J. Appl. Math. & Informatics Vol. **32**(2014), No. 1 - 2, pp. 113 - 128
http://dx.doi.org/10.14317/jami.2014.113

ON SOME $p(x)$-KIRCHHOFF TYPE EQUATIONS WITH WEIGHTS

NGUYEN THANH CHUNG

Abstract. Consider a class of $p(x)$-Kirchhoff type equations of the form

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = \lambda V(x)|u|^{q(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $p(x), q(x) \in C(\overline{\Omega})$ with $1 < p^- := \inf_\Omega p(x) \le p^+ := \sup_\Omega p(x) < N$, $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function that may be degenerate at zero, λ is a positive parameter. Using variational method, we obtain some existence and multiplicity results for such problem in two cases when the weight function $V(x)$ may change sign or not.

AMS Mathematics Subject Classification : 34B27, 35J60, 35B05.
Key words and phrases : $p(x)$-Kirchhoff type equations, Mountain pass theorem, Ekeland variational principle, Weight functions.

1. Introduction

In this paper, we are concerned with the following $p(x)$-Kirchhoff type equations

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = \lambda f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with boundary $\partial\Omega$, $p(x) \in C(\overline{\Omega})$ with $1 < p^- := \inf_\Omega p(x) \le p^+ := \sup_\Omega p(x) < N$, $M : \mathbb{R}^+ \to \mathbb{R}+$ is a continuous function, f is a Carathéodory function having special structures, and λ is a paramter.

Since the first equation in (1.1) contains an integral over Ω, it is no longer a pointwise identity, and therefore it is often called nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see

Received October 10, 2012. Revised May 2, 2013. Accepted May 17, 2013.

[4]. Problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.2}$$

presented by Kirchhoff in 1883, see [19]. This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross section, E is the Young modulus of thematerial, ρ is themass density, and P_0 is the initial tension.

In recent years, elliptic problems involving p-Kirchhoff type operators have been studied in many papers, we refer to some interesting works [2, 5, 9, 21, 22, 25, 26], in which the authors have used different methods to get the existence of solutions for (1.1) in the case when $p(x) = p$ is a constant. To our knowledge, the study of $p(x)$-Kirchhoff type problems was firstly done by G. Dai et al. in the papers [11, 12]. It is not difficult to see that the $p(x)$-Laplacian possesses more complicated nonlinearities than p-Laplacian, for example it is inhomogeneous. The study of differential equations and variational problems involving $p(x)$-growth conditions is a consequence of their applications. Materials requiring such more advanced theory have been studied experimentally since the middle of last century. In [11], the authors established the existence of infinitely many distinct positive solutions for problem (1.1) in the special case $M(t) = a + bt$. In [12], the authors considered the problem in the case when $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous and non-descreasing function, satisfying the well-known condition:

(M_0) there exists $m_0 > 0$ such that $M(t) \geq m_0$ for all $t \geq 0$,

which plays an enssential role in the arguments, see further papers [2, 3, 6, 10, 21, 22]. There have been some authors improving (M_0) in the sense that the Kirchhoff function M may be degenerate at zero, see for example [7, 8, 9, 15]. In this paper, we assume that the Kirchhoff function M satisfies the following hypotheses:

(M_1) There exist $m_2 \geq m_1 > 0$ and $1 < \alpha \leq \beta < \min\{\frac{N}{p^+}, \frac{Np^-}{p^+(N-p^-)}\}$ such that

$$m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\beta-1}$$

for all $t \in \mathbb{R}^+$;

(M_2) For all $t \in \mathbb{R}^+$, it holds that

$$\widehat{M}(t) \geq M(t)t$$

where $\widehat{M}(t) = \int_0^t M(s)\,ds$.

Motivated by the ideas in [7, 8, 9, 15] and the results in [18, 23] for the $p(x)$-Laplacian, i.e., $M(t) \equiv 1$, in this paper, we consider problem (1.1) with $f(x, u) = \lambda V(x)|u|^{q(x)-2}u$ in two cases when the weight function $V(x)$ may change sign or not. The results in this work suplement or complement our earlier ones in [7], in which we studied the problem in the case when the concave and convex nonlinearities were combined and the weight function did not change sign.

First, we consider the case when the parameter $\lambda = 1$ and $f(x,u) = V(x)|u|^{q(x)-2}u$ in which the weight function $V(x)$ does not change sign. Problem (1.1) then becomes

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = V(x)|u|^{q(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.3}$$

More exactly, $V : \Omega \to [0, +\infty)$ belongs to $L^\infty(\Omega)$ and satisfies

(V_1) There exist an $x_0 \in \Omega$ and two positive constants r and R with $0 < r < R$ such that $\overline{B_R(x_0)} \subset \Omega$ and $V(x) = 0$ for $x \in \overline{B_R(x_0)\backslash B_r(x_0)}$ while $V(x) > 0$ for $x \in \Omega\backslash\overline{B_R(x_0)\backslash B_r(x_0)}$,

and the function q is assumed to satisfy

(Q_1) $q \in C_+(\overline{\Omega})$, $1 < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$;

(Q_2) Either

$$\max_{x \in \overline{B_r(x_0)}} q(x) < p^-\alpha \le p^-\beta \le p^+\alpha \le p^+\beta < \min_{x \in \overline{\Omega\backslash B_R(x_0)}} q(x)$$

or

$$\max_{x \in \overline{\Omega\backslash B_R(x_0)}} q(x) < p^-\alpha \le p^-\beta \le p^+\alpha \le p^+\beta < \min_{x \in \overline{B_r(x_0)}} q(x),$$

where the numbers α and β are given by (M_1).

Definition 1.1. A function $u \in X = W_0^{1,p(x)}(\Omega)$ is said to be a weak solution of problem (1.3) if and only if

$$M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx - \int_\Omega V(x)|u|^{q(x)-2}uv\,dx = 0$$

for all $v \in X$.

Our main result concerning problem (1.3) is given by the following theorem.

Theorem 1.2. *Assume that the conditions (M_1)-(M_2), (V_1) and (Q_1)-(Q_2) are satisfied. Then there exists a positive constant ϵ_0 such that problem (1.3) has at least two non-trivial non-negative weak solutions, provided that $|V|_{L^\infty(\Omega)} < \epsilon_0$.*

It should be noticed that Theorem 1.2 is only true when $q(x)$ is a non-constant function while $p(x)$ may be a constant. If $p(x) = p$ is a constant then it follows from (Q_2) that $\alpha = \beta$.

Next, we consider problem (1.1) in the case when $f(x,u) = \lambda V(x)|u|^{q(x)-2}u$, in which $V(x)$ is a sign changing weight function, that is,

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda V(x)|u|^{q(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.4}$$

More exactly, we study the existence of solutions for (1.4) under the hypotheses (M_1), (M_2) and

(V_2) $V(x) \in L^{\frac{s(x)}{\alpha}}(\Omega)$ with $s(x) \in C_+(\overline{\Omega})$, $s(x) > \alpha N$ for all $x \in \overline{\Omega}$, and $V(x) > 0$ in $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$, α is given by (M_1);

and the function $q(x) \in C_+(\overline{\Omega})$ is assumed to satisfy the following condition

(Q_3) $1 < q(x) < p(x) < N$ for all $x \in \overline{\Omega}$.

As we shall see in Section 4, due to the hypothesis (Q_3), we cannot use the mountain pass theorem [1] in order to get the solutions for problem (1.4) as in Theorem 1.2. We emphasize that this is the main different point between two problems (1.3) and (1.4).

Definition 1.3. A function $u \in X = W_0^{1,p(x)}(\Omega)$ is said to be a weak solution of problem (1.4) if and only if

$$M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx - \lambda\int_\Omega V(x)|u|^{q(x)-2}uv\,dx = 0$$

for all $v \in X$.

Our main result concerning problem (1.4) in this case is given by the following theorem.

Theorem 1.4. *Assume that the conditions* (M_1)-(M_2), (V_2) *and* (Q_3) *are satisfied. Then there exists a positive constant* λ^* *such that for any* $\lambda \in (0, \lambda^*)$, *problem (1.4) has at least one non-trivial non-negative weak solution, i.e., any* $\lambda \in (0, \lambda^*)$ *is an eigenvalue of eigenvalue problem (1.4).*

Our paper is organized as follows. In the next section, we shall recall some useful concepts and properties on the generalized Lebesgue-Sobolev spaces. Section 3 is devoted to the proof of Theorem 1.2 while we shall present the proof of Theorem 1.4 in Section 4.

2. Preliminaries

We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to the book of Musielak [24] and the papers of Kováčik and Rákosník [20] and Fan et al. [16, 17]. Set

$$C_+(\overline{\Omega}) := \{h : \quad h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define $h^+ = \sup_{x\in\Omega} h(x)$ and $h^- = \inf_{x\in\Omega} h(x)$. For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{u : \text{ a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}.$$

We recall the following so-called *Luxemburg norm* on this space defined by the formula

$$|u|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \le p^+ < \infty$ and continuous functions are dense if $p^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \le p_2(x)$ a.e. $x \in \Omega$ then there exists a continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequality

$$\left|\int_\Omega uv\,dx\right| \le \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)|u|_{p(x)}|v|_{p'(x)}$$

holds true.

Moreover, if h_1, h_2 and $h_3 : \overline{\Omega} \to (1, \infty)$ are three Lipschitz continuous functions such that $\frac{1}{h_1(x)} + \frac{1}{h_2(x)} + \frac{1}{h_3(x)} = 1$, then for any $u \in L^{h_1(x)}(\Omega)$, $v \in L^{h_2(x)}(\Omega)$ and $w \in L^{h_3(x)}(\Omega)$, the following inequality holds:

$$\left|\int_\Omega uvw\,dx\right| \le \left(\frac{1}{h_1^-} + \frac{1}{h_2^-} + \frac{1}{h_3^-}\right)|u|_{h_1(x)}|v|_{h_2(x)}|w|_{h_3(x)}.$$

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

Proposition 2.1 ([17]). *If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold*

$$|u|^{p^-}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^+}_{p(x)} \tag{2.1}$$

provided that $|u|_{p(x)} > 1$ while

$$|u|^{p^+}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^-}_{p(x)} \tag{2.2}$$

provided that $|u|_{p(x)} < 1$ and

$$|u_n - u|_{p(x)} \to 0 \ \Leftrightarrow\ \rho_{p(x)}(u_n - u) \to 0. \tag{2.3}$$

Proposition 2.2 ([18]). *Let p and q be measurable functions such that $p \in L^\infty(\Omega)$ and $1 \le p(x)q(x) \le \infty$ for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \ne 0$. Then the following relations hold*

$$|u|^{p^+}_{p(x)q(x)} \le \left||u|^{p(x)}\right|_{q(x)} \le |u|^{p^-}_{p(x)q(x)} \tag{2.4}$$

provided that $|u|_{p(x)} \leq 1$ *while*

$$|u|^{p^-}_{p(x)q(x)} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)} \tag{2.5}$$

provided that $|u|_{p(x)} \geq 1$. *In particular, if* $p(x) = p$ *is a constant, then*

$$\left||u|^{p}\right|_{q(x)} = |u|^{p}_{pq(x)}. \tag{2.6}$$

In this paper, we assume that $p \in C_+^{\log}(\overline{\Omega})$, where $C_+^{\log}(\overline{\Omega})$ is the space of all the functions of $C_+(\overline{\Omega})$ which are *logarithmic Hölder continuous*, that is, there exists $R > 0$ such that for all $x, y \in \Omega$ with $0 < |x-y| \leq \frac{1}{2}$, $|p(x) - p(y)| \leq -\frac{R}{\log|x-y|}$, see [13, 16]. We define the space $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(x)}.$$

Proposition 2.3 ([17, 18])**.** *The space* $\left(W_0^{1,p(x)}(\Omega), \|.\|\right)$ *is a separable and Banach space. Moreover, if* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$ *then the embedding* $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ *is compact and continuous, where* $p^*(x) = \frac{Np(x)}{N-p(x)}$ *if* $p(x) < N$ *or* $p^*(x) = \infty$ *if* $p(x) \geq N$.

3. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2, which is essentially based on the mountain pass theorem [1] combined with the Ekeland variational principle [14].

Let us define the functional $J : X = W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ by the formula

$$J(u) = \Phi(u) - \Psi_1(u), \tag{3.1}$$

where

$$\Phi(u) = \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right), \quad \Psi_1(u) = \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}\,dx, \tag{3.2}$$

where $\widehat{M}(t) = \int_0^t M(s)\,ds$. Then, the functional J associated with problem (1.1) is well defined and of C^1 class on X. Moreover, we have

$$\begin{aligned}
&J'(u)(v)\\
&= M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v\,dx - \int_\Omega V(x)|u|^{q(x)-2}uv\,dx\\
&= \Phi'(u)(v) - \Psi_1'(u)(v)
\end{aligned}$$

for all $u, v \in X$. Thus, weak solutions of problem (1.3) are exactly the ciritical points of the functional J. Due to the conditions (M_1) and (Q_1), we can show that J is weakly lower semi-continuous in X. The following lemma plays an essential role in our arguments.

Lemma 3.1. *The following assertions hold:*

(*i*) *There exists* $\epsilon_0 > 0$ *such that for any* $|V|_{L^\infty(\Omega)} < \epsilon_0$, *there exist* $\rho_1, \gamma_1 > 0$ *for which* $J(u) \geq \gamma_1$, $\forall u \in X$ *with* $\|u\| = \rho_1$;

(*ii*) *There exists* $\varphi_1 \in X$, $\varphi_1 \geq 0$, $\varphi_1 \neq 0$ *such that* $\lim_{t\to\infty} J(t\varphi_1) = -\infty$

(*iii*) *There exists* $\psi_1 \in X$, $\psi_1 \geq 0$, $\psi_1 \neq 0$ *such that* $J(t\psi_1) < 0$ *for all* $t > 0$ *small enough.*

Proof. We shall prove Lemma 3.1 in details for the case

$$\max_{x\in\overline{B_r(x_0)}} q(x) < p^-\alpha \leq p^-\beta \leq p^+\alpha \leq p^+\beta < \min_{x\in\overline{\Omega\backslash B_R(x_0)}} q(x),$$

the remaining case is similarly proved.

(*i*) Let us define the function $q_1 : \overline{B_r(x_0)} \to (1, +\infty)$ by $q_1(x) = q(x)$ for all $x \in \overline{B_r(x_0)}$ and the function $q_2 : \overline{\Omega\backslash B_R(x_0)} \to (1, +\infty)$ by $q_2(x) = q(x)$ for all $x \in \overline{\Omega\backslash B_R(x_0)}$. We denote $q_1^- = \min_{x\in\overline{B_r(x_0)}} q_1(x)$, $q_1^+ = \max_{x\in\overline{B_r(x_0)}} q_1(x)$, $q_2^- = \min_{x\in\overline{\Omega\backslash B_R(x_0)}} q_2(x)$ and $q_2^+ = \max_{x\in\overline{\Omega\backslash B_R(x_0)}} q_2(x)$. By the conditions (Q_1) and (Q_2),

$$1 < q_1^- \leq q_1^+ < p^-\alpha \leq p^-\beta \leq p^+\alpha \leq p^+\beta < q_2^- \leq q_2^+ < p^\star(x) \text{ for all } x \in \overline{\Omega}, \quad (3.3)$$

which helps us to deduce that X is continuously embedded in $L^{q_i^\pm}(\Omega)$ for $i = 1, 2$. Then there exists a positive constant c_1 such that

$$\int_\Omega |u|^{q_i^\pm} dx \leq c_1 \|u\|^{q_i^\pm}, \quad \forall u \in X \text{ and } i = 1, 2. \quad (3.4)$$

From (3.4), there exist two positive constants c_2, c_3 such that

$$\begin{aligned}\int_{B_r(x_0)} |u|^{q_1(x)} dx &\leq \int_{B_r(x_0)} |u|^{q_1^-} dx + \int_{B_r(x_0)} |u|^{q_1^+} dx \\ &\leq \int_\Omega |u|^{q_1^-} dx + \int_\Omega |u|^{q_1^+} dx \\ &\leq c_2\Big(\|u\|^{q_1^-} + \|u\|^{q_1^+}\Big), \quad \forall u \in X,\end{aligned} \quad (3.5)$$

and

$$\begin{aligned}\int_{\Omega\backslash B_R(x_0)} |u|^{q_2(x)} dx &\leq \int_{\Omega\backslash B_R(x_0)} |u|^{q_2^-} dx + \int_{\Omega\backslash B_R(x_0)} |u|^{q_2^+}| \, dx \\ &\leq \int_\Omega |u|^{q_2^-} dx + \int_\Omega |u|^{q_2^+} dx \\ &\leq c_3\Big(\|u\|^{q_2^-} + \|u\|^{q_2^+}\Big), \quad \forall u \in X.\end{aligned} \quad (3.6)$$

Using the hypothesis (M_1), relations (3.5) and (3.6) give us

$$\begin{aligned} J(u) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right) - \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}\,dx \\ &\geq \frac{m_1}{\alpha}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)^\alpha - \int_{B_r(x_0)} \frac{V(x)}{q(x)}|u|^{q(x)}\,dx - \int_{\Omega\setminus B_R(x_0)} \frac{V(x)}{q(x)}|u|^{q(x)}\,dx \\ &\geq \frac{m_1}{\alpha p^+}\|u\|^{\alpha p^+} - \frac{c_4|V|_{L^\infty(\Omega)}}{q^-}\left(\|u\|^{q_1^-} + \|u\|^{q_1^+} + \|u\|^{q_2^-} + \|u\|^{q_2^+}\right) \\ &\geq \left[\frac{m_1}{2\alpha p^+}\|u\|^{\alpha p^+} - \frac{c_4}{q^-}|V|_{L^\infty(\Omega)}\left(\|u\|^{q_1^-} + \|u\|^{q_1^+}\right)\right] \\ &\qquad + \left[\frac{m_1}{2\alpha p^+}\|u\|^{\alpha p^+} - \frac{c_4}{q^-}|V|_{L^\infty(\Omega)}\left(\|u\|^{q_2^-} + \|u\|^{q_2^+}\right)\right] \end{aligned} \tag{3.7}$$

for all $u \in X$ with $\|u\| < 1$. Since the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(t) = \frac{m_1}{2\alpha p^+} - \frac{c_4}{q^-}t^{q_2^+ - \alpha p^+} - \frac{c_4}{q^-}t^{q_2^- - \alpha p^+} \tag{3.8}$$

is positive in a neighbourhood of the origin, it follows that there exists $\rho_1 \in (0,1)$ such that $g(\rho_1) > 0$. On the other hand, defining

$$\epsilon_0 = \min\left\{1, \frac{m_1 q^-}{4\alpha p^+ c_4}\min\{\rho_1^{\alpha p^+ - q_1^-}, \rho_1^{\alpha p^+ - q_1^+}\}\right\} \tag{3.9}$$

we deduce that, for any $|V|_{L^\infty(\Omega)} < \epsilon_0$, there exists $\gamma_1 > 0$ such that for any $u \in X$ with $\|u\| = \rho_1$ we have $J(u) \geq \gamma_1$. (ii) Let $\psi_1 \in C_0^\infty(\Omega)$, $\psi_1 \geq 0$ and there exist $x_1 \in \Omega\setminus B_R(x_0)$ and $\epsilon > 0$ such that for any $x \in B_\epsilon(x_1) \subset (\Omega\setminus B_R(x_0))$ we have $\psi_1(x) > 0$. For any $t > 1$, we have

$$\begin{aligned} J(t\psi_1) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla t\psi_1|^{p(x)}\,dx\right) - \int_\Omega \frac{V(x)}{q(x)}|t\psi_1|^{q(x)}\,dx \\ &\leq \frac{m_2}{\beta}\left(\int_\Omega \frac{1}{p(x)}|\nabla t\psi_1|^{p(x)}\,dx\right)^\beta - \int_{\Omega\setminus B_R(x_0)} \frac{V(x)}{q(x)}|t\psi_1(x)|^{q(x)}\,dx \\ &\leq \frac{m_2}{\beta}t^{\beta p^+}\left(\int_\Omega \frac{1}{p(x)}|\nabla \psi_1|^{p(x)}\,dx\right)^\beta - t^{q_2^-}\int_{\Omega\setminus B_R(x_0)} \frac{V(x)}{q(x)}|\psi_1(x)|^{q(x)}\,dx. \end{aligned} \tag{3.10}$$

Since $\beta p^+ < q_2^-$ we infer that $\lim_{t\to\infty} J(t\psi_1) = -\infty$. (iii) Let $\varphi_1 \in C_0^\infty(\Omega)$, $\varphi_1 \geq 0$ and there exist $x_2 \in B_r(x_0)$ and $\epsilon > 0$ such that for any $x \in B_\epsilon(x_2) \subset B_r(x_0)$ we have $\varphi_1(x) > 0$. Letting $0 < t < 1$ we find

$$\begin{aligned} J(t\varphi_1) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla t\varphi_1|^{p(x)}\,dx\right) - \int_\Omega \frac{V(x)}{q(x)}|t\varphi_1|^{q(x)}\,dx \\ &\leq \frac{m_2}{\beta}\left(\int_\Omega \frac{1}{p(x)}|\nabla t\varphi_1|^{p(x)}\,dx\right)^\beta - \int_{B_r(x_0)} \frac{V(x)}{q(x)}|t\varphi_1|^{q(x)}\,dx \\ &\leq \frac{m_2}{\beta}t^{\beta p^-}\left(\int_\Omega \frac{1}{p(x)}|\nabla \varphi_1|^{p(x)}\,dx\right)^\beta - t^{q_1^+}\int_{B_r(x_0)} \frac{V(x)}{q(x)}|\varphi_1|^{q(x)}\,dx. \end{aligned} \tag{3.11}$$

Obviously, we have $J(t\varphi_1) < 0$ for any $0 < t < \delta^{\frac{1}{\beta p^- - q_1^+}}$, where

$$0 < \delta < \min\left\{1, \frac{\int_{B_r(x_0)} \frac{V(x)}{q(x)}|\varphi_1|^{q(x)}dx}{\frac{m_2}{\beta}\left(\int_\Omega \frac{1}{p(x)}|\nabla\varphi_1|^{p(x)}dx\right)^\beta}\right\}.$$

The proof of Lemma 3.1 is complete. □

Lemma 3.2. *The functional J satisfies the Palais-Smale condition in X.*

Proof. Let $\{u_m\} \subset X$ be such that

$$J(u_m) \to \bar{c}, \quad J'(u_m) \to 0 \text{ in } X^* \text{ as } m \to \infty, \tag{3.12}$$

where X^* is the dual space of X.

We shall prove that $\{u_m\}$ is bounded in X. In order to do that, we assume by contradiction that passing if necessary to a subsequence, still denoted by $\{u_m\}$, we have $\|u_m\| \to \infty$ as $m \to \infty$. By (3.12) and (M_1)-(M_2), for m large enough and $|V|_{L^\infty(\Omega)} < \epsilon_0$, we have

$$\begin{aligned}
&1 + \bar{c} + \|u_m\| \\
&\geq J(u_m) - \frac{1}{q_2^-}J'(u_m)(u_m) \\
&\geq M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_m|^{p(x)}\,dx\right)\int_\Omega \frac{1}{p(x)}|\nabla u_m|^{p(x)}\,dx - \int_\Omega \frac{V(x)}{q(x)}|u_m|^{q(x)}\,dx \\
&\qquad - \frac{1}{q_2^-}M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_m|^{p(x)}\,dx\right)\int_\Omega |\nabla u_m|^{p(x)}dx + \frac{1}{q_2^-}\int_\Omega V(x)|u_m|^{q(x)}\,dx \\
&\geq \frac{m_1}{\alpha(p^+)^{\alpha-1}}\Big(\frac{1}{p^+} - \frac{1}{q_2^-}\Big)\left(\int_\Omega |\nabla u_m|^{p(x)}\,dx\right)^\alpha + \int_{B_r(x_0)} V(x)\Big(\frac{1}{q_2^-} - \frac{1}{q_1(x)}\Big)|u_m|^{q_1(x)}\,dx \\
&\geq \frac{m_1}{\alpha(p^+)^{\alpha-1}}\Big(\frac{1}{p^+} - \frac{1}{q_2^-}\Big)\|u_m\|^{\alpha p^-} - \epsilon_0\Big(\frac{1}{q_1^-} - \frac{1}{q_2^-}\Big)\Big(\|u_m\|^{q_1^-} + \|u_m\|^{q_1^+}\Big)
\end{aligned} \tag{3.13}$$

Dividing the above inequality by $\|u_m\|^{\alpha p^-}$ taking into account that (3.3) holds true and passing to the limit as $m \to \infty$ we obtain a contradiction. It follows that $\{u_m\}$ is bounded in X. Thus, there exists $u_1 \in X$ such that passing to a subsequence, still denoted by $\{u_m\}$, it converges weakly to u_1 in X. Then $\{\|u_m - u\|\}$ is bounded. By (3.3), the embedding from X to the space $L^{q(x)}(\Omega)$ is compact. Then, using the Hölder inequality, Propositions 2.1-2.3, we have

$$\lim_{m\to\infty}\int_\Omega V(x)|u_m|^{q(x)-2}u_m(u_m - u)\,dx = 0. \tag{3.14}$$

This fact and relation (3.12) yield

$$\lim_{m\to\infty}\Phi'(u_m)(u_m - u) = 0. \tag{3.15}$$

Since $\{u_m\}$ is bounded in X, passing to a subsequence, if necessary, we may assume that

$$\int_\Omega \frac{1}{p(x)}|\nabla u_m|^{p(x)}\,dx \to t_0 \geq 0 \text{ as } m \to \infty.$$

If $t_0 = 0$ then $\{u_m\}$ converges strongly to $u = 0$ in X and the proof is finished. If $t_0 > 0$ then we deduce by the continuity of M that

$$M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_m|^{p(x)}\,dx\Big) \to M(t_0) \text{ as } m \to \infty.$$

Thus, by (M_1), for sufficiently large m, we have

$$0 < c_5 \leq M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_m|^{p(x)}\,dx\Big) \leq c_6. \tag{3.16}$$

From (3.15), (3.16), it follows that

$$\lim_{m\to\infty}\int_\Omega |\nabla u_m|^{p(x)-2}(\nabla u_m - \nabla u)\,dx = 0.$$

Thus, $\{u_m\}$ converges strongly to u in X and the functional J satisfies the Palais-Smale condition. $\square$

Proof of Theorem 1.2. By Lemmas 3.1 and 3.2, all assumptions of the mountain pass theorem in [1] are satisfied. Then we deduce u_1 as a non-trivial critical point of the functional J with $J(u_1) = \overline{c}$ and thus a non-trivial weak solution of problem (1.3).

We now prove that there exists a second weak solution $u_2 \in X$ such that $u_2 \neq u_1$. Indeed, let ϵ_0 as in the proof of Lemma 3.1(i) and assume that $|V|_{L^\infty(\Omega)} < \epsilon_0$. By Lemma 3.1(i), it follows that on the boundary of the ball centered at the origin and of radius ρ_1 in X, denoted by $B_{\rho_1}(0) = \{u \in X : \ \|u\| < \rho_1\}$, we have

$$\inf_{u\in\partial B_{\rho_1}(0)} J(u) > 0. \tag{3.17}$$

On the other hand, by Lemma 3.1(ii), there exists $\varphi_1 \in X$ such that $J(t\varphi_1) < 0$ for all $t > 0$ small enough. Moreover, from (3.7), the functional J is bouned from below on $B_{\rho_1}(0)$. It follows that

$$-\infty < \underline{c} := \inf_{u\in\overline{B}_{\rho_1}(0)} J(u) < 0.$$

Applying the Ekeland variational principle in [14] to the functional $J : \overline{B}_{\rho_1}(0) \to \mathbb{R}$, it follows that there exists $u_\epsilon \in \overline{B}_{\rho_1}(0)$ such that

$$J(u_\epsilon) < \inf_{u\in\overline{B}_{\rho_1}(0)} J(u) + \epsilon,$$

$$J(u_\epsilon) < J(u) + \epsilon\|u - u_\epsilon\|, \quad u \neq u_\epsilon.$$

By Lemma 3.1, we have

$$\inf_{u\in\partial B_{\rho_1}(0)} J(u) \geq \gamma_1 > 0 \text{ and } \inf_{u\in\overline{B}_{\rho_1}(0)} J(u) < 0.$$

Let us choose $\epsilon > 0$ such that

$$0 < \epsilon < \inf_{u\in\partial B_{\rho_1}(0)} J(u) - \inf_{u\in\overline{B}_{\rho_1}(0)} J(u).$$

Then, $J(u_\epsilon) < \inf_{u\in\partial B_{\rho_1}(0)} J(u)$ and thus, $u_\epsilon \in B_{\rho_1}(0)$.

Now, we define the functional $I : \overline{B}_{\rho_1}(0) \to \mathbb{R}$ by $I(u) = J(u) + \epsilon\|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of I and thus

$$\frac{I(u_\epsilon + tv) - I(u_\epsilon)}{t} \geq 0$$

for all $t > 0$ small enough and all $v \in B_{\rho_1}(0)$. The above information shows that

$$\frac{J(u_\epsilon + tv) - J(u_\epsilon)}{t} + \epsilon\|v\| \geq 0.$$

Letting $t \to 0^+$, we deduce that

$$\langle J'(u_\epsilon), v\rangle \geq -\epsilon\|v\|.$$

It should be noticed that $-v$ also belongs to $B_{\rho_1}(0)$, so replacing v by $-v$, we get

$$\langle J'(u_\epsilon), -v\rangle \geq -\epsilon\| - v\|$$

or

$$\langle J'(u_\epsilon), v\rangle \leq \epsilon\|v\|,$$

which helps us to deduce that $\|J'(u_\epsilon)\|_{X^*} \leq \epsilon$. Therefore, there exists a sequence $\{u_m\} \subset B_{\rho_1}(0)$ such that

$$J(u_m) \to \underline{c} = \inf_{u \in \overline{B}_{\rho_1}(0)} J(u) < 0 \text{ and } J'(u_m) \to 0 \text{ in } X^* \text{ as } m \to \infty. \tag{3.18}$$

From Lemma 3.2, the sequence $\{u_m\}$ converges strongly to u_2 as $m \to \infty$. Moreover, since $J \in C^1(X, \mathbb{R})$, by (3.17) it follows that $J(u_2) = \underline{c}$ and $J'(u_2) = 0$. Thus, u_2 is a non-trivial weak solution of problem (1.2).

Finally, we point out the fact that $u_1 \neq u_2$ since $J(u_1) = \overline{c} > 0 > \underline{c} = J(u_2)$. Moreover, since $J(u) = J(|u|)$, problem (1.3) has at least two non-trivial non-negative weak solutions. The proof of Theorem 1.2 is complete. □

4. Proof of Theorem 1.4

In this section, assume that we are under the hypotheses of Theorem 1.4, we shall prove Theorem 1.4 using the Ekeland variational principle [14]. For each $\lambda \in \mathbb{R}$, define the functional $J_\lambda : X = W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$J_\lambda(u) = \Phi(u) - \lambda\Psi_2(u),$$

where

$$\Phi(u) = \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right), \quad \Psi_2(u) = \int_\Omega \frac{V(x)}{q(x)}|u|^{q(x)}\,dx. \tag{4.1}$$

From (V_2), (2.4) and (2.5), it is clear that for all $u \in X$,

$$\begin{aligned}|\Psi_2(u)| &\leq \frac{1}{q^-}|V|_{\frac{s(x)}{\alpha}}\left||u|^{q(x)}\right|_{\frac{s(x)}{s(x)-\alpha}} \\ &\leq \begin{cases} \frac{1}{q^-}|V|_{\frac{s(x)}{\alpha}}|u|^{q^-}_{\frac{s(x)q(x)}{s(x)-\alpha}} & \text{if } |u|_{q(x)} \leq 1, \\ \frac{1}{q^-}|V|_{\frac{s(x)}{\alpha}}|u|^{q^+}_{\frac{s(x)q(x)}{s(x)-\alpha}} & \text{if } |u|_{q(x)} \geq 1. \end{cases}\end{aligned}$$

On the other hand, by (V_2) and (Q_3), we have $\frac{s(x)q(x)}{s(x)-\alpha} < p^*(x)$ and $\frac{s(x)q(x)}{s(x)-\alpha q(x)} < p^*(x)$ for all $x \in \overline{\Omega}$ and thus the embeddings $X \hookrightarrow L^{\frac{s(x)q(x)}{s(x)-\alpha}}(\Omega)$ and $X \hookrightarrow L^{\frac{s(x)q(x)}{s(x)-\alpha q(x)}}(\Omega)$ are continuous and compact. For these reasons, we can use the similar arguments as in [18, Proposition 2] in order to show that the functional J_λ is well-defined. Moreover, J_λ is of C^1 class in X and

$$\begin{aligned} J'_\lambda(u)(v) &= M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v\,dx - \lambda\int_\Omega V(x)|u|^{q(x)-2}uv\,dx \\ &= \Phi'(u)(v) - \lambda\Psi'_2(u)(v) \end{aligned}$$

for all $u, v \in X$. Thus, weak solutions of problem (1.4) are exactly the ciritical points of the functional J_λ.

Lemma 4.1. *For any $\rho_2 \in (0,1)$, there exist $\lambda^* > 0$ and $\gamma_2 > 0$ such that for all $u \in X$ with $\|u\| = \rho_2$,*

$$J_\lambda(u) \geq \gamma_2 > 0 \text{ for all } \lambda \in (0, \lambda^*).$$

Proof. Since the embedding $X \hookrightarrow L^{\frac{s(x)q(x)}{s(x)-\alpha}}(\Omega)$ is continuous, there exists a positive constant c_7 such that

$$|u|_{\frac{s(x)q(x)}{s(x)-\alpha}} \leq c_7\|u\|, \quad \forall u \in X. \tag{4.2}$$

Now, let us assume that $\|u\| < \min\{1, \frac{1}{c_7}\}$, where c_7 is the positive constant from above. Then we have $|u|_{\frac{s(x)q(x)}{s(x)-\alpha}} < 1$. Using relations (2.2), (4.2), the condition (M_1) and the Hölder inequality, we deduce that for any $u \in X$ with $\|u\| = \rho_2 \in (0,1)$ the following inequalities hold true

$$\begin{aligned} J_\lambda(u) &\geq \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right) - \frac{\lambda}{q^-}\int_\Omega V(x)|u|^{q(x)}\,dx \\ &\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} - \frac{\lambda}{q^-}|V|_{\frac{s(x)}{\alpha}}\,\big||u|^{q(x)}\big|_{\frac{s(x)}{s(x)-\alpha}} \\ &\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} - \frac{\lambda}{q^-}|V|_{\frac{s(x)}{\alpha}}|u|^{q^-}_{\frac{s(x)q(x)}{s(x)-\alpha}} \\ &\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} - \frac{\lambda}{q^-}|V|_{\frac{s(x)}{\alpha}}c_7^{q^-}\|u\|^{q^-} \\ &= \rho_2^{q^-}\Big(\frac{m_1}{\alpha(p^+)^\alpha}\rho_2^{\alpha p^+-q^-} - \frac{\lambda}{q^-}c_7^{q^-}|V|_{\frac{s(x)}{\alpha}}\Big). \end{aligned} \tag{4.3}$$

By (Q_3) we have $q^- \leq q^+ < p^- \leq p^+ < \alpha p^+$. So, if we take

$$\lambda^* := \frac{m_1\rho_2^{\alpha p^+-q^-}}{2\alpha(p^+)^\alpha}.\frac{q^-}{c_7^{q^-}|V|_{\frac{s(x)}{\alpha}}}, \tag{4.4}$$

then for any $\lambda \in (0, \lambda^*)$ and $u \in X$ with $\|u\| = \rho_2$, there exists $\gamma_2 > 0$ such that $J_\lambda(u) \geq \gamma_2 > 0$. The proof of the Lemma 4.1 is complete. □

Lemma 4.2. *For any $\lambda \in (0, \lambda^*)$, where λ^* is given by (4.4), there exists $\psi_2 \in X$ such that $\psi_2 \geq 0$, $\psi_2 \neq 0$ and $J_\lambda(t\psi_2) < 0$ for all $t > 0$ smaller than a certain value depending on λ.*

Proof. From (Q_3) we have $q(x) < \beta p(x)$ for all $x \in \overline{\Omega}_0$, where Ω_0 is given by (V_2). In the sequel, we use the notation $q_0^- = \inf_{\Omega_0} q(x)$, $q_0^+ = \sup_{\Omega_0} q(x)$, $p_0^- = \inf_{\Omega_0} p(x)$, and $p_0^+ = \sup_{\Omega_0} p(x)$. Let $\delta_0 > 0$ be such that $q_0^- + \delta_0 < \beta p_0^-$. Since $q \in C(\overline{\Omega}_0)$, there exists an open set $\Omega_1 \subset \Omega_0$ such that $|q(x) - q_0^-| < \delta_0$ for all $x \in \Omega_1$. It follows that $q(x) < q_0^- + \delta_0 < \beta p_0^-$ for all $x \in \Omega_1$.

Let $\psi_2 \in C_0^\infty(\Omega)$ such that $\mathrm{supp}(\psi_2) \subset \Omega_1 \subset \Omega_0$, $\psi_2 = 1$ in a subset $\Omega_1' \subset \mathrm{supp}(\psi_2)$, $0 \leq \psi_2 \leq 1$ in Ω_1. Then, using (M_1) we have

$$\begin{aligned} J_\lambda(t\psi_2) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla t\psi_2|^{p(x)}\,dx\right) - \lambda\int_\Omega \frac{V(x)}{q(x)}|t\psi_2|^{q(x)}\,dx \\ &\leq \frac{m_2}{\beta}\left(\int_\Omega \frac{1}{p(x)}|\nabla t\psi_2|^{p(x)}\,dx\right)^\beta - \lambda\int_{\Omega_1}\frac{V(x)}{q(x)}t^{q(x)}|\psi_2|^{q(x)}\,dx \qquad (4.5)\\ &\leq \frac{t^{\beta p_0^-}m_2}{\beta(p_0^-)^\beta}\left(\int_{\Omega_0}|\nabla\psi_2|^{p(x)}\,dx\right)^\beta - \frac{\lambda t^{q_0^-+\delta_0}}{q_0^+}\int_{\Omega_1}V(x)|\psi_2|^{q(x)}\,dx. \end{aligned}$$

Therefore, $J_\lambda(t\psi_2) < 0$ for $0 < t < \delta^{\frac{1}{\beta p_0^- - q_0^- - \delta_0}}$ with

$$0 < \delta < \min\left\{1, \frac{\lambda\beta(p_0^-)^\beta}{m_2 q_0^+}.\frac{\int_{\Omega_1}V(x)|\psi_2|^{q(x)}\,dx}{\left(\int_{\Omega_0}|\nabla\psi_2|^{p(x)}\,dx\right)^\beta}\right\}.$$

Finally, we shall point that

$$\int_{\Omega_0}|\nabla\psi_2|^{p(x)}\,dx > 0.$$

In fact, due to the choice of ψ_2, if $\int_{\Omega_0}|\nabla\psi_2|^{p(x)}\,dx = 0$ then $\int_\Omega|\nabla\psi_2|^{p(x)}dx = 0$. Using (2.3), we deduce that $|\nabla\psi_2| = 0$ and consequently $\psi_2 = 0$ in Ω, which is a contradiction. The proof of Lemma 4.2 is complete. □

Proof of Theorem 1.4. Let $\lambda^* > 0$ be defined by (4.4) and $\lambda \in (0, \lambda^*)$. By Lemma 4.1, it follows that on the boundary of the ball centered at the origin and of radius ρ_2 in X, denoted by $B_{\rho_2}(0)$, we have

$$\inf_{\partial B_{\rho_2}(0)} J_\lambda(u) > 0.$$

On the other hand, by Lemma 4.2, there exists $\psi_2 \in X$ such that $J_\lambda(t\psi_2) < 0$ for all $t > 0$ small enough. Moreover, relation (4.3) implies that for any $u \in B_{\rho_2}(0)$ we have

$$J_\lambda(u) \geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} - \frac{\lambda}{q^-}c_7^{q^-}|V|_{\frac{s(x)}{\alpha}}\|u\|^{q^-}.$$

It follows that

$$-\infty < \underline{c} := \inf_{\overline{B_{\rho_2}(0)}} J_\lambda(u) < 0.$$

Using the Ekeland variational principle [14] and the similar arguments as those used in the proof of Theorem 1.1, we can deduce that there exists a sequence $\{u_m\} \subset B_{\rho_2}(0)$ such that

$$J_\lambda(u_m) \to \underline{c}, \quad J'_\lambda(u_m) \to 0. \tag{4.6}$$

It is clear that $\{u_m\}$ is bounded in X. Thus, there exists $u \in X$ such that, up to a subsequence, $\{u_m\}$ converges weakly to u in X. Since $\frac{s(x)q(x)}{s(x)-\alpha q(x)} < p^*(x)$ for all $x \in \overline{\Omega}$ we deduce that X is compactly embedded in $L^{\frac{s(x)q(x)}{s(x)-\alpha q(x)}}(\Omega)$, hence the sequence $\{u_m\}$ converges strongly to u in $L^{\frac{s(x)q(x)}{s(x)-\alpha q(x)}}(\Omega)$. Using the Hölder inequality, we have

$$\begin{aligned}\int_\Omega V(x)|u_m|^{q(x)-2}u_m(u_m-u)\,dx &\leq |V|_{\frac{s(x)}{\alpha}}\,\big||u_m|^{q(x)-2}u_m(u_m-u)\big|_{\frac{s(x)q(x)}{s(x)-\alpha}} \\ &\leq |V|_{\frac{s(x)}{\alpha}}\,\big||u_m|^{q(x)-2}u_m\big|_{\frac{q(x)}{q(x)-1}}|u_m-u|_{\frac{s(x)q(x)}{s(x)-\alpha q(x)}}.\end{aligned} \tag{4.7}$$

Now, if $\big||u_m|^{q(x)-2}u_m\big|_{\frac{q(x)}{q(x)-1}} > 1$, then we get $\big||u_m|^{q(x)-2}u_m\big|_{\frac{q(x)}{q(x)-1}} \leq |u_m|_{q(x)}^{q^+}$. The compact embedding $X \hookrightarrow L^{\frac{s(x)q(x)}{s(x)-\alpha q(x)}}(\Omega)$ ensures that

$$\lim_{m\to\infty}\int_\Omega V(x)|u_m|^{q(x)-2}u_m(u_m-u)\,dx = 0. \tag{4.8}$$

Relation (4.6) yields

$$\lim_{m\to\infty} J'_\lambda(u_m)(u_m-u) = 0.$$

Using the above information, we also obtain relation (3.15) and thus, $\{u_m\}$ converges strongly to some u in X. So, by (4.6), $J_\lambda(u) = \underline{c} < 0$ and $J'_\lambda(u) = 0$. It is clear that $J_\lambda(|u|) = J_\lambda(u)$. Therefore, u is a non-trivial non-negative weak solution of problem (1.4). Theorem 1.4 is completely proved. □

Remark 4.3. We cannot use the mountain pass argument in the proof of Theorem 1.4 since the functional J_λ does not satisfy the geometry of the mountain pass theorem. More exactly, we cannot find a function $\varphi_2 \geq 0$ such that $J_\lambda(t\varphi_2) \to -\infty$ as $t \to \infty$ as in Lemma 3.1.

Acknowledgments

The author would like to thank the referees for their helpful comments and suggestions which improved the presentation of the original manuscript. This paper was done when the author was working at the Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore as a Research Fellow.

References

1. A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical points theory and applications*, J. Funct. Anal., **04** (1973), 349-381.
2. A. Bensedik and M. Bouchekif, *On an elliptic equation of Kirchhoff-type with a potential asymptotically linear at infinity*, Math. Comput. Modelling, **49** (2009), 1089-1096.
3. F. Cammaroto and L. Vilasi, *Multiple solutions for a Kirchhoff-type problem involving the $p(x)$-Laplacian operator*, Nonlinear Anal., **74** (2011), 1841-1852.
4. M. Chipot and B. Lovat, *Some remarks on nonlocal elliptic and parabolic problems*, Nonlinear Anal., **30** (7) (1997), 4619-4627.
5. B. Cheng, X. Wu and J. Liu, *Multiplicity of nontrivial solutions for Kirchhoff type problems*, Boundary Value Problems, Volume 2010, Article ID 268946, 13 pages.
6. N.T. Chung, *Multiple solutions for a $p(x)$-Kirchhoff-type equation with sign-changing nonlinearities*, Complex Variables and Elliptic Equations, 2012, 1-10, iFirst.
7. N.T. Chung, *Multiplicity results for a class of $p(x)$-Kirchhoff type equations with combined nonlinearities*, E. J. Qualitative Theory of Diff. Equ., Volume 2012 (42) (2012), 1-13.
8. F. Colasuonno and P. Pucci, *Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations*, Nonlinear Analysis, **74** (2011), 5962-5974.
9. F.J.S.A. Corrêa, G.M. Figueiredo, *On a p-Kirchhoff equation via Krasnoselskii's genus*, Appl. Math. Letters, **22** (2009), 819-822.
10. G. Dai, *Three solutions for a nonlocal Dirichlet boundary value problem involving the $p(x)$-Laplacian*, Applicable Analysis, **92**(1) (2013), 191-210.
11. G. Dai and D. Liu, *Infinitely many positive solutions for a $p(x)$-Kirchhoff-type equation*, J. Math. Anal. Appl., **359** (2009) 704-710.
12. G. Dai and R. Hao, *Existence of solutions for a $p(x)$-Kirchhoff-type equation*, J. Math. Anal. Appl., **359** (2009), 275-284.
13. L. Diening, P. Harjulehto, P. Hasto and M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes, vol. 2017, Springer-Verlag, Berlin, 2011, ix+509 pp.
14. I. Ekeland, *On the variational principle*, J. Math. Anal. Appl., **47** (1974), 324-353.
15. X.L. Fan, *On nonlocal $p(x)$-Laplacian Dirichlet problems*, Nonlinear Anal., **72** (2010), 3314-3323.
16. X.L. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl., **263** (2001), 424-446.
17. X.L. Fan, J. Shen and D. Zhao, *Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$*, J. Math. Anal. Appl., **262** (2001), 749-760.
18. K. Kefi, *$p(x)$-Laplacian with indefinite weight*, Proc. Amer. Math. Soc., **139**(12) (2011), 4351-4360.
19. G. Kirchhoff, *Mechanik*, Teubner, Leipzig, Germany, 1883.
20. O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. J., **41** (1991), 592-618.
21. D. Liu, *On a p-Kirchhoff equation via fountain theorem and dual fountain theorem*, Nonlinear Analysis, **72** (2010), 302-308.
22. T.F. Ma, *Remarks on an elliptic equation of Kirchhoff type*, Nonlinear Anal., **63** (2005),1967-1977.
23. M. Mihăilescu and G. Morosanu, *Existence and multiplicity of solutions for an anisotropic elliptic problem involving variable exponent growth conditions*, Applicable Analysis, **89**(2) (2010), 257-271.
24. J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Vol. 1034, Springer, Berlin, 1983.
25. B. Ricceri, *On an elliptic Kirchhoff-type problem depending on two parameters*, Journal of Global Optimization, **46**(4) 2010, 543-549.
26. J.J. Sun and C.L. Tang, *Existence and multiplicity of solutions for Kirchhoff type equations*, Nonlinear Anal., **74** (2011), 1212-1222.

Nguyen Thanh Chung

Dep. Science Management & International Cooperation, Quang Binh University, 312 Ly Thuong Kiet, Dong Hoi, Quang Binh, Vietnam.
e-mail: ntchung82@@yahoo.com