J. Appl. Math. & Informatics Vol. **32**(2014), No. 1 - 2, pp. 27 - 38
http://dx.doi.org/10.14317/jami.2014.027

ON GENERALIZED (α, β)-DERIVATIONS IN BCI-ALGEBRAS†

ABDULLAH M. AL-ROQI

ABSTRACT. *The notion of generalized (regular) (α, β)- derivations of a BCI-algebra is introduced, some useful examples are discussed, and related properties are investigated. The condition for a generalized (α, β)- derivation to be regular is provided. The concepts of a generalized F-invariant (α, β)- derivation and α-ideal are introduced, and their relations are discussed. Moreover, some results on regular generalized (α, β)- derivations are proved..*

AMS Mathematics Subject Classification : 03G25, 06F35.
Key words and phrases : BCI-algebras, (α, β)- derivations, generalized(α, β)- derivations, regular generalized (α, β)- derivations.

1. Introduction

Throughout our discussion X will denote a BCI-algebra unless otherwise mentioned. In the year 2004, Jun and Xin [1] applied the notion of derivation in ring and near-ring theory to BCI-algebras, and as a result they introduced a new concept, called a (regular) derivation, in BCI-algebras. Using this concept as defined they investigated some of its properties. Using the notion of a regular derivation, they also established characterizations of a p-semisimple BCI-algebra. For a self map d of a BCI-algebra, they defined a d-invariant ideal, and gave conditions for an ideal to be d-invariant. According to Jun and Xin, a self map $d : X \to X$ is called a left-right derivation (briefly (l, r)-derivation)of X if $d(x * y) = d(x) * y \wedge x * d(y)$ holds for all $x, y \in X$. Similarly, a self map $d : X \to X$ is called a right-left derivation (briefly (r, l)-derivation) of X if $d(x * y) = x * d(y) \wedge d(x) * y$ holds for all $x, y \in X$. Moreover, if d is both $(l, r)-$ and $(r, l)-$derivation, it is a derivation on X. After the work of Jun and Xin [1], many research articles have appeared on the derivations of BCI-algebras and a greater interest have been devoted to the study of derivations in BCI algebras

Received March 1, 2013. Revised June 6, 2013. Accepted June 21, 2013.
†This work was supported by the research grant of the University University of Tabuk, P. O. Box 741, Tabuk 71491, Saudi Arabia.

on various aspects (see [2, 3, 4, 5, 6, 7]).
Recently in [5], Muhiuddin and Al-roqi introduced the notion of (α, β)-derivations of a BCI-algebra, and investigated some related properties. Using the idea of *regular* (α, β)*-derivations*, they gave characterizations of a p-semisimple BCI-algebra. In the present paper, we consider a more general version of the paper [5]. We first introduce the notion of *generalized (regular)* (α, β)*-derivations* of a BCI-algebra, and investigate related properties. We provide a condition for a *generalized* (α, β)*-derivation* to be regular. We also introduce the concepts of a *generalized* F*-invariant* (α, β)*-derivation* and α-ideal, and then we investigate their relations. Furthermore, we obtain some results on regular *generalized* (α, β)*- derivations.*

2. Preliminaries

We begin with the following definitions and properties that will be needed in this paper.

A nonempty set X with a constant 0 and a binary operation $*$ is called a BCI*-algebra* if for all $x, y, z \in X$ the following conditions hold:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Define a binary relation $\leq$ on X by letting $x * y = 0$ if and only if $x \leq y$. Then $(X, \leq)$ is a partially ordered set. A BCI-algebra X satisfying $0 \leq x$ for all $x \in X$, is called BCK-algebra.

A BCI-algebra X has the following properties: for all $x, y, z \in X$

(a1) $x * 0 = x$.
(a2) $(x * y) * z = (x * z) * y$.
(a3) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.
(a4) $(x * z) * (y * z) \leq x * y$.
(a5) $x * (x * (x * y)) = x * y$.
(a6) $0 * (x * y) = (0 * x) * (0 * y)$.
(a7) $x * 0 = 0$ implies $x = 0$.

For a BCI-algebra X, denote by X_+ (resp. $G(X)$) the BCK-part (resp. the BCI-G part) of X, i.e., X_+ is the set of all $x \in X$ such that $0 \leq x$ (resp. $G(X) := \{x \in X \mid 0 * x = x\}$). Note that $G(X) \cap X_+ = \{0\}$ (see [8]). If $X_+ = \{0\}$, then X is called a p*-semisimple* BCI*-algebra.* In a p-semisimple BCI-algebra X, the following hold:

(a8) $(x * z) * (y * z) = x * y$.
(a9) $0 * (0 * x) = x$ for all $x \in X$.
(a10) $x * (0 * y) = y * (0 * x)$.
(a11) $x * y = 0$ implies $x = y$.
(a12) $x * a = x * b$ implies $a = b$.
(a13) $a * x = b * x$ implies $a = b$.

(a14) $a * (a * x) = x$.
(a15) $(x * y) * (w * z) = (x * w) * (y * z)$.

Let X be a p-semisimple BCI-algebra. We define addition "+" as $x + y = x * (0 * y)$ for all $x, y \in X$. Then $(X, +)$ is an abelian group with identity 0 and $x - y = x * y$. Conversely let $(X, +)$ be an abelian group with identity 0 and let $x * y = x - y$. Then X is a p-semisimple BCI-algebra and $x + y = x * (0 * y)$ for all $x, y \in X$ (see [9]).

For a BCI-algebra X we denote $x \wedge y = y * (y * x)$, in particular $0 * (0 * x) = a_x$, and $L_p(X) := \{a \in X \mid x * a = 0 \;\Rightarrow\; x = a, \forall x \in X\}$. We call the elements of $L_p(X)$ the *p-atoms* of X. For any $a \in X$, let $V(a) := \{x \in X \mid a * x = 0\}$, which is called the *branch* of X with respect to a. It follows that $x * y \in V(a * b)$ whenever $x \in V(a)$ and $y \in V(b)$ for all $x, y \in X$ and all $a, b \in L_p(X)$. Note that $L_p(X) = \{x \in X \mid a_x = x\}$, which is the p-semisimple part of X, and X is a p-semisimple BCI-algebra if and only if $L_p(X) = X$ (see [10],[Proposition 3.2]). Note also that $a_x \in L_p(X)$, i.e., $0 * (0 * a_x) = a_x$, which implies that $a_x * y \in L_p(X)$ for all $y \in X$. It is clear that $G(X) \subset L_p(X)$, and $x * (x * a) = a$ and $a * x \in L_p(X)$ for all $a \in L_p(X)$ and all $x \in X$. For more details, refer to [11, 12, 1, 10, 8, 9].

Definition 2.1 ([6])**.** A BCI-algebra X is said to be *torsion free* if it satiafies:

$$(\forall x \in X)\,(x + x = 0 \;\Rightarrow\; x = 0)\,.$$

Definition 2.2 ([5])**.** Let α and β are two endomorphisms of a BCI-algebra X. Then a self map $d_{(\alpha,\beta)} : X \to X$ is called a (α, β)-derivation of X if it satisfies:

$$(\forall x, y \in X)\,\big(d_{(\alpha,\beta)}(x * y) = \big(d_{(\alpha,\beta)}(x) * \alpha(y)\big) \wedge \big(d_{(\alpha,\beta)}(y) * \beta(x)\big)\big)\,.$$

3. Main results

In what follows, α and β are endomorphisms of a BCI-algebra X unless otherwise specified.

Definition 3.1. Let X be a BCK/BCI-algebra. Then a self map F on X is called a *generalized* (α, β)*-derivation* if there exists an (α, β)*-derivation* $d_{(\alpha,\beta)}$ of X such that

$$(\forall x, y \in X)\,\big(F(x * y) = (F(x) * \alpha(y)) \wedge \big(d_{(\alpha,\beta)}(y) * \beta(x)\big)\big) \tag{3.1}$$

Clearly, the notion of *generalized* (α, β)*-derivation* covers the concept of (α, β)*-derivation* when $F = d_{(\alpha,\beta)}$ and the concept of *generalized derivation* when $F = d_{(\alpha,\beta)} = D$, and $\alpha = \beta = I_X$ where I_X is the identity map on X.

Example 3.2. Consider a BCI-algebra $X = \{0, a, b\}$ with the following Cayley table:

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

(1) Define a map

$$d_{(\alpha,\beta)} : X \to X,\ x \mapsto \begin{cases} b & \text{if } x \in \{0, a\}, \\ 0 & \text{if } x = b, \end{cases}$$

and define two endomorphisms

$$\alpha : X \to X,\ x \mapsto \begin{cases} 0 & \text{if } x \in \{0, a\}, \\ b & \text{if } x = b, \end{cases}$$

and

$$\beta : X \to X,\ x \mapsto \begin{cases} 0 & \text{if } x \in \{0, b\}, \\ a & \text{if } x = a. \end{cases}$$

Then $d_{(\alpha,\beta)}$ is a (α, β)-derivation of X [5].
Again, define a self map

$$F : X \to X,\ x \mapsto \begin{cases} 0 & \text{if } x \in \{0, a\}, \\ b & \text{if } x = b. \end{cases}$$

It is routine to verify that F is a *generalized* (α, β)*-derivation* of X.
(2) Define a map

$$d_{(\alpha,\beta)} : X \to X,\ x \mapsto \begin{cases} 0 & \text{if } x \in \{0, b\}, \\ a & \text{if } x = a, \end{cases}$$

and define two endomorphisms

$$\alpha : X \to X,\ x \mapsto \begin{cases} 0 & \text{if } x \in \{a, b\}, \\ b & \text{if } x = 0, \end{cases}$$

and

$$\beta : X \to X,\ x \mapsto \begin{cases} 0 & \text{if } x \in \{0, a\}, \\ a & \text{if } x = b. \end{cases}$$

Then $d_{(\alpha,\beta)}$ is a (α, β)-derivation of X [5].
Again, define a self map $F : X \to X$ by $F(x) = b$ for all $x \in X$. It is routine to verify that F is a *generalized* (α, β)*-derivation* of X.

Lemma 3.3 ([12]). *Let X be a BCI-algebra. For any $x, y \in X$, if $x \le y$, then x and y are contained in the same branch of X.*

Lemma 3.4 ([12]). *Let X be a BCI-algebra. For any $x, y \in X$, if x and y are contained in the same branch of X, then $x * y,\ y * x \in X_+$.*

Proposition 3.5. *Let X be a commutative BCI-algebra. Then every generalized (α, β)-derivation F of X satisfies the following assertion:*

$$(\forall x, y \in X)\,(x \le y \ \Rightarrow\ F(x) \le F(y))\,, \tag{3.2}$$

that is, every generalized (α, β)-derivation of X is isotone.

Proof. Let $x, y \in X$ be such that $x \leq y$. Since X is commutative, we have $x = x \wedge y$. Hence

$$\begin{aligned} F(x) &= F(x \wedge y) \\ &= (F(y) * \alpha(y * x)) \wedge \big(d_{(\alpha,\beta)}(y * x) * \beta(y)\big) \\ &\leq (F(y) * \alpha(y * x)) \end{aligned} \tag{3.3}$$

Since every endomorphism of X is isotone, we have $\alpha(x) \leq \alpha(y)$. It follows from Lemma 3.3 that $0 = \alpha(x) * \alpha(y) \in X_+$ and $\alpha(y) * \alpha(x) \in X_+$ so that there exists $a(\neq 0) \in X_+$ such that $\alpha(y * x) = \alpha(y) * \alpha(x) = a$. Hence (3.3) implies that $F(x) \leq F(y) * a$. Using (a3), (a2) and (III), we have

$$\begin{aligned} F(x) * F(y) &\leq (F(y) * a) * F(y) \\ &= (F(y) * F(y)) * a = 0 * a = 0, \end{aligned}$$

and so $F(x) * F(y) = 0$, that is, $F(x) \leq F(y)$ by (a7). □

If we take $F = d_{(\alpha,\beta)}$, then we have the following corollary.

Corollary 3.6 ([5])**.** *Let X be a commutative BCI-algebra. Then every (α, β)-derivation $d_{(\alpha,\beta)}$ of X satisfies the following assertion:*

$$(\forall x, y \in X) \left(x \leq y \;\Rightarrow\; d_{(\alpha,\beta)}(x) \leq d_{(\alpha,\beta)}(y)\right), \tag{3.4}$$

that is, every (α, β)-derivation of X is isotone.

Proposition 3.7. *Every generalized (α, β)-derivation F of a BCI-algebra X satisfies the following assertion:*

$$(\forall x \in X) \left(F(x) = F(x) \wedge d_{(\alpha,\beta)}(0)\right). \tag{3.5}$$

Proof. Let F be a generalized (α, β)-derivation of X. Using (a2) and (a4), we have

$$\begin{aligned} F(x) &= F(x * 0) = (F(x) * \alpha(0)) \wedge \big(d_{(\alpha,\beta)}(0) * \beta(x)\big) \\ &= (F(x) * 0) \wedge \big(d_{(\alpha,\beta)}(0) * \beta(x)\big) \\ &= F(x) \wedge \big(d_{(\alpha,\beta)}(0) * \beta(x)\big) \\ &= \big(d_{(\alpha,\beta)}(0) * \beta(x)\big) * \big(\big(d_{(\alpha,\beta)}(0) * \beta(x)\big) * F(x)\big) \\ &= \big(d_{(\alpha,\beta)}(0) * \beta(x)\big) * \big(\big(d_{(\alpha,\beta)}(0) * F(x)\big) * \beta(x)\big) \\ &\leq d_{(\alpha,\beta)}(0) * \big(d_{(\alpha,\beta)}(0) * F(x)\big) \\ &= F(x) \wedge d_{(\alpha,\beta)}(0) \end{aligned}$$

Obviously $F(x) \wedge d_{(\alpha,\beta)}(0) \leq F(x)$ by (II). Therefore the equality (3.5) is valid. □

If we take $F = d_{(\alpha,\beta)}$, then we have the following corollary.

Corollary 3.8 ([5])**.** *Every (α, β)-derivation $d_{(\alpha,\beta)}$ of a BCI-algebra X satisfies the following assertion:*

$$(\forall x \in X)\left(d_{(\alpha,\beta)}(x) = d_{(\alpha,\beta)}(x) \wedge d_{(\alpha,\beta)}(0)\right). \tag{3.6}$$

Theorem 3.9. *Let F be a generalized (α, β)-derivation on a BCI-algebra X. Then*

(1) $(\forall a \in Lp(X), x \in X)\,(F(a * x) = F(a) * \alpha(x))$.
(2) $(\forall a \in Lp(X), x \in X)\,(F(a + x) = F(a) + \alpha(x))$.
(3) $(\forall a, b \in Lp(X))\,(F(a + b) = F(a) + \alpha(b))$.

Proof. (1) For any $a \in Lp(X)$, we have $a * x \in Lp(X)$ for all $x \in X$. Thus $F(a * x) = F(a) * \alpha(x) \wedge d_{(\alpha,\beta)}(x) * \beta(a) = F(a) * \alpha(x)$.

(2) For any $a \in Lp(X)$ and $x \in X$, it follows from (1) that

$$\begin{aligned} F(a + x) &= F(a * (0 * x)) = F(a) * \alpha(0 * x) \\ &= F(a) * (\alpha(0) * \alpha(x)) = F(a) * (0 * \alpha(x)) \\ &= F(a) + \alpha(x). \end{aligned}$$

(3) The proof follows directly from (2). □

If we take $F = d_{(\alpha,\beta)}$, then we have the following corollary.

Corollary 3.10 ([5])**.** *Let $d_{(\alpha,\beta)}$ be an (α, β)-derivation on a BCI-algebra X. Then*

(1) $(\forall a \in Lp(X), x \in X)\left(d_{(\alpha,\beta)}(a * x) = d_{(\alpha,\beta)}(a) * \alpha(x)\right)$.
(2) $(\forall a \in Lp(X), x \in X)\left(d_{(\alpha,\beta)}(a + x) = d_{(\alpha,\beta)}(a) + \alpha(x)\right)$.
(3) $(\forall a, b \in Lp(X))\left(d_{(\alpha,\beta)}(a + b) = d_{(\alpha,\beta)}(a) + \alpha(b)\right)$.

Definition 3.11. Let X be a BCI-algebra and F, F' be two self maps of X, we define $F \circ F' : X \to X$ by $(F \circ F')(x) = F(F'(x))$ for all $x \in X$.

Theorem 3.12. *Let X be a p-semisimple BCI-algebra. Let F and F' be two generalized (α, β)-derivations associated with $d_{(\alpha,\beta)}$ and $d'_{(\alpha,\beta)}$ (α, β)-derivations respectively on X such that $\alpha^2 = \alpha$. Then $F \circ F'$ is an (α, β)-derivation on X.*

Proof. For any $x, y \in X$, it follows from (a14) that

$$\begin{aligned}
(F \circ F')(x * y) &= F(F'(x * y)) \\
&= F\left((F'(x) * \alpha(y)) \wedge \left(d'_{(\alpha,\beta)}(y) * \beta(x)\right)\right) \\
&= F\left(F'(x) * \alpha(y)\right) \\
&= (F(F'(x)) * \alpha(\alpha(y))) \wedge \left(d_{(\alpha,\beta)}(\alpha(y)) * \beta(F'(x))\right) \\
&= F(F'(x)) * \alpha(y) \\
&= \left(d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(y) * \beta(x))\right) * \\
&\qquad \left(\left(d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(y) * \beta(x)\right) * (F(F'(x)) * \alpha(y))\right) \\
&= (F(F'(x)) * \alpha(y)) \wedge \left(d_{(\alpha,\beta)}(d'_{(\alpha,\beta)}(y) * \beta(x))\right) \\
&= ((F \circ F')(x) * \alpha(y)) \wedge \left((d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)})(y) * \beta(x)\right).
\end{aligned}$$

This completes the proof. □

If we take $F = d_{(\alpha,\beta)}$, then we have the following corollary.

Corollary 3.13 ([5]). *Let X be a p-semisimple BCI-algebra. If $d_{(\alpha,\beta)}$ and $d'_{(\alpha,\beta)}$ are two (α, β)-derivations on X such that $\alpha^2 = \alpha$, then $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)}$ is an (α, β)-derivation on X.*

Theorem 3.14. *Let α, β be two endomorphisms and F be a self map on a p-semisimple BCI-algebra X such that $F(x) = \alpha(x)$ for all $x \in X$. Then F is a generalized (α, β)-derivation on X.*

Proof. Let us take $F(x) = \alpha(x)$ for all $x \in X$. Since $x, y \in X \implies x * y \in X$, by using (a14) we have

$$\begin{aligned}
F(x * y) = \alpha(x * y) &= \alpha(x) * \alpha(y) = F(x) * \alpha(y) \\
&= \left(d_{(\alpha,\beta)}(y) * \beta(x)\right) * \left(\left(d_{(\alpha,\beta)}(y) * \beta(x)\right) * (F(x) * \alpha(y))\right) \\
&= (F(x) * \alpha(y)) \wedge \left(d_{(\alpha,\beta)}(y) * \beta(x)\right).
\end{aligned}$$

This completes the proof. □

If we take $F = d_{(\alpha,\beta)}$, then we have the following corollary.

Corollary 3.15 ([5]). *Let α, β be two endomorphisms and $d_{(\alpha,\beta)}$ be a self map on a p-semisimple BCI-algebra X such that $d_{(\alpha,\beta)}(x) = \alpha(x)$ for all $x \in X$. Then $d_{(\alpha,\beta)}$ is an (α, β)-derivation on X.*

Definition 3.16. A generalized (α, β)-derivation F of a BCI-algebra X is said to be *regular* if $F(0) = 0$.

Example 3.17. (1) The generalized (α, β)-derivation F of X in Example 3.2(1) is regular.

(2) The generalized (α, β)-derivation F of X in Example 3.2(2) is not regular.

We provide a condition for a generalized (α, β)-derivation to be regular.

Theorem 3.18. *Let F be a generalized (α, β)-derivation of a BCI-algebra X. If there exists $a \in X$ such that $F(x) * \alpha(a) = 0$ for all $x \in X$, then F is regular.*

Proof. Assume that there exists $a \in X$ such that $F(x) * \alpha(a) = 0$ for all $x \in X$. Then

$$\begin{aligned} 0 = F(x * a) * a &= \big((F(x) * \alpha(a)) \wedge \big(d_{(\alpha,\beta)}(a) * \beta(x)\big)\big) * a \\ &= \big(0 \wedge \big(d_{(\alpha,\beta)}(a) * \beta(x)\big)\big) * a = 0 * a, \end{aligned}$$

and so $F(0) = F(0 * a) = (F(0) * \alpha(a)) \wedge \big(d_{(\alpha,\beta)}(a) * \beta(0)\big) = 0$. Hence F is regular. □

If we take $F = d_{(\alpha,\beta)}$, then we have the following corollary.

Corollary 3.19 ([5])**.** *Let $d_{(\alpha,\beta)}$ be an (α, β)-derivation of a BCI-algebra X. If there exists $a \in X$ such that $d_{(\alpha,\beta)}(x) * \alpha(a) = 0$ for all $x \in X$, then $d_{(\alpha,\beta)}$ is regular.*

Definition 3.20. For a generalized (α, β)-derivation F of a BCI-algebra X, we say that an ideal A of X is an *α-ideal* (resp. *β-ideal*) if $\alpha(A) \subseteq A$ (resp. $\beta(A) \subseteq A$).

Definition 3.21. For a generalized (α, β)-derivation F of a BCI-algebra X, we say that an ideal A of X is *F-invariant* if $F(A) \subseteq A$.

Example 3.22. (1) Let F be a generalized (α, β)-derivation of X which is described in Example 3.2(1). We know that $A := \{0, a\}$ is both an α-ideal and a β-ideal of X. Furthermore, $A := \{0, a\}$ is also F-invariant.
(2) Let F be a generalized (α, β)-derivation of X which is described in Example 3.2(2). We know that $A := \{0, a\}$ is a β-ideal of X. But $A := \{0, a\}$ is an ideal of X which is neither α-ideal nor F-invariant.

Next, we prove some results on regular generalized (α, β)-derivations in a BCI-algebra. In our further discussion, we shall assume that for every regular generalized (α, β)-derivation $F : X \to X$ there exists a regular (α, β)-derivation $d_{(\alpha,\beta)} : X \to X$ i.e. $d_{(\alpha,\beta)}(0) = 0$.

Theorem 3.23. *Let F be a regular generalized (α, β)-derivation of a BCI-algebra X. Then*

(1) $(\forall a \in X)\,(a \in L_p(X) \;\Rightarrow\; F(a) \in L_p(X))$.
(2) $(\forall a \in X)\,(a \in L_p(X) \;\Rightarrow\; \alpha(a),\ \beta(a) \in L_p(X))$.
(3) $(\forall a \in L_p(X))\,(F(a) = F(0) + \alpha(a))$.
(4) $(\forall a, b \in L_p(X))\,(F(a + b) = F(a) + F(b) - F(0))$.

Proof. (1) Let F be a regular generalized (α, β)-derivation. Then the proof follows directly from Proposition 3.7.

(2) Let $a \in L_p(X)$. Then $a = 0 * (0 * a)$, and so $\alpha(a) = \alpha(0 * (*0 * a)) = 0 * (*0 * \alpha(a))$. Thus $\alpha(a) \in L_p(X)$. Similarly, $\beta(a) \in L_p(X)$.

(3) Let $a \in L_p(X)$. Using (2), (a1) and (a14), we have

$$\begin{aligned} F(a) &= F(0 * (0 * a)) \\ &= (F(0) * \alpha(0 * a)) \wedge \big(d_{(\alpha,\beta)}(0 * a) * \beta(0)\big) \\ &= (F(0) * \alpha(0 * a)) \wedge \big(d_{(\alpha,\beta)}(0 * a) * 0\big) \\ &= (F(0) * \alpha(0 * a)) \wedge d_{(\alpha,\beta)}(0 * a) \\ &= d_{(\alpha,\beta)}(0 * a) * \big(d_{(\alpha,\beta)}(0 * a) * (F(0) * \alpha(0 * a))\big) \\ &= F(0) * \alpha(0 * a) \\ &= F(0) * (0 * \alpha(a)) \\ &= F(0) + \alpha(a). \end{aligned}$$

(4) Let $a, b \in L_p(X)$. Then $a + b \in L_p(X)$. Using (3), we have

$$\begin{aligned} F(a + b) &= F(0) + \alpha(a + b) = F(0) + \alpha(a) + \alpha(b) \\ &= F(0) + \alpha(a) + F(0) + \alpha(b) - F(0) \\ &= F(a) + F(b) - F(0). \end{aligned}$$

This completes the proof. □

If we take $F = d_{(\alpha,\beta)}$, then we have the following corollary.

Corollary 3.24 ([5])**.** *Let $d_{(\alpha,\beta)}$ be a regular (α, β)-derivation of a BCI-algebra X. Then*

(1) $(\forall a \in X)\left(a \in L_p(X) \;\Rightarrow\; d_{(\alpha,\beta)}(a) \in L_p(X)\right)$.
(2) $(\forall a \in X)\left(a \in L_p(X) \;\Rightarrow\; \alpha(a),\ \beta(a) \in L_p(X)\right)$.
(3) $(\forall a \in L_p(X))\left(d_{(\alpha,\beta)}(a) = d_{(\alpha,\beta)}(0) + \alpha(a)\right)$.
(4) $(\forall a, b \in L_p(X))\left(d_{(\alpha,\beta)}(a + b) = d_{(\alpha,\beta)}(a) + d_{(\alpha,\beta)}(b) - d_{(\alpha,\beta)}(0)\right)$.

Theorem 3.25. *Let X be a torsion free BCI-algebra and F be a regular generalized (α, β)-derivation on X such that $\alpha \circ F = F$. If $F^2 = 0$ on $Lp(X)$, then $F = 0$ on $Lp(X)$.*

Proof. Let us suppose $F^2 = 0$ on $Lp(X)$. If $x \in Lp(X)$, then $x + x \in Lp(X)$ and so by using Theorem 3.23 (3) and (4), we have

$$\begin{aligned} 0 &= F^2(x + x) = F(F(x + x)) \\ &= F(0) + \alpha(F(x + x)) = F(0) + F(x + x) \\ &= F(0) + F(x) + F(x) - F(0) \\ &= F(x) + F(x). \end{aligned}$$

Since X is a torsion free, therefore $F(x) = 0$ for all $x \in X$ implying thereby $F = 0$. This completes the proof. $\square$

If we take $F = d_{(\alpha,\beta)}$, then we have the following corollary.

Corollary 3.26 ([5])**.** *Let X be a torsion free BCI-algebra and $d_{(\alpha,\beta)}$ be a regular (α, β)-derivation on X such that $\alpha \circ d_{(\alpha,\beta)} = d_{(\alpha,\beta)}$. If $d^2_{(\alpha,\beta)} = 0$ on $Lp(X)$, then $d_{(\alpha,\beta)} = 0$ on $Lp(X)$.*

Theorem 3.27. *Let X be a torsion free BCI-algebra and F, F' be two regular generalized (α, β)-derivations on X such that $\alpha \circ F' = F'$. If $F \circ F' = 0$ on $Lp(X)$, then $F' = 0$ on $Lp(X)$.*

Proof. Let us suppose $F \circ F' = 0$ on $Lp(X)$. If $x \in Lp(X)$, then $x + x \in Lp(X)$ and so by using Theorem 3.23 (1) and (2), we have

$$\begin{aligned}
0 &= (F \circ F')\,(x + x) = F(F'(x + x)) = F(0) + \alpha(F'(x + x)) \\
&= F(0) + F'(x + x) = F(0) + (F'(x) + F'(x) - F'(0)) \\
&= (F(0) - F'(0)) + (F'(x) + F'(x)) \\
&= ((F(0) * F'(0))) + (F'(x) + F'(x)) \\
&= (F(0) * (0 * F'(0))) + (F'(x) + F'(x)) \\
&= (F(0) + F'(0)) + (F'(x) + F'(x)) \\
&= (F(0) + \alpha F'(0)) + (F'(x) + F'(x)) \\
&= F\,(F'(0)) + (F'(x) + F'(x)) \\
&= (F \circ F')(0) + (F'(x) + F'(x)) = F'(x) + F'(x).
\end{aligned}$$

Since X is a torsion free, therefore $F'(x) = 0$ for all $x \in X$ and so $F' = 0$. This completes the proof. $\square$

If we take $F = d_{(\alpha,\beta)}$, then we have the following corollary.

Corollary 3.28 ([5])**.** *Let X be a torsion free BCI-algebra and $d_{(\alpha,\beta)}$, $d'_{(\alpha,\beta)}$ be two regular (α, β)-derivations on X such that $\alpha \circ d'_{(\alpha,\beta)} = d'_{(\alpha,\beta)}$. If $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} = 0$ on $Lp(X)$, then $d'_{(\alpha,\beta)} = 0$ on $Lp(X)$.*

Proposition 3.29. *Let F be a regular generalized (α, β)-derivation of a BCI-algebra X. If $F^2 = 0$ on $L_p(X)$, then $(\alpha \circ F)(x) = \frac{1}{2}\left((\alpha \circ F)(0) - F(0)\right)$ for all $x \in L_p(X)$.*

Proof. Assume that $F^2 = 0$ on $L_p(X)$. If $x \in L_p(X)$, then $x + x \in L_p(X)$ and so by using Theorem 3.23 (3) and (4), we have

$$\begin{aligned}
0 &= F^2(x + x) = F(F(x + x)) = F(0) + \alpha(F(x + x)) \\
&= F(0) + \alpha(F(x) + F(x) - F(0)) \\
&= F(0) + 2\alpha(F(x)) - \alpha(F(0)).
\end{aligned}$$

Hence $(\alpha \circ F)(x) = \frac{1}{2}((\alpha \circ F)(0) - F(0))$ for all $x \in L_p(X)$.
This completes the proof. □

If we take $F = d_{(\alpha,\beta)}$, then we have the following corollary.

Corollary 3.30 ([5])**.** *Let $d_{(\alpha,\beta)}$ be a regular (α, β)-derivation of a BCI-algebra X. If $d^2_{(\alpha,\beta)} = 0$ on $L_p(X)$, then $(\alpha \circ d_{(\alpha,\beta)})(x) = \frac{1}{2}\left((\alpha \circ d_{(\alpha,\beta)})(0) - d_{(\alpha,\beta)}(0)\right)$ for all $x \in L_p(X)$.*

Proposition 3.31. *Let F and F' be two regular generalized (α, β)-derivations of a BCI-algebra X. If $F \circ F' = 0$ on $L_p(X)$, then $(\alpha \circ F')(x) = \frac{1}{2}((\alpha \circ F')(0) - F(0))$ for all $x \in L_p(X)$.*

Proof. Let $x \in L_p(X)$. Then $x + x \in L_p(X)$, and so $F'(x + x) \in L_p(X)$ by Theorem 3.23 (1). It follows from Theorem 3.23 (3) and (4) that

$$\begin{aligned} 0 &= (F \circ F')(x + x) = F(F'(x + x)) \\ &= F(0) + \alpha(F'(x + x)) \\ &= F(0) + \alpha(F'(x) + F'(x) - F'(0)) \\ &= F(0) + 2\alpha(F'(x)) - \alpha(F'(0)) \end{aligned}$$

so that $\alpha(F'(x)) = \frac{1}{2}((\alpha \circ F')(0) - F(0))$ for all $x \in L_p(X)$.
This completes the proof. □

If we take $F = d_{(\alpha,\beta)}$, then we have the following corollary.

Corollary 3.32 ([5])**.** *Let $d_{(\alpha,\beta)}$ and $d'_{(\alpha,\beta)}$ be two regular (α, β)-derivations of a BCI-algebra X. If $d_{(\alpha,\beta)} \circ d'_{(\alpha,\beta)} = 0$ on $L_p(X)$, then $(\alpha \circ d'_{(\alpha,\beta)})(x) = \frac{1}{2}\left((\alpha \circ d'_{(\alpha,\beta)})(0) - d_{(\alpha,\beta)}(0)\right)$ for all $x \in L_p(X)$.*

References

1. Y. B. Jun and X. L. Xin, *On derivations of BCI-algebras,* Inform. Sci. 159 (2004), 167–176.
2. H.A.S. Abujabal and N.O. Al-Shehri, *On Left Derivations of BCI-algebras,* Soochow J. Math. 33(3) (2007), 435–444.
3. S. Ilbira, A. Firat and B. Y. Jun : *On Symmetric bi-Derivations of BCI-algebra*, Applied Math. Sci. 5 (60) (2011), 2957-2966.
4. G. Mudiuddin and A. M. Al-roqi, *O*n t-derivations of BCI-algebras, Abstract and Applied Analysis, Volume 2012, Article ID 872784, (2012), 12 pages.
5. G. Mudiuddin and A. M. Al-roqi, *O*n (α, β)-derivations in BCI-algebras, Discrete Dynamics in Nature and Society, Volume 2012, Article ID 403209, (2012), 11 pages.
6. M. A. Ozturk, Y. Ceven and Y. B. Jun : *Generalized Derivations of BCI-algebras*, Honam Math. J. 31 (4) (2009), 601-609.
7. J. Zhan and Y. L. Liu, *On f-derivations of BCI-algebras,* Int. J. Math. Math. Sci. (11) (2005), 1675–1684.
8. Y. B. Jun and E. H. Roh, *On the BCI-G part of BCI-algebras,* Math. Japon. 38(4) (1993), 697–702.
9. D. J. Meng, *BCI-algebras and abelian groups,* Math. Japon. 32(5) (1987), 693–696.

10. Y. B. Jun, X. L. Xin and E. H. Roh, *The role of atoms in BCI-algebras,* Soochow J. Math. 30(4) (2004), 491–506.
11. M. Aslam and A.B. Thaheem : *A note on p-semisimple BCI-algebras*, Math. Japon. 36 (1991), 39-45.
12. S. A. Bhatti, M. A. Chaudhry and B. Ahmad, *On classification of BCI-algebras,* Math. Japon. 34(6) (1989), 865–876.

Abdullah M. Al-roqi received M.Sc. from University of Missouri, Kansas city, United States and Ph.D at School of Mathematics and Statistics, University of Birmingham , United Kingdom. His mathematical research areas are Algebras related to logic, Finite Group Theory, Soluble groups, Classification of finite simple groups and Representation Theory.

Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia.
e-mail: aalroqi@kau.edu.sa