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p})^T \nabla_{pp} h(\bar{x}, \bar{y}, \bar{p})(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p}) - \delta(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p})^T[\nabla_y f(\bar{x}, \bar{y}) + \nabla_p h(\bar{x}, \bar{y}, \bar{p})] = 0,$$

which along with equations (15), (16) yields

$$(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p})^T \nabla_{pp} h(\bar{x}, \bar{y}, \bar{p})(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p}) + \alpha\delta\bar{y}^T[\nabla_y f(\bar{x}, \bar{y}) + \nabla_p h(\bar{x}, \bar{y}, \bar{p})] = 0, \tag{20}$$

Now from equations (1), (19) and (20), we obtain

$$(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p})^T \nabla_{pp} h(\bar{x}, \bar{y}, \bar{p})(\beta - \alpha\bar{p} - \alpha\bar{y} - \delta\bar{p}) \geqq 0. \tag{21}$$

Using hypothesis (i) in inequality (21), we have

$$\beta = \alpha\bar{p} + \alpha\bar{y} + \delta\bar{p}. \tag{22}$$

This, together with hypothesis (ii) and equation (14), yields

$$\delta = 0. \tag{23}$$

Now, we claim that $\alpha \neq 0$. Indeed if $\alpha = 0$, then equations (22) and (23) give

$$\beta = 0.$$

Therefore equations (12), (13) and (23), imply $\mu = 0$ and $\xi = 0$. Hence $(\alpha, \beta, \delta, \mu, \xi)$, a contradiction to (19). Thus

$$\alpha > 0. \tag{24}$$

Using equations (13), (22) and (24), we have

$$[\nabla_y h(\bar{x}, \bar{y}, \bar{p}) - \nabla_p h(\bar{x}, \bar{y}, \bar{p}) + \nabla_{yy} f(\bar{x}, \bar{y})\bar{p}] = \frac{\xi}{\alpha},$$

or

$$\bar{y}^T[\nabla_y h(\bar{x},\bar{y},\bar{p}) - \nabla_p h(\bar{x},\bar{y},\bar{p}) + \nabla_{yy} f(\bar{x},\bar{y})\bar{p}] = \frac{\bar{y}^T\xi}{\alpha} = 0, \text{ using (18)} \quad (25)$$

Therefore hypothesis (iii) implies

$$\bar{p} = 0. \quad (26)$$

Moreover, equation (12) along with (22), (26) and hypothesis (iv) yields

$$\alpha[\nabla_y f(\bar{x},\bar{y}) + \nabla_r g(\bar{x},\bar{y},\bar{p})] = \mu,$$

or

$$\nabla_y f(\bar{x},\bar{y}) + \nabla_r g(\bar{x},\bar{y},\bar{p}) = \frac{\mu}{\alpha} \geqq 0, \quad (27)$$

Also using equation (17)

$$\bar{x}^T[\nabla_y f(\bar{x},\bar{y}) + \nabla_r g(\bar{x},\bar{y},\bar{p})] = \frac{\bar{x}^T\mu}{\alpha} = 0. \quad (28)$$

Thus $(\bar{x},\bar{y},\bar{r}=0)$ satisfies the constraints (4)-(6), that is, it is a feasible solution for the dual problem (WHD). Now using equations (19), (26), (27) and hypothesis (iv), we get

$$f(\bar{x},\bar{y}) + h(\bar{x},\bar{y},\bar{p}) - \bar{p}^T\nabla_p h(\bar{x},\bar{y},\bar{p}) - \bar{y}^T[\nabla_y f(\bar{x},\bar{y}) + \nabla_p h(\bar{x},\bar{y},\bar{p})$$

$$= f(\bar{x},\bar{y}) + g(\bar{x},\bar{y},\bar{p}) - \bar{r}^T\nabla_r g(\bar{x},\bar{y},\bar{p}) - \bar{x}^T[\nabla_y f(\bar{x},\bar{y}) + \nabla_r g(\bar{x},\bar{y},\bar{p})],$$

i.e

$$L(\bar{x},\bar{y},\bar{p}) = M(\bar{x},\bar{y},\bar{r}).$$

Finally, by Theorem (3.1), $(\bar{x},\bar{y},\bar{p}=0)$ and $(\bar{x},\bar{y},\bar{r}=0)$ are global optimal solutions of the respective problems. □

Theorem 3.3 (Strong Duality)**.** *Let $(\bar{u},\bar{v},\bar{r})$ be a local optimal solution of (WHP). Assume that*
(i) $\nabla_{rr} g(\bar{u},\bar{v},\bar{r})$ is positive definite,
(ii)$\nabla_u f(\bar{u},\bar{v}) + \nabla_r g(\bar{u},\bar{v},\bar{r}) \neq 0$,
(iii) $\bar{r}^T[\nabla_u f(\bar{u},\bar{v}) + \nabla_r g(\bar{u},\bar{v},\bar{r})] = 0 \Rightarrow \bar{r} = 0$,
(iv) $g(\bar{u},\bar{v},0) = g(\bar{u},\bar{v},0)$, $\nabla_u g(\bar{u},\bar{v},0) = \nabla_r g(\bar{u},\bar{v},0)$, $\nabla_v g(\bar{x},\bar{y},0) = \nabla_p h(\bar{x},\bar{y},0)$.
Then (I) $(\bar{u},\bar{v},\bar{p}=0)$ is feasible for (WHP) and
(II) $L(\bar{u},\bar{v},\bar{p}) = M(\bar{u},\bar{v},\bar{r})$.
Also, if the hypotheses of Theorem (3.1) *are satisfied for all feasible solutions of (WHP) and (WHD), then $(\bar{u},\bar{v},\bar{r}=0)$ and $(\bar{u},\bar{v},\bar{p}=0)$ are global optimal solutions of (WHD) and (WHP), respectively.*

Proof. Follows on the line of Theorem (3.2). □

4. Self Duality

A mathematical problem is said to be self dual if it formally identical with its dual, that is, the dual can be rewritten in the form of the primal. In general, (WHP) is not self-dual without some added restrictions on f, g and h. If $f : R^n \times R^m \to R$ and $g : R^n \times R^m \times R^n \to R$ are skew symmetric, i.e

$$f(u,v) = -f(v,u),\ g(u,v,r) = -g(v,u,r)$$

as shown below. By recasting the dual problem (WHD) as a minimization problem, we have Minimize

$$M(u,v,r) = -\{f(u,v)+g(u,v,r)-r^T\nabla_r g(u,v,r)-u^T[\nabla_u f(u,v) + \nabla_r g(u,v,r)]\}$$

subject to

$$\nabla_u f(u,v) + \nabla_r g(u,v,r) \geqq 0,$$
$$r^T[\nabla_u f(u,v) + \nabla_r g(u,v,r)] \leqq 0,$$
$$u,\ v \geqq 0.$$

Now as f and g are skew symmetric, i.e

$$\nabla_u f(u,v) = -\nabla_u f(v,u)$$
$$\nabla_r g(u,v,r) = -\nabla_r g(v,u,r),$$

then the above problem rewritten as :
Minimize

$$M(u,v,r) = f(v,u) + g(v,u,r) - r^T\nabla_r g(v,u,r) - u^T[\nabla_u f(v,u) + \nabla_r g(v,u,r)]$$

subject to

$$\nabla_u f(v,u) + \nabla_r g(v,u,r) \leqq 0,$$
$$r^T[\nabla_u f(v,u) + \nabla_r g(v,u,r)] \geqq 0,$$
$$u,\ v \geqq 0.$$

Which is identical to primal problem, i.e., the objective and the constraint functions are identical. Thus, the problem (WHP) is self-dual.
It is obvious that (x,y,p) is feasible for (WHP), then (y,x,p) is feasible for (WHD) and vice versa.

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Khushboo Verma is currently pursuing her Ph.D. degree in Mathematics at Indian Institute of Technology Roorkee, India. She received her Master of Science in Mathematics from Banaras Hindu University, Varanasi, India.

Department of Mathematics, Indian Institute of Technology, Roorkee-247 667, India.
`e-mail: 1986khushi@gmail.com`

T.R. Gulati is a Professor in Department of Mathematics, Indian Institute of Technology Roorkee, India. His research interests are in the areas of single and multiobjective mathematical programming and generalized convexity. He has published more than 75 research papers in journal of intermational repute.

Department of Mathematics, Indian Institute of Technology, Roorkee-247 667, India.
`e-mail: trgmaiitr@rediffmail.com`