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DUALITY IN MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS INVOLVING (H_p, r)-INVEX FUNCTIONS†

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Abstract. In this paper, we have taken step in the direction to establish weak, strong and strict converse duality theorems for three types of dual models related to multiojective fractional programming problems involving (H_p, r)-invex functions.

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1. Introduction

Generalized convexity plays an important role in many aspects of optimization, such as optimality conditions, duality theorems, variational inequalities, saddle point theory and convergence of optimization algorithms, so the research on generalized convexity is one of the important aspects of mathematical programming problems.

The problem in which objective functions are ratio of two functions are termed as fractional programming problems. Such problems are studied in various fields like economics [3], information theory [12], heat exchange networking [24] and others. Duality in multiobjective fractional programming problems involving generalized convex functions have been of much interest in recent past, (see [4, 5, 8, 14, 16, 18, 22]) and the references cited therein. For more information about fractional programming problems, the reader may consult the research bibliography compiled by Stancu-Minasian [19, 20, 21].

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Mukherjee [13] considered a multiobjective fractional programming problem and discussed the Mond-Weir type duality results under generalized convexity. Gulati and Ahmad [6] proved the duality results using Fritz John conditions for multiobjective programming problem involving generalized convex functions. Kaul *et al.* [9] derived duality results for a Mond-Weir type dual problem related to multiobjective fractional programming problem involving pseudolinear and η-pseudolinear functions. Osuna-Gómez *et al.* [15] focus his study to establish the optimality condition and duality theorems for a class of multiobjective fractional programs under generalized convexity assumptions by applying parametric approach.

The notion of convexity was not enough to meet the challenging demand of some problems on Economics and Engineering. To meet this demand the notion of invexity was introduced by Hanson [7] by substituting the linear term $(x-y)$ appearing in the definition of convex functions with an arbitrary vector valued function.

Antczak [2] introduced a new class of functions named (p, r)-invex function, which is an extension of invex function. Recently, Jayswal *et al.* [8] focus his study on multiobjective fractional programming problems and derived sufficient optimality conditions and duality theorems involving $(p, r) - \rho - (\eta, \theta)$-invex functions [11].

Yuan *et al.* [23] introduced new types of generalized convex functions and sets, which are called locally $(H_p, r, \alpha)-$pre-invex and locally H_p-invex sets. They obtained also optimality conditions and duality theorems for a scalar nonlinear programming problem. Recently, Liu *et al.* [10] proposed the concept of (H_p, r)-invex function and focus his study to discuss sufficient optimality conditions to multiple objective programming problem and multiobjective fractional programming problem involving the aforesaid class of functions but no step was taken to prove the duality results involving (H_p, r)-invex functions.

In this paper, viewing the importance of duality theorems in optimization theory, we establish weak, strong and strict converse duality theorems involving (H_p, r)-invex function to three types of dual models related to mulitiobjective fractional programming problems. The organization of the remainder of this paper is as follows. The formulation of multiobjective fractional programming problem along with some definitions and notations related to (H_p, r)-invexity is given in Section 2. Weak, strong and strict converse duality theorems for three types of dual models related to multiobjective fractional programming problem under (H_p, r)-invexity are derived in Section 3 to Section 5. Finally, conclusions and further developments are given in Section 6.

2. Notation and Preliminaries

Throughout the paper, let R^n be the n-dimensional Euclidean space, $R^n_+ = \{x \in R^n \mid x \geq 0\}$ and $\dot{R}^n_+ = \{x \in R^n \mid x > 0\}$. Let $x, y \in R^n$. Then $x \preceq y \Leftrightarrow x_i \leq y_i, i = 1, 2, \cdots, n$ and $x \neq y$.

Definition 2.1 ([2])**.** Let $a_1, a_2 > 0, \lambda \in (0, 1)$ and $r \in R$. Then the weighted r-mean of a_1 and a_2 is given by

$$M_r(a_1, a_2; \lambda) = \begin{cases} (\lambda {a_1}^r + (1-\lambda) a_2^r)^{\frac{1}{r}}, & \text{for } r \neq 0 \\ a_1^{\lambda} a_2^{(1-\lambda)}, & \text{for } r = 0. \end{cases}$$

Definition 2.2 ([23])**.** $X \subset R^n$ is locally H_p-invex set if and only if, for any x,$u \in X$, there exist a maximum positive number $a(x, u) \leq 1$ and a vector function $H_p : X \times X \times [0, 1] \to R^n$, such that

$$H_p(x, u; 0) = e^u, \ H_p(x, u; \lambda) \in \dot{R}^n_+$$

$$\ln(H_p(x, u; \lambda)) \in X, \ \forall \ 0 < \lambda < a(x, u) \text{ for } p \ \in R,$$

and $H_p(x, u; \lambda)$ is continuous on the interval $(0, a(x, u))$, where the logarithm and the exponentials appearing in the relation are understood to be taken componentwise.

Definition 2.3 ([23])**.** A function $f : X \to R$ defined on a locally H_p-invex set $X \subset R^n$ is said to be locally (H_p, r)-pre-invex on X if, for any $x, u \in X$, there exists a maximum positive number $a(x, u) \leq 1$ such that

$$f(\ln(H_p(x, u; \lambda))) \leq \ln(M_r(e^{f(x)}, e^{f(u)}; \lambda^{\alpha})), \forall \ 0 < \lambda < a(x, u) \text{ for } p \in R,$$

where the logarithm and the exponentials appearing in the left-hand side of the inequality are understood to be taken componentwise. If u is fixed, then f is said to be (H_p, r)-pre-invex at u. Correspondingly, if the direction of above inequality is changed to the opposite one, then f is said to (H_p, r)-pre-incave on S or at u.

For convenience, we assume that X be a H_p-invex set, H_p is right differentiable at 0 with respect to the variable λ for each given pair $x, u \in X$, and $f : X \to R$ is differential on X. The symbol $H_p^{'}(x, u; 0+) \triangleq (H_{p1}^{'}(x, u; 0+), ...,$ $H_{pn}^{'}(x, u; 0+))^T$ denotes the right derivative of H_p at 0 with respect to the variable λ for each given pair $x, u \in X$; $\nabla f(x) \triangleq (\nabla_1 f(x), ..., \nabla_n f(x))^T$ denotes the differential of f at x, and so $\frac{\nabla f(u)}{e^u}$ denotes $(\frac{\nabla_1 f(u)}{e^{u_1}}, ..., \frac{\nabla_n f(u)}{e^{u_n}})^T$.

Definition 2.4 ([10])**.** Let X be a H_p-invex set, H_p is right differentiable at 0 with respect to the variable λ for each given pair $x, u \in X$, and $f : X \to R$ is differentiable on X. If for all $x \in X$, one of the relations

$$\frac{1}{r} e^{rf(x)} \geq \frac{1}{r} e^{rf(u)} \left[1 + r \frac{\nabla f(u)^T}{e^u} H_p^{'}(x, u; 0+) \right] \ (>) \text{ for } r \neq 0,$$

$$f(x) - f(u) \geq \frac{\nabla f(u)^T}{e^u} H_p^{'}(x, u; 0+) \ (>) \text{ for } r = 0,$$

hold, then f is said to be (H_p, r)-invex (strictly (H_p, r)-invex) at $u \in X$. If the above inequalities are satisfied at any point $u \in X$ then f is said to be (H_p, r)-invex (strictly (H_p, r)-invex) on X.

We now consider the following multiobjective fractional programming problems:

(FP) Minimize $\frac{f(x)}{g(x)} \triangleq (\frac{f_1(x)}{g_1(x)}, ..., \frac{f_k(x)}{g_k(x)})$

subject to

$$h(x) \leq 0, \tag{1}$$
$$x \in X \subset R^n,$$

where $f,\ g : X \to R^k$ and $h : X \to R^m, f = (f_1, f_2, ..., f_k), g = (g_1, g_2, ..., g_k), h = (h_1, h_2, ..., h_m)$, are differentiable functions on a (nonempty) H_p-invex set X. Without loss of generality, we can assume that $f_i(x) \geq 0,\ g_i(x) > 0,\ i = 1, 2, ..., k$ for all $x \in X$. Let $X^0 = \{x \in X : h(x) \leq 0\}$ be the set of all feasible solutions to (FP).

We denote $\phi_i(x) = \frac{f_i(x)}{g_i(x)}$ and $\phi(x) = (\phi_1(x), \phi_2(x), ..., \phi_k(x))$.

Definition 2.5. A feasible solution $x^* \in X^0$ of (FP) is said to be an efficient solution of (FP) if there exist no other feasible solution $x \in X^0$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)} \text{ for all } i = 1, 2, ..., k,$$

and

$$\frac{f_t(x)}{g_t(x)} < \frac{f_t(x^*)}{g_t(x^*)} \text{ for some } t \in \{1, 2, ..., k\}.$$

It is well known (see, for example [17]) that, if $x^* \in X^0$ is an efficient solution of a multiobjective fractional programming problem (FP), then the following necessary optimality conditions are satisfied:

Theorem 2.1 (Necessary optimality conditions)**.** *Let $x^* \in X^0$ be an efficient solution to a multiobjective fractional programming problem* (FP) *and h satisfies the constraints qualification* [17] *at x^*. Then, there exist $y^* \in R^k_+,\ z^* \in R^m,\ v^* \in R^k$ such that*

$$\sum_{i=1}^{k} y_i^*[\nabla f_i(x^*) - v_i^* \nabla g_i(x^*)] + \sum_{j=1}^{m} z_j^* \nabla h_j(x^*) = 0, \tag{2}$$

$$f_i(x^*) - v_i^* g_i(x^*) = 0, \text{ for all } i = 1, 2, ..., k, \tag{3}$$

$$z_j^* h_j(x^*) = 0, \text{ for all } j = 1, 2, ..., m, \tag{4}$$

$$h_j(x^*) \leq 0, \text{ for all } j = 1, 2, ..., m, \tag{5}$$

$$y^* \in \Omega,\ z^* \in R^m_+, \tag{6}$$

where $\Omega = \{y \in R^k : y = (y_1, y_2, ..., y_k) > 0 \text{ and } \sum_{i=1}^{k} y_i = 1\}$.

The above conditions will be needed in the present analysis.

Remark 2.1 All the theorems in the subsequent parts of this paper will be proved only in the the case when $r \neq 0$. The proofs in other cases are easier than in this one, since the differences arise only the form of inequality. Moreover, without loss of generality, we shall assume that $r > 0$ (in the case when $r < 0$, the direction some of the inequalities in the proof of the theorems should be changed to the opposite one).

3. Parametric duality

We consider the following parametric dual of (FP) as follows:
(DI) Maximize $v = (v_1, v_2, ..., v_k)$
subject to

$$\sum_{i=1}^{k} y_i[\nabla f_i(u) - v_i \nabla g_i(u)] + \sum_{j=1}^{m} z_j \nabla h_j(u) = 0, \tag{7}$$

$$f_i(u) - v_i g_i(u) \geq 0, \text{ for all } i = 1, 2, ..., k, \tag{8}$$

$$\sum_{j=1}^{m} z_j h_j(u) = 0, \tag{9}$$

$$y \in \Omega, \ z \geq 0, \ v \geq 0. \tag{10}$$

Theorem 3.1 (Weak duality)**.** *Let* $x \in X^0$ *be a feasible solution for* (FP), *and let* (u, y, z, v) *be a feasible solution for* (DI). *Moreover, we assume that any one of the following conditions holds:*

(*a*) $S(.) = \sum_{i=1}^{k} y_i[f_i(.) - v_i g_i(.)] + \sum_{j=1}^{m} z_j h_j(.)$ *is* (H_p, r)*-invex at* u,

(*b*) $P(.) = \sum_{i=1}^{k} y_i[f_i(.) - v_i g_i(.)]$ and $Q(.) = \sum_{i=1}^{k} z_j h_j(.)$ *are* (H_p, r)*-invex at* u.

Then $\phi(x) \npreceq v$.

Proof. If the condition (a) holds, then (H_p, r)-invexity of $S(.)$ at u, we have

$$\frac{1}{r} e^{rS(x)} \geq \frac{1}{r} e^{rS(u)} \left[1 + r \frac{\nabla S(u)^T}{e^u} H_p'(x, u; 0+) \right].$$

Using the fundamental property of exponential functions, the above inequality together with (7), imply that

$$S(x) \geq S(u). \tag{11}$$

Now suppose contrary to the result that $\phi(x) \preceq v$. Then

$$\frac{f_i(x)}{g_i(x)} \leq v_i \text{ for } i = 1, 2, ..., k,$$

$$\text{and } \frac{f_t(x)}{g_t(x)} < v_t \text{ for some } t \in \{1, 2, ..., k\}.$$

That is,

$$f_i(x) - v_i g_i(x) \leq 0 \leq f_i(u) - v_i g_i(u) \text{ for } i = 1, 2, ..., k,$$

$$f_t(x) - v_t g_t(x) < 0 \leq f_t(u) - v_t g_t(u) \text{ for some } t \in \{1, 2, ..., k\}.$$

The above inequalities along with (10) give

$$\sum_{i=1}^{k} y_i[f_i(x) - v_i g_i(x)] < \sum_{i=1}^{k} y_i[f_i(u) - v_i g_i(u)]. \tag{12}$$

By the feasibility of x and from (9) and (10), we have

$$\sum_{j=1}^{m} z_j h_j(x) \leq \sum_{j=1}^{m} z_j h_j(u). \tag{13}$$

On adding (12) and (13), we obtain

$$\sum_{i=1}^{k} y_i[f_i(x) - v_i g_i(x)] + \sum_{j=1}^{m} z_j h_j(x) < \sum_{i=1}^{k} y_i[f_i(u) - v_i g_i(u)] + \sum_{j=1}^{m} z_j h_j(u),$$

i.e.,

$$S(x) < S(u),$$

which contradicts (11).
If condition (b) holds, the from the (H_p, r)-invexity of $Q(.)$ at u,

$$\frac{1}{r} e^{rQ(x)} \geq \frac{1}{r} e^{rQ(u)}[1 + r \frac{\nabla Q(u)^T}{e^u} H_p^{'}(x, u; 0+)],$$

equivalently

$$\frac{1}{r}[e^{r(Q(x)-Q(u))} - 1] \geq \frac{\nabla Q(u)^T}{e^u} H_p{}'(x, u; 0+). \tag{14}$$

From (13) and (14), we get

$$\frac{\nabla Q(u)}{e^u}^T H_p^{'}(x, u; 0+) \leq 0.$$

The above inequality together with (7) yields

$$\frac{\nabla P(u)^T}{e^u} H_p^{'}(x, u; 0+) \geq 0. \tag{15}$$

From the (H_p, r)-invexity of $P(.)$ at u, we have

$$\frac{1}{r} e^{rP(x)} \geq \frac{1}{r} e^{rP(u)}[1 + r \frac{\nabla P(u)^T}{e^u} H_p^{'}(x, u, 0+)]. \tag{16}$$

The inequalities (15) and (16), and the fundamental property of exponential functions imply that

$$P(x) \geq P(u).$$

That is,

$$\sum_{i=1}^{k} y_i[f_i(x) - v_i g_i(x)] \geq \sum_{i=1}^{k} y_i[f_i(u) - v_i g_i(u)]. \tag{17}$$

Again if $\phi(x) \preceq v$, then we get (12) in the same way. But (12) contradicts (17). Therefore, $\phi(x) \npreceq v$. This completes the proof. □

Theorem 3.2 (Strong duality). *Let x^* be an efficient solution for* (FP) *and let h satisfy the constraints qualification* [17] *at x^*. Then there exist $y^* \in \Omega$, $z^* \in R^m$ and $v^* \in R^k$ such that (x^*, y^*, z^*, v^*) is feasible for* (DI).

Also, if the weak duality theorem 3.1 holds for all feasible solutions of the problems (FP) *and* (DI), *then* (x^*, y^*, z^*, v^*) *is an efficient solution for* (DI) *and the two objectives are equal at these points.*

Proof. Since x^* is an efficient solution for (FP) and h satisfy the constraints qualification at x^*, there exist $y^* \in \Omega,\ z^* \in R^m$ and $v^* \in R^k$ such that (x^*, y^*, z^*, v^*) satisfies (2)-(6). This, in turn, imply that (x^*, y^*, z^*, v^*) is feasible for (DI). From the weak duality theorem, for any feasible points (x, y, z, v) to (DI), the inequality $\phi(x^*) \succeq v$ holds. Hence we conclude that (x^*, y^*, z^*, v^*) is an efficient solution to (DI) and the objective functions of (FP) and (DI) are equal at these points. This completes the proof. $\square$

Theorem 3.3 (Strict converse duality). *Assume that* x^* *and* (u^*, y^*, z^*, v^*) *be an efficient solution for* (FP) *and* (DI), *respectively with* $v_i^* = \frac{f_i(x^*)}{g_i(x^*)}$ *for all* $i = 1, 2, ..., k$. *Assume that*

$$A(.) = \sum_{i=1}^{k} y_i^*[f_i(.) - v_i^* g_i(.)] + \sum_{j=1}^{m} z_j^* h_j(.)$$

is strictly (H_p, r)*-invex at* u^*. *Then* $x^* = u^*$; *that is,* u^* *is an efficient solution for* (FP).

Proof. Suppose on the contrary that $x^* \neq u^*$. From (8), (9) and (10), we get

$$A(u^*) = \sum_{i=1}^{k} y_i^*[f_i(u^*) - v_i^* g_i(u^*)] + \sum_{j=1}^{m} z_j^* h_j(u^*) \geq 0. \tag{18}$$

From the strictly (H_p, r)-invexity of $A(.)$, we have

$$\frac{1}{r}[e^{r(A(x^*)-A(u^*))} - 1] > r\frac{\nabla A(u^*)^T}{e^{u^*}} {H_p}'(x^*, u^*; 0+).$$

Using the fundamental property of exponential functions, the above inequality together with (7), imply that

$$A(x^*) > A(u^*). \tag{19}$$

Since

$$v_i^* = \frac{f_i(x^*)}{g_i(x^*)} \text{ for all } i = 1, 2, ..., k,$$

i.e.,

$$f_i(x^*) - v_i^* g_i(x^*) = 0 \text{ for all } i = 1, 2, ..., k. \tag{20}$$

By the feasibility of x^* and (10), we have

$$\sum_{j=1}^{m} z_j^* h(x^*) \leq 0. \tag{21}$$

Therefore, from (10), (20) and (21), we conclude that

$$A(x^*) = \sum_{i=1}^{k} y_i^*[f_i(x^*) - v_i^* g_i(x^*)] + \sum_{j=1}^{m} z_j^* h_j(x^*) \leq 0. \quad (22)$$

Hence from (19) and (22), we have $A(u^*) < 0$ which contradicts (18). Hence $x^* = u^*$. This completes the proof. □

Remark 3.1 The function $A(.)$ in Theorem 3.3 is expressed by the sum of the modified objective part $B(.) = \sum_{i=1}^{k} y_i^*[f_i(.) - v_i^* g_i(.)]$ of (FP) and its constraint part $C(.) = \sum_{j=1}^{m} z_j^* h_j(.)$. If $B(.)$ is strictly (H_p, r)-invex and $C(.)$ is (H_p, r)-invex then the Theorem 3.3 is still holds.

4. Parameter free duality

In this section, we take the following form of theorem 2.1:

Theorem 4.1 *Let x^* be an efficient solution to* (FP). *Assume that h satisfies the constraints qualification at x^*. Then there exist $y^* \in R_+^k$, $z^* \in R^m$, such that*

$$\sum_{i=1}^{k} y_i^* g_i(x^*)[\nabla f_i(x^*) + \sum_{j=1}^{m} z_j^* \nabla h_j(x^*)] + \sum_{i=1}^{k} y_i^*(-\nabla g_i(x^*))[f_i(x^*) + \sum_{j=1}^{m} z_j^* h_j(x^*)] = 0, \quad (23)$$

$$z_j^* h_j(x^*) = 0, \text{ for all } j = 1, 2, ..., m, \quad (24)$$

$$h_j(x^*) \leq 0, \text{ for all } j = 1, 2, ..., m, \quad (25)$$

$$y^* \in \Omega, \ z^* \geq 0. \quad (26)$$

Now we consider the following parameter free dual problem to (FP):

(DII) Maximize $\left(\frac{f_1(u)+\sum_{j=1}^{m} z_j h_j(u)}{g_1(u)}, ..., \frac{f_k(u)+\sum_{j=1}^{m} z_j h_j(u)}{g_k(u)}\right)$

subject to

$$\sum_{i=1}^{k} y_i g_i(u)[\nabla f_i(u) + \sum_{j=1}^{m} z_j \nabla h_j(u)] + \sum_{i=1}^{k} y_i [f_i(u) + \sum_{j=1}^{m} z_j h_j(u)](-\nabla g_i(u)) = 0, \quad (27)$$

$$y \in \Omega, \ z \geq 0. \quad (28)$$

Denote $\Psi_i(u, z) = \frac{f_i(u)+\sum_{j=1}^{m} z_j h_j(u)}{g_i(u)}$ and $\Psi(u, z) = (\Psi_1(u, z), \Psi_2(u, z), ..., \Psi_k(u, z))$. Throughout this section, we assume $f_i(u) + \sum_{j=1}^{m} z_j h_j(u) \geq 0$ and $g_i(u) > 0$, for all $i = 1, 2, ..., k$.

Theorem 4.2 (Weak duality)**.** *Let $x \in X^0$ be a feasible solution for* (FP) *and let (u, y, z) be a feasible solution for* (DII). *Assume that*

$$\Theta(.) = \sum_{i=1}^{k} y_i g_i(u)[f_i(.) + \sum_{j=1}^{m} z_j h_j(.)] - \sum_{i=1}^{k} y_i g_i(.)[f_i(u) + \sum_{j=1}^{m} z_j h_j(u)]$$

is (H_p, r)-invex at u . Then $\phi(x) \nleq \Psi(u, z)$.

Proof. From (H_p, r)-invexity of $\Theta(.)$ at u, we have

$$\frac{1}{r}e^{r\Theta(x)} \geq \frac{1}{r}e^{r\Theta(u)}[1 + r\frac{\nabla\Theta(u)^T}{e^u}H_p^{'}(x, u; 0+)]$$

Using the fundamental property of exponential functions, the above inequality together with (27), imply that

$$\Theta(x) \geq \Theta(u) = 0,$$

i.e,

$$\Theta(x) \geq 0. \tag{29}$$

Suppose contrary to the result that $\phi(x) \preceq \Psi(u, z)$. Then

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(u) + \sum_{j=1}^m z_j h_j(u)}{g_i(u)} \text{ for } i = 1, 2, ..., k,$$

$$\frac{f_t(x)}{g_t(x)} < \frac{f_t(u) + \sum_{j=1}^m z_j h_j(u)}{g_t(u)} \text{ for some } t \in \{1, 2, ..., k\}.$$

It follows that

$$\sum_{i=1}^k y_i[f_i(x)g_i(u)] < \sum_{i=1}^k y_i g_i(x)[f_i(u) + \sum_{j=1}^m z_j h_j(u)],$$

equivalently,

$$\sum_{i=1}^k y_i[f_i(x) + \sum_{j=1}^m z_j h_j(x)]g_i(u) - \sum_{i=1}^k y_i g_i(x)[f_i(u) + \sum_{j=1}^m z_j h_j(u)]$$
$$< \sum_{i=1}^k y_i g_i(u) \sum_{j=1}^m z_j h_j(x). \tag{30}$$

From the feasibility of x, $g_i(u) > 0$ and (28), we have

$$\sum_{i=1}^k y_i g_i(u) \sum_{j=1}^m z_j h_j(x) \leq 0.$$

Therefore (30), implies

$$\sum_{i=1}^k y_i g_i(u)[f_i(x) + \sum_{j=1}^m z_j h_j(x)] - \sum_{i=1}^k y_i g_i(x)[f_i(u) + \sum_{j=1}^m z_j h_j(u)] < 0,$$

i.e,

$$\Theta(x) < 0,$$

which contradicts (29). This completes the proof. □

Theorem 4.3 (Strong duality)**.** *Let x^* be an efficient solution for* (FP) *and let h satisfy the constraints qualification* [17] *at x^*. Then there exist $y^* \in \Omega$ and $z^* \in R^m$ such that (x^*, y^*, z^*) is feasible to* (DII)*.*

Also, If the weak duality theorem 5.2 holds for all feasible solutions of the problem (FP) *and* (DII)*, then (x^*, y^*, z^*) is an efficient solution for* (DII) *and the two objectives are equal at these points.*

Proof. Since x^* is an efficient solution for (FP) and h satisfy the constraints qualification at x^*, there exist $y^* \in \Omega$ and $z^* \in R^m$ such that (x^*, y^*, z^*) satisfies (23)-(26). This, in turn, imply that (x^*, y^*, z^*) is feasible for (DII). From the weak duality theorem 4.2, for any feasible points (x, y, z) to (DII)), the inequality $\phi(x^*) \preceq \Psi(x, z)$ holds. Hence we conclude that (x^*, y^*, z^*) is an efficient solution to (DII) and the objective functions of (FP) and (DII) are equal at these points. This completes the proof. □

Theorem 4.4 (Strict converse duality)**.** *Assume that x^* and (u^*, y^*, z^*) be an efficient solution for* (FP) *and* (DII), *respectively. Assume that*

$$U(.) = \sum_{i=1}^{k} y_i^* g_i(u^*)[f_i(.) + \sum_{j=1}^{m} z_j^* h_j(.)]$$

$$- \sum_{i=1}^{k} y_i^* g_i(.)[f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)]$$

is strictly (H_p, r)-invex at u^* Then $x^* = u^*$; that is, u^* is an efficient solution for (FP).

Proof. Suppose on the contrary that $x^* \neq u^*$. From Theorem 4.3, we know that there exist $\bar{y}$ and $\bar{z}$ such that $(x^*, \bar{y}, \bar{z})$ is an efficient solution for (DII) and

$$\frac{f_i(x^*) + \sum_{j=1}^{m} z_j^* h_j(x^*)}{g_i(x^*)} = \frac{f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)}{g_i(u^*)}. \tag{31}$$

By (24), (26) and (31), we obtain

$$\frac{f_i(x^*)}{g_i(x^*)} = \frac{f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)}{g_i(u^*)}. \tag{32}$$

Hence

$$f_i(x^*) g_i(u^*) = [f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)] g_i(x^*). \tag{33}$$

From (28) and (33), we have

$$U(x^*) = \sum_{i=1}^{k} y_i^* g_i(u^*) \sum_{j=1}^{m} z_j^* h_j(x^*).$$

By the feasibility of x^*, $g_i(u^*) > 0$, from (28) and the above inequality, we have

$$U(x^*) \leq 0.$$

Therefore,

$$U(x^*) \leq 0 = U(u^*).$$

That is,

$$U(x^*) \leq U(u^*). \tag{34}$$

On the other hand, from strictly (H_p, r)-invexity of $U(.)$ at u^*, we have

$$\frac{1}{r}e^{rU(x^*)} > \frac{1}{r}e^{rU(u^*)}[1 + r\frac{\nabla U(u^*)^T}{e^u}H_p^{'}(x^*, u^*; 0+)].$$

The above inequality together with (27) and the fundamental property of the exponential functions yields

$$U(x^*) > U(u^*),$$

which contradicts inequality (34). Hence $x^* = u^*$; that is, u^* is an efficient solution for (FP). This completes the proof. □

5. Mond-Weir duality

In this section, we consider the following Mond-Weir dual to (FP):
(DIII) Maximize $(\frac{f_1(u)}{g_1(u)}, ..., \frac{f_k(u)}{g_k(u)})$
subject to

$$\sum_{i=1}^{k} y_i g_i(u)[\nabla f_i(u) + \sum_{j=1}^{m} z_j \nabla h_j(u)] + \sum_{i=1}^{k} y_i(-\nabla g_i(u))[f_i(u) + \sum_{j=1}^{m} z_j h_j(u)] = 0, \tag{35}$$

$$\sum_{j=1}^{m} z_j h_j(u) \geq 0, \tag{36}$$

$$y > 0, \ z \geq 0. \tag{37}$$

Denote $\Phi_i(u) = \frac{f_i(u)}{g_i(u)}$ and $\Phi(u) = (\Phi_1(u), \Phi_2(u), ..., \Phi_k(u))$.

Now we shall state weak, strong and strict converse duality theorems without proof as they can be proved in light of the Theorem 4.2, Theorem 4.3 and Theorem 4.4, proved in previous section.

Theorem 5.1 (Weak duality)**.** *Let $x \in X^0$ be a feasible solution for* (FP) *and let (u, y, z) be a feasible solution for* (DIII). *Assume that*

$$\sum_{i=1}^{k} y_i g_i(u)[f_i(.) + \sum_{j=1}^{m} z_j h_j(.)] - \sum_{i=1}^{k} y_i g_i(.)[f_i(u) + \sum_{j=1}^{m} z_j h_j(u)].$$

is (H_p, r)-invex at u . Then

$$\phi(x) \nleq \Phi(u).$$

Theorem 5.2 (Strong duality)**.** *Let x^* be an efficient solution for* (FP) *and let h satisfy the constraints qualification* [17] *at x^*. Then there exist $y^* \in I$ and $z^* \in R^m$ such that (x^*, y^*, z^*) is feasible to* (DIII).

Also, If the weak duality theorem 5.1 holds for all feasible solutions of the problem (FP) *and* (DIII), *then (x^*, y^*, z^*) is an efficient solution for* (DIII) *and the two objectives are equal at these points.* **Theorem 5.3** (Strict converse duality). *Assume that* x^* *and* (u^*, y^*, z^*) *be an efficient solution for* (FP) *and* (DIII), *respectively. Assume that*

$$\sum_{i=1}^{k} y_i^* g_i(u^*)[f_i(.) + \sum_{j=1}^{m} z_j^* h_j(.)]$$
$$- \sum_{i=1}^{k} y_i^* g_i(.)[f_i(u^*) + \sum_{j=1}^{m} z_j^* h_j(u^*)]$$

is strictly (H_p, r)*-invex at* u^* . *Then* $x^* = u^*$*; that is,* u^* *is an efficient solution for* (FP).

6. Conclusion

In this paper, we have used the concept of (H_p, r)-invex functions to established duality results for three type of dual models related to multiobjective fractional programming problem. The question arise whether optimality and duality theorems established in this paper also holds under the assumption of (H_p, r)-invexity for a class of minimax fractional programming problem considered in [1].

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