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QUOTIENT MOMENTS OF THE ERLANG-TRUNCATED EXPONENTIAL DISTRIBUTION BASED ON RECORD VALUES AND A CHARACTERIZATION

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Abstract. Erlang-truncated exponential distribution is widely used in the field of queuing system and stochastic processes. This family of distribution include exponential distribution. In this paper we establish some exact expression and recurrence relations satisfied by the quotient moments and conditional quotient moments of the upper record values from the Erlang-truncated exponential distribution. Further a characterization of this distribution based on recurrence relations of quotient moments of record values is presented.

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1. Introduction

A random variable X is said to have Erlang-truncated exponential distribution if its probability density function (pdf) is of the form

$$f(x) = \beta(1 - e^{-\lambda})e^{-\beta x(1-e^{-\lambda})}, \ x \geq 0, \ \beta, \lambda > 0 \tag{1.1}$$

and the corresponding survival function is

$$\bar{F}(x) = e^{-\beta x(1-e^{-\lambda})}. \tag{1.2}$$

Therefore, in view of (1.1) and (1.2), we have

$$f(x) = \beta(1 - e^{-\lambda})\bar{F}(x). \tag{1.3}$$

The relation in (1.3) will be used to derive some recurrence relations for the quotient moments of record values from the Erlang-truncated exponential distribution. More details on this distribution can be found in Ei-Alosey [1].

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Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recording them: for example, Olympic records or world records in sport. Record values are used in reliability theory. Moreover, these statistics are closely connected with the occurrences times of some corresponding non homogeneous Poisson process used in shock models. The statistical study of record values started with Chandler [9], he formulated the theory of record values as a model for successive extremes in a sequence of independently and identically random variables. Feller [23] gave some examples of record values with respect to gambling problems. Resnick [17] discussed the asymptotic theory of records. Theory of record values and its distributional properties have been extensively studied in the literature, for example, see, Ahsanullah [10], Arnold *et al.* [2],[3], Nevzorov [21] and Kamps [19] for reviews on various developments in the area of records. We shall now consider the situations in which the record values (e.g. successive largest insurance claims in non-life insurance, highest water-levels or highest temperatures) themselves are viewed as "outliers" and hence the second or third largest values are of special interest. Insurance claims in some non life insurance can be used as one of the examples. Observing successive k largest values in a sequence, Dziubdziela and Kopocinski [22] proposed the following model of k record values, where k is some positive integer.

Let $\{X_n, n \geq 1\}$ be a sequence of identically independently distributed ($i.i.d$) random variables with *pdf* $f(x)$ and distribution function (*df*) $F(x)$. The $j-$th order statistics of a sample $(X_1, X_2, \ldots, X_n)$ is denoted by $X_{j:n}$. For a fix $k \geq 1$ we define the sequence $\{U_n^{(k)}, n \geq 1\}$ of k upper record times of $\{X_n, n \geq 1\}$ as follows

$$U_1^{(k)} = 1,$$
$$U_{n+1}^{(k)} = min\{j > U_n^{(k)} : X_j : j+k+1 > X_{U_n^{(k)}:U_n^{(k)}+k-1}\}.$$

The sequence $\{Y_n^{(k)}, n \geq 1\}$ with $Y_n^{(k)} = X_{U_n^{(k)}:U_n^{(k)}+k-1}, n = 1, 2, \ldots$ are called the sequences of k upper record values of $\{X_n, n \geq 1\}$.

For $k = 1$ and $n = 1, 2, \ldots$ we write $U_1^{(1)} = U_n$. Then $\{U_n, n \geq 1\}$ is the sequence of record times of $\{X_n, n \geq 1\}$. The sequence $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$ is called the sequence of k upper record values of $\{X_n, n \geq 1\}$. For convenience, we shall also take $Y_0^{(k)} = 0$. Note that $k = 1$ we have $Y_n^{(1)} = X_{U_n}, n \geq 1$, which are record value of $\{X_n, n \geq 1\}$. Moreover $Y_1^{(k)} = min\{X_1, X_2, \ldots, X_k = X_{1:k}\}$.

Let $\{X_n^{(k)}, n \geq 1\}$ be the sequence of k upper record values then the *pdf* of $X_n^{(k)}$, $n \geq 1$ is given by

$$f_{X_n^{(k)}}(x) = \frac{k^n}{(n-1)!}[-ln(\bar{F}(x))]^{n-1}[\bar{F}(x)]^{k-1}f(x) \tag{1.4}$$

and the joint *pdf* of $X_m^{(k)}$ and $X_n^{(k)}$, $1 \leq m < n$, $n > 2$ is given by

$$f_{X_m^{(k)},X_n^{(k)}}(x,y) = \frac{k^n}{(m-1)!(n-m-1)!}[-ln(\bar{F}(x))]^{m-1}$$

$$\times[-ln\bar{F}(y) + ln\bar{F}(x)]^{n-m-1}[\bar{F}(y)]^{k-1}\frac{f(x)}{\bar{F}(x)}f(y),\ x < y. \tag{1.5}$$

where $\bar{F}(x) = 1 - F(x)$.

Recurrence relations for single and product moments of record values from Weibull, Pareto, generalized Pareto, Burr, exponential and Gumble distribution are derived by Pawalas and Szynal [14], [15] and [16]. Kumar [4], Kumar and Khan [6] are established recurrence relations for moments of record values from exponentiated log-logistic and generalized beta II distributions respectively. And similar results for this paper have been done by Lee and Chang [11], [13] and [13], Chang [18] and Kumar [5] for exponential distribution, Pareto distribution, power function distribution, Weibull distribution and generalized Pareto distribution respectively. Kamps [20] investigated the importance of recurrence relations of order statistics in characterization.

In this paper, we established some explicit expressions and recurrence relations satisfied by the quotient moments and conditional quotient moments of the upper record values from the Erlang-truncated exponential distribution. A characterization of this distribution based on recurrence relations of quotient moments of record values.

2. Relations for the quotient moment

Theorem 2.1. *For the Erlang-truncated exponential distribution as given in (1.1) and* $1 \leq m \leq n-2,\ k = 1,2,\dots,\ s = 1,2,\dots$

$$E\Big(\frac{(X_{U(m)}^{(k)})^r}{(X_{U(n)}^{(k)})^{s+1}}\Big) = \frac{1}{(m-1)!(n-m-1)!}\sum_{u=0}^{n-m-1}\sum_{v=0}^{n-m-s-u-2}(-1)^u$$

$$\times\left(\begin{array}{c} n-m-1 \\ u \end{array}\right)\frac{\Gamma(n-m-u-s-1)\Gamma(r+u+v+m)}{v![\beta k(1-e^{-\lambda})]^{r-s-1}}. \tag{2.1}$$

Proof. From (1.5), we have

$$E\Big(\frac{(X_{U(m)}^{(k)})^r}{(X_{U(n)}^{(k)})^{s+1}}\Big) = \frac{k^n}{(m-1)!(n-m-1)!}$$

$$\times\int_0^\infty x^r[-ln(\bar{F}(x))]^{m-1}\frac{f(x)}{[\bar{F}(x)]}G(x)dx, \tag{2.2}$$

where

$$G(x) = \int_x^\infty y^{-(s+1)}[-ln(\bar{F}(y)) + ln(\bar{F}(x))]^{n-m-1}[\bar{F}(x)]^{k-1}f(y)dy. \tag{2.3}$$

On using the equations (1.2) and (1.3) in equation (2.3), we get

$$G(x) = [\beta(1-e^{-\lambda})]^{n-m} \sum_{v=0}^{n-m-1} (-1)^u \begin{pmatrix} n-m-1 \\ u \end{pmatrix} x^u$$

$$\times \int_x^\infty y^{n-m-u-s-2} e^{-\beta k(1-e^{-\lambda})y} dy$$

$$= [\beta(1-e^{-\lambda})]^{n-m} \sum_{v=0}^{n-m-1} \sum_{v=0}^{n-m-s-u-2} (-1)^u \begin{pmatrix} n-m-1 \\ u \end{pmatrix}$$

$$\times \frac{x^{u+v} e^{\beta k(1-e^{-\lambda})x} \Gamma(n-m-u-s-1)}{v![\beta k(1-e^{-\lambda})]^{n-m-s-u-v-1}}$$

(Gradshteyn and Ryzhik, [7], p-346). Upon substituting this expression for $G(x)$ in (2.2) and then integrating the resulting expression, we establish the result given in (2.1). □

Theorem 2.2. *For the Erlang-truncated exponential distribution as given in (1.1) and* $1 \leq m \leq n-2,\ k = 1, 2, \ldots$

$$E\Big(\frac{(X_{U(m)}^{(k)})^{r+1}}{(X_{U(n)}^{(k)})^s}\Big) = \frac{1}{(m-1)!(n-m-1)!} \sum_{u=0}^{n-m-1} \sum_{v=0}^{n-m-s-u-2} (-1)^u$$

$$\times \begin{pmatrix} n-m-1 \\ u \end{pmatrix} \frac{\Gamma(n-m-u-s)\Gamma(r+u+v+m+1)}{v![\beta k(1-e^{-\lambda})]^{r-s+v+1}}. \tag{2.4}$$

Proof. Proof can be established on line of Theorem 2.1. □

Remark 2.1. Setting $k = 1$ in (2.1) and (2.4) we deduce the explicit expression for the quotient moments of record values from the Erlang-truncated exponential distribution.

Making use of (1.3), we can derive recurrence relations for the quotient moments of k upper record values.

Theorem 2.3. *For* $1 \leq m \leq n-2,\ k = 1, 2, \ldots,\ r = 0, 1, 2, \ldots,$ *and* $s = 1, 2, \ldots$

$$E\Big(\frac{(X_{U(m)}^{(k)})^r}{(X_{U(n)}^{(k)})^{s+1}}\Big) = E\Big(\frac{(X_{U(m)}^{(k)})^r}{(X_{U(n-1)}^{(k)})^{s+1}}\Big) - \frac{s+1}{\beta k(1-e^{-\lambda})} E\Big(\frac{(X_{U(m)}^{(k)})^r}{(X_{U(n)}^{(k)})^{s+2}}\Big). \tag{2.5}$$

Proof. From equation (1.5) $1 \leq m \leq n-1,\ r = 0, 1, 2, \ldots,$ and $s = 1, 2, \ldots$

$$E\Big(\frac{(X_{U(m)}^{(k)})^r}{(X_{U(n)}^{(k)})^{s+1}}\Big) = \frac{k^n}{(m-1)!(n-m-1)!}$$

$$\times \int_0^\infty x^r [-ln(\bar{F}(x))]^{m-1} \frac{f(x)}{[\bar{F}(x)]} I_1(x) dx, \tag{2.6}$$

where

$$I_1(x) = \int_x^{\infty} y^{-(s+1)}[-ln(\bar{F}(y)) + ln(\bar{F}(x))]^{n-m-1}[\bar{F}(x)]^{k-1} f(y)dy.$$

Integrating $I_1(x)$ by parts treating $[\bar{F}(x)]^{k-1}f(y)$ for integration and the rest of the integrand for differentiation, and substituting the resulting expression in (2.6), we get

$$E\Big(\frac{(X_{U(m)}^{(k)})^r}{(X_{U(n)}^{(k)})^{s+1}}\Big) - E\Big(\frac{(X_{U(m)}^{(k)})^r}{(X_{U(n-1)}^{(k)})^{s+1}}\Big) = -\frac{(s+1)k^n}{k(m-1)!(n-m-1)!}\int_0^{\infty}\int_x^{\infty}\frac{x^r}{y^{s+2}}$$
$$\times[-ln(\bar{F}(x))]^{m-1}[-ln(\bar{F}(y)) + ln(\bar{F}(x))]^{n-m-1}[\bar{F}(y)]^k\frac{f(x)}{\bar{F}(x)}dydx$$

the constant of integration vanishes since the integral in $I_1(x)$ is a definite integral. On using the relation (1.3), we obtain

$$E\Big(\frac{(X_{U(m)}^{(k)})^r}{(X_{U(n)}^{(k)})^{s+1}}\Big) - E\Big(\frac{(X_{U(m)}^{(k)})^r}{(X_{U(n-1)}^{(k)})^{s+1}}\Big) = -\frac{(s+1)k^n}{[\beta k(1-e^{-\lambda})](m-1)!(n-m-1)!}$$
$$\times\int_0^{\infty}\int_x^{\infty}\frac{x^r}{y^{s+1}}[-ln(\bar{F}(x))]^{m-1}[-ln(\bar{F}(y)) + ln(\bar{F}(x))]^{n-m-1}$$
$$\times[\bar{F}(y)]^{k-1}\frac{f(x)}{\bar{F}(x)}f(y)dydx$$

and hence the result given in (2.5). □

Theorem 2.4. *For $1 \le m \le n-2,\ r, s =, 1, 2, \ldots,$*

$$E\Big(\frac{(X_{U(m)}^{(k)})^{r+1}}{(X_{U(n)}^{(k)})^{s}}\Big) = E\Big(\frac{(X_{U(m)}^{(k)})^{r+1}}{(X_{U(n-1)}^{(k)})^{s}}\Big) - \frac{s}{\beta k(1-e^{-\lambda})}E\Big(\frac{(X_{U(m)}^{(k)})^{r+1}}{(X_{U(n)}^{(k)})^{s+1}}\Big). \quad (2.7)$$

Proof. Proof follows on the line of Theorem 2.3. □

Corollary 2.5. *For $m \ge 1$, $r = 0, 1, 2, \ldots$ and $s = 1, 2, \ldots$*

$$E\Big(\frac{(X_{U(m)}^{(k)})^{r}}{(X_{U(n)}^{(k)})^{s+1}}\Big) = E[(X_{U(m)}^{(k)})^{r-s-1}] - \frac{s+1}{\beta k(1-e^{-\lambda})}E\Big(\frac{(X_{U(m)}^{(k)})^{r}}{(X_{U(n)}^{(k)})^{s+2}}\Big). \quad (2.8)$$

Proof. Upon substituting $n = m+1$ in (2.5) and simplifying, then we get the result given in (2.8). □

Corollary 2.6. *For $m \ge 1$, $r, s = 0, 1, 2, \ldots,$*

$$E\Big(\frac{(X_{U(m)}^{(k)})^{r+1}}{(X_{U(n)}^{(k)})^{s}}\Big) = E[(X_{U(m)}^{(k)})^{r-s+1}] - \frac{s}{\beta k(1-e^{-\lambda})}E\Big(\frac{(X_{U(m)}^{(k)})^{r+1}}{(X_{U(n)}^{(k)})^{s+1}}\Big). \quad (2.9)$$

Proof. Upon substituting $n = m+1$ in (2.7) and simplifying, then we get the result given in (2.9). □

Remark 2.2. Setting $k = 1$ in (2.5) and (2.7) we deduce the recurrence relation for the quotient moments of record values from the Erlang-truncated exponential distribution.

3. Relation of quotient conditional expectation

Let $\{X_n^{(k)}, n \geq 1\}$ be a sequence of *i.i.d* continuous random variables with *df* $F(x)$ and *pdf* $f(x)$. Let $X_{U(m)}$ and $X_{U(n)}$ be the $m-$th and $n-$th upper record values, then the conditional *pdf* of $X_{U(m)}$ given $X_{U(n)} = y$, $1 \leq m < n$ in view of (1.4) and (1.5), is

$$f_{m|n}(x|y) = \frac{(n-1)!}{(m-1)!(n-m-1)!}[-ln(\bar{F}(x))]^{m-1}[-ln\bar{F}(y) + ln\bar{F}(x)]^{n-m-1}$$

$$\times \frac{f(x)}{\bar{F}(x)[-ln(\bar{F}(y))]^{n-1}}, \ x < y \tag{3.1}$$

and the conditional *pdf* of $X_{U(n)}$ give $X_{U(m)} = x$, $1 \leq m < n$ is

$$f_{n|m}(y|x) = \frac{1}{(n-m-1)!}[-ln\bar{F}(y) + ln\bar{F}(x)]^{n-m-1}\frac{f(y)}{\bar{F}(x)}, \ x > y. \tag{3.2}$$

Theorem 3.1. *For the Erlang-truncated exponential distribution as given in (1.1) and* $1 \leq m \leq n-2, \ k = 1, 2, \ldots,$

$$E\Big(\frac{X_{U(m)}^{(r)}}{X_{U(n)}^{(s)}}|X_{U(m)} = x\Big) = \frac{1}{(n-m-1)!}\sum_{u=0}^{n-m-1}\sum_{v=0}^{n-m-s-u-1}(-1)^u$$

$$\times \begin{pmatrix} n-m-1 \\ u \end{pmatrix} \frac{x^{r+u+v}\Gamma(n-m-s-u)}{v![\beta(1-e^{-\lambda})]^{-(s+u+v)}}. \tag{3.3}$$

Proof. From (3.2), we have

$$E\Big(\frac{X_{U(m)}^{(r)}}{X_{U(n)}^{(s)}}|X_{U(m)} = x\Big) = \frac{x^r}{\bar{F}(x)(n-m-1)!}$$

$$\times \int_x^{\infty} y^{-s}[-ln\bar{F}(y) + ln\bar{F}(x)]^{n-m-1}f(y)dy.$$

On using the (1.2) and (1.3), we have

$$E\Big(\frac{X_{U(m)}^{(r)}}{X_{U(n)}^{(s)}}|X_{U(m)} = x\Big) = \frac{x^r[\beta(1-e^{-\lambda})]^{n-m}}{\bar{F}(x)(n-m-1)!}\sum_{u=0}^{n-m-1}(-1)^u \begin{pmatrix} n-m-1 \\ u \end{pmatrix} x^u$$

$$\times \int_x^{\infty} y^{n-m-s-u-1}e^{-\beta(1-e^{-\lambda})y}dy \tag{3.4}$$

integrating the equation (3.4), we established the result given in (3.3). $\square$

Theorem 3.2. *For the Erlang-truncated exponential distribution as given in (1.1) and* $1 \le m \le n-2,\ k = 1, 2, \ldots,$

$$E\Big(\frac{X_{U(m)}^{(r)}}{X_{U(n)}^{(s)}}|X_{U(n)} = y\Big) = \frac{[\beta(1-e^{-\lambda})]^{n-2}(n-1)!}{(m-1)!(n-m-1)!}\sum_{u=0}^{n-m-1}(-1)^u$$

$$\times \binom{n-m-1}{u}\frac{y^{n-s+r-2}}{(r+m+p)}. \tag{3.5}$$

Proof. Proof follows on the line of Theorem 3.1. □

Making use of (1.3), we can derive recurrence relations for the quotient conditional moments of upper record values.

Theorem 3.3. *For* $1 \le m \le n-2,\ r = 0, 1, 2, \ldots$ *and* $s = 1, 2, \ldots$

$$\Big(\frac{s+1}{\beta(1-e^{-\lambda})}\Big)E\Big(\frac{X_{U(m)}^{(r)}}{X_{U(n)}^{(s)}}|X_{U(m)} = x\Big)$$

$$= E\Big(\frac{X_{U(m)}^{(r)}}{X_{U(n-1)}^{(s+1)}}|X_{U(m)} = x\Big) - E\Big(\frac{X_{U(m)}^{(r)}}{X_{U(n)}^{(s+1)}}|X_{U(m)} = x\Big). \tag{3.6}$$

Proof. From equation (3.1), we have

$$E\Big(\frac{X_{U(m)}^{(r)}}{X_{U(n)}^{(s)}}|X_{U(m)} = x\Big) = \frac{x^r}{\bar{F}(x)(n-m-1)!}I_2(x) \tag{3.7}$$

where

$$I_2(x) = \int_x^{\infty} y^{-s}[-ln(\bar{F}(y)) + ln(\bar{F}(x))]^{n-m-1}f(y)dy.$$

$$= \beta(1-e^{-\lambda})\int_x^{\infty} y^{-s}[-ln(\bar{F}(y)) + ln(\bar{F}(x))]^{n-m-1}\bar{F}(y)dy$$

Integrating $I_2(x)$ by parts treating y^{-s} for integration and the rest of the integrand for differentiation, and substituting the resulting expression in (3.7), we get the result given in (3.6). □

Theorem 3.4. *For* $1 \le m \le n-2,\ r, s = 0, 1, 2, \ldots$

$$\Big(\frac{r+1}{\beta(1-e^{-\lambda})}\Big)E\Big(\frac{X_{U(m)}^{(r)}}{X_{U(m+1)}^{(s)}}|X_{U(n)} = y\Big)$$

$$= E\Big(\frac{X_{U(m)}^{(r+1)}}{X_{U(n-1)}^{(s)}}|X_{U(n)} = y\Big) - E\Big(\frac{X_{U(m-1)}^{(r+1)}}{X_{U(m-1)}^{(s)}}|X_{U(n)} = y\Big). \tag{3.8}$$

Proof. Proof follows on the line of Theorem 3.3. □

Corollary 3.5. *For* $m \geq 1,\ r = 0, 1, 2, \ldots$ *and* $s = 1, 2, \ldots$

$$\Big(\frac{s+1}{\beta(1-e^{-\lambda})}\Big) E\Big(\frac{X^{(r)}_{U(m)}}{X^{(s)}_{U(m+1)}} | X_{U(m)} = x\Big)$$

$$= E[X^{(r-s-1)}_{U(m)} | X_{U(m)}] - E\Big(\frac{X^{(r)}_{U(m)}}{X^{(s+1)}_{U(m+1)}} | X_{U(m)} = x\Big). \tag{3.9}$$

Proof. Upon substituting $n = m + 1$ in (3.6) and simplifying, then we get the result given in (3.9). □

Corollary 3.6. *For* $1 \leq m \leq n - 2,\ r, s = 0, 1, 2, \ldots$

$$\Big(\frac{r+1}{\beta(1-e^{-\lambda})}\Big) E\Big(\frac{X^{(r)}_{U(m)}}{X^{(s)}_{U(m+1)}} | X_{U(m+1)} = y\Big)$$

$$= E[X^{r-s+1}_{U(m)} | X_{U(m+1)} = y] - E\Big(\frac{X^{(r+1)}_{U(m-1)}}{X^{(s)}_{U(m)}} | X_{U(m+1)} = y\Big). \tag{3.10}$$

Proof. Upon substituting $n = m + 1$ in (3.8) and simplifying, then we get the result given in (3.10). □

4. Characterization

Theorem 4.1. *Let* $k \geq 1$ *is a fix positive integer,* r *be a non- negative integer and* y *be an absolutely continuous random variable with pdf* $f(y)$ *and df* $F(y)$ *on the support* $(0, \infty)$*, then*

$$E\Big(\frac{(X^{(k)}_{U(m)})^r}{(X^{(k)}_{U(n)})^{s+1}}\Big) = E\Big(\frac{(X^{(k)}_{U(m)})^r}{(X^{(k)}_{U(n-1)})^{s+1}}\Big) - \frac{s+1}{\beta k(1-e^{-\lambda})} E\Big(\frac{(X^{(k)}_{U(m)})^r}{(X^{(k)}_{U(n)})^{s+2}}\Big). \tag{4.1}$$

if and only if

$$\bar{F}(y) = e^{-\beta y(1-e^{-\lambda})},\ y \geq 0,\ \beta, \lambda > 0.$$

Proof. The necessary part follows immediately from equation (2.5). On the other hand if the recurrence relation in equation (4.1) is satisfied, then on using equation (1.5), we have

$$\frac{k^n}{(m-1)!(n-m-1)!} \int_0^\infty \int_x^\infty \frac{x^r}{y^{s+1}} [-ln(\bar{F}(x))]^{m-1} [-ln(\bar{F}(y)) + ln(\bar{F}(x))]^{n-m-1}$$
$$\times [\bar{F}(y)]^{k-1} \frac{f(x)}{\bar{F}(x)} f(y) dy dx$$
$$= \frac{k^n(n-m-1)}{k(m-1)!(n-m-1)!} \int_0^\infty \int_x^\infty \frac{x^r}{y^{s+1}} [-ln(\bar{F}(x))]^{m-1} [-ln(\bar{F}(y)) + ln(\bar{F}(x))]^{n-m-2}$$
$$\times [\bar{F}(y)]^{k-1} \frac{f(x)}{\bar{F}(x)} f(y) dy dx$$

$$-\frac{(s+1)k^n}{[\beta k(1-e^{-\lambda})](m-1)!(n-m-1)!}\int_0^\infty\int_x^\infty \frac{x^r}{y^{s+2}}[-ln(\bar{F}(x))]^{m-1}$$
$$\times[-ln(\bar{F}(y))+ln(\bar{F}(x))]^{n-m-1}[\bar{F}(y)]^{k-1}\frac{f(x)}{\bar{F}(x)}f(y)dydx. \tag{4.2}$$

Integrating the first integral on the right hand side of equation (4.2) by parts and simplifying the resulting expression, we find that

$$\frac{(s+1)k^n}{k(m-1)!(n-m-1)!}\int_0^\infty\int_x^\infty \frac{x^r}{y^{s+2}}[-ln(\bar{F}(x))]^{m-1}[-ln(\bar{F}(y))+ln(\bar{F}(x))]^{n-m-1}$$
$$\times[\bar{F}(y)]^{k-1}\frac{f(x)}{\bar{F}(x)}\Big\{\bar{F}(y)-\frac{1}{\beta(1-e^{-\lambda})}f(y)\Big\}dydx=0. \tag{4.3}$$

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, [8]) to equation (4.3), we get

$$\frac{f(y)}{\bar{F}(y)}=\beta(1-e^{-\lambda})$$

which proves that

$$\bar{F}(y)=e^{-\beta y(1-e^{-\lambda})},\ y\geq 0,\ \beta,\lambda>0.$$

□

5. Conclusion

In this study some exact expressions and recurrence relations for the quotient moments and conditional quotient moments of record values from the Erlang-truncated exponential distribution have been established. Further, recurrence relation of the quotient moments of record values has been utilized to obtain a characterization of the Erlang-truncated exponential distribution.

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