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PRACTICAL OBSERVER FOR IMPULSIVE SYSTEMS

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Abstract. In this paper, we deal with the problem of practical observer design and the practical stabilization for a class of perturbed impulsive systems. We show that, under the classical conditions of uniform complete controllability and uniform complete observability of the nominal system without impulsive effects, it is possible to design an observer controller for a class of perturbed linear impulsive system when the origin is not an equilibrium point.

1. Introduction

Impulsive systems describing evolution processes exhibit impulsive dynamical behaviors due to abrupt changes at certain instants during the continuous dynamical processes. Systems of such type have increasingly been at the center of attention in recent years due to their wide range of applications in practice, for examples in population dynamics in relation to impulsive vaccination [11], population ecology [10], drug distribution in the human body [5], management of renewable resources. . ..

Unlike the stability problem that has been extensively studied in the literature ([7], [8], [9]), the observer design and stabilization problems for impulsive systems has attracted less attention and not as many works are available in this area ([1], [2], [3], [12]).

One attempt to implement the feedback law is to construct an observer for estimating the state and then to feedback the estimated state using the separation principle. Hence the construction of observers is useful and essential for control purposes.

Taking into account that mathematical models of complex systems usually contain model errors and that exogenous perturbations are ubiquitous it is natural to consider systems with time varying perturbations and look the state estimation and the stabilization problems.

In this paper we derive a practical impulsive observer which yields performance equivalent to the Kalman's one (see [4]). Indeed, we propose a Kalman

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type observer for a class of linear time-varying impulsive systems perturbed by a nonlinearity, in the case of continuous-time observation function. We prove that it is possible, under the assumptions of uniform complete controllability, uniform complete observability of the associated continuous system and some others assumptions made in impulsive matrices D_k, to construct a practical observer.

This kind of observer is very useful for designing practical controller of such systems since, in many cases, controlling a system to an idealized point is either expensive or impossible in the presence of disruptions.

An outline of this paper is as follows. Some definitions and lemma which are useful for stating the main results, are given in Section 2. Section 3, focuses on the problem of practical observer design for linear time-varying impulsive perturbed systems. Section 4, has the objective of practical stabilization for the same class. Finally, in Section 5, a separation principle is established.

Notation. Throughout this paper, I denotes the identity matrix. The notation $P > 0$ is that the matrix P is positive definite, P^T is its transpose matrix, $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the smallest eigenvalue of the symmetric matrix and the largest one respectively. For $x \in \mathbb{R}^n$, $X \in \mathbb{R}^{n\times n}$, let $\| x \|$ be the Euclidean vector norm and $\| X \|= \sqrt{\lambda_{\max}(X^T X)}$ the induced matrix norm. $B_\rho = \{x \in \mathbb{R}^n / \| x \|< \rho\}$. We denote by

- $\mathcal{C}[\mathbb{R}^+, \mathbb{R}]$ the set of functions $\psi : \mathbb{R}^+ \to \mathbb{R}$ which are continuous;
- $\mathcal{PC}^1[\mathbb{R}^+, \mathbb{R}]$ the set of functions $\psi : \mathbb{R}^+ \to \mathbb{R}$ which are piecewise continuous differentiable.

2. Preliminaries

Consider the following time-varying impulsive system:

$$\begin{aligned} \dot{x} &= F(t, x), \quad t \neq t_k \\ \Delta x &= I_k(x), \\ y &= h(t, x), \end{aligned} \tag{2.1}$$

where

- $F, h : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous.
- $I_k : \mathbb{R}^n \to \mathbb{R}^n$ is continuous.
- $t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k\to+\infty} t_k = +\infty$;
- $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$, $x(t_k^-) = \lim_{h\to 0^+} x(t_k - h)$, and $x(t_k^-) = x(t_k)$.

Definition 1. (1) The system (2.1) is said to be uniformly practically exponentially stable (UPES) with respect to B_ρ with region of attraction Ω, if there exists a constant $\lambda > 0$, such that for all $t_0 \geq 0$, $x_0 \in \Omega$, there exists $K \geq 0$ such that

$$\| x(t) \|\leq \rho + K \| x_0 \| e^{-\lambda(t-t_0)}.$$

(2) The system (2.1) is said to be globally uniformly practically exponentially stable (GUPES) with respect to B_ρ if it is (UEPS) with respect to B_ρ and with $\mathbb{R}^n$ as the region of attraction.

Lemma 1 ([6]). *Assume that*

(1) $v \in \mathcal{PC}^1[\mathbb{R}^+, \mathbb{R}]$ *and* $v(t)$ *is left continuous at* t_k, $k = 1, 2, \ldots$.

(2) *for* $k = 1, 2, \ldots$, $t \geq t_0$,

$$D^+ v(t) \leq a(t)v(t) + b(t), \qquad v(t_k^+) \leq c_k v(t_k) + d_k,$$

where $a, b \in \mathcal{C}[\mathbb{R}^+, \mathbb{R}]$, $c_k > 0$, *and* d_k *are constants.*

Then

$$v(t) \leq v(t_0)\Big(\prod_{t_0 < t_k < t} c_k\Big) e^{\int_{t_0}^t a(s)ds} + \sum_{t_0 < t_k < t} \Big(\prod_{t_0 < t_j < t} c_j\Big) e^{\int_{t_k}^t a(s)ds} d_k$$
$$+ \int_{t_0}^t \Big(\prod_{s < t_k < t} c_k\Big) e^{\int_s^t a(u)du} b(s)ds.$$

3. State estimation

In this subsection, we are interested in designing a practical exponential observer for a certain class of perturbed impulsive systems. We use the second Lyapunov method to prove the practical exponential stability of the estimation error. Let us consider the linear impulsive system described by

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A(t)x(t) + B(t)u(t), \quad t \neq t_k, \\ \Delta x(t_k) & = & D_k x(t_k^-), \\ y(t) & = & C(t)x(t), \end{array} \right. \tag{3.1}$$

where

- $t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k\to+\infty} t_k = +\infty$;
- $A(t) = (a_{ij}(t)) \in \mathbb{R}^{n\times n}$, $B(t) = (b_{ij}(t)) \in \mathbb{R}^{n\times p}$, $C(t) = (c_{ij}(t)) \in \mathbb{R}^{q\times n}$, where a_{ij}, b_{ij}, c_{ij} are uniformly bounded and piecewise continuous functions from $\mathbb{R}^+$ to $\mathbb{R}$, with discontinuities of the first kind only at $t = t_k$, $k = 1, 2, \ldots$;
- $u(t) \in \mathbb{R}^p$ is the input vector. As usual, the admissible control input are limited to piecewise continuous functions;
- $y(t) \in \mathbb{R}^q$ is the output vector;
- for all $k \geq 0$, $D_k \in \mathbb{R}^{n\times n}$ are known constant matrices;
- $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$, $x(t_k^-) = \lim_{h\to 0^+} x(t_k - h)$, and $x(t_k^-) = x(t_k)$, which implies that the solution of (2.1) is left continuous at t_k.

Under some perturbation which can be caused by errors in measuring or modeling, the perturbed impulsive system has the form:

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A(t)x(t) + g(t, x(t)) + B(t)u(t), \quad t \neq t_k, \\ \Delta x(t_k) & = & D_k x(t_k^-), \\ y(t) & = & C(t)x(t), \end{array} \right. \tag{3.2}$$

where the perturbation function $g(t,x)$ is piecewise continuous in t with discontinuities of the first kind at $t = t_k$ such that $\lim_{(t,y)\to(t_k,x)} g(t,y)$ exists; and globally Lipschitz with respect to x. In the general case, we suppose that $g(t,0) \neq 0,\ \forall t \geq 0$. This means that the origin is not necessarily an equilibrium point.

In the remainder of this subsection, the following assumptions are introduced to design the proposed observer.

$(\mathcal{A}_1)$ System (3.1) is uniformly completely observable.
$(\mathcal{A}_2)$ Suppose that $\| I + D_k \| \leq d_k < 1$ and $\sum_{k\geq 0} d_k < +\infty$.
$(\mathcal{A}_3)$ Suppose that the perturbation term $g(t,x)$ satisfies the following estimate

$$\| g(t,x) \| \leq \lambda(t), \quad \forall t \geq 0,\ \forall x \in \mathbb{R}^n,$$

where $\lambda : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function on $[0,+\infty[$.

For system (3.2) we establish the following result.

Theorem 1. *Let assumptions* $(\mathcal{A}_1)-(\mathcal{A}_2)$ *and* $(\mathcal{A}_3)$ *hold. In addition, assume that* $\lambda(t)$ *in* $(\mathcal{A}_3)$ *is integrable on* $[0,+\infty[$. *Then there exists a practical observer for the perturbed impulsive system* (3.2), *of the form*

$$\begin{aligned} &\dot{\hat{x}}(t) = A(t)\hat{x}(t) + g(t,\hat{x}(t)) + B(t)u(t) - L(t)(C(t)\hat{x}(t) - y(t)),\ t \neq t_k, \\ &\Delta\hat{x}(t_k) = D_k\hat{x}(t_k), \end{aligned} \tag{3.3}$$

with $L(t)$ *given by*

$$\begin{aligned} &\dot{M}(t) = A(t)M(t) + M(t)A^T(t) - M(t)C^T(t)W^{-1}C(t)M(t) + P,\ t \neq t_k, \\ &M(0) = M_0 = M_0^T > 0,\ W = W^T > 0, \\ &L(t) = M(t)C^T(t)W^{-1},\ \textit{with}\ P = P^T > 0. \end{aligned} \tag{3.4}$$

Proof. Let $e(t) = \hat{x}(t) - x(t)$ denote the estimation error. Then its dynamic is described by

$$\begin{aligned} &\dot{e}(t) = \big(A(t) - L(t)C(t)\big)e(t) + \Big(g(t,\hat{x}) - g(t,x)\Big), \quad t \neq t_k, \\ &\Delta e(t_k) = D_k e(t_k^-). \end{aligned} \tag{3.5}$$

Let us consider the following Lyapunov function candidate

$$V(t,e) = e^T(t)M^{-1}(t)e(t).$$

On one hand, the Dini derivative of V along the trajectories of (3.5) for $t \neq t_k$ is given by

$$\begin{aligned} D^+V_{(3.5)}(t,e) &= e(t)^T\big(-M^{-1}(t)PM^{-1}(t) - C^T(t)W^{-1}C(t)\big)e(t) \\ &\quad + 2\Big(g(t,\hat{x}) - g(t,x)\Big)^T M^{-1}e(t) \\ &\leq -e(t)^T M^{-1}(t)PM^{-1}(t)e(t) + 4\beta\lambda(t)e(t). \end{aligned}$$

Recall that there exist positive constants α, β and t_0 such that for all $t \geq t_0$: $0 < \alpha I \leq M^{-1}(t) \leq \beta I$ basically from the condition of uniform complete observability (see [4]). Let $\lambda_{\min}(P) > 0$ be the smallest eigenvalue of the symmetric matrix P. Then for all $t \geq t_0$, we have

$$-M^{-1}(t)PM^{-1}(t) \leq -\lambda_{\min}(P)\alpha M^{-1}(t).$$

This implies that

$$D^+V_{(3.5)}(t) \leq -\lambda_{\min}(P)\alpha V(t) + \frac{4\beta\lambda(t)}{\sqrt{\alpha}}\sqrt{V(t)}.$$

Let $w(t) = \sqrt{V(t)}$. We use the fact that

$$D^+w_{(3.5)}(t) = \frac{D^+V_{(3.5)}(t)}{2\sqrt{V(t)}}$$

if $V(t) \neq 0$ to obtain the inequality

$$D^+w_{(3.5)}(t) \leq -\Big(\frac{\lambda_{\min}(P)\alpha}{2}\Big)w(t) + \frac{2\beta}{\sqrt{\alpha}}\lambda(t)$$

which holds for $V(t) = 0$ ($D^+w_{(3.5)}(t) \leq \frac{2\beta}{\sqrt{\alpha}}\lambda(t)$).
Now, for $t = t_k$ we have

$$\begin{aligned} V(t_k^+, e(t_k^+)) &= \big(e(t_k) + D_k e(t_k)\big)^T M^{-1}(t)\big(e(t_k) + D_k e(t_k)\big) \\ &\leq \| I + D_k \|^2 \frac{\beta}{\alpha} V(t_k, e(t_k)) \\ &\leq \frac{\beta d_k^2}{\alpha} V(t_k, e(t_k)). \end{aligned}$$

Then we obtain

$$w(t_k^+) \leq d_k\sqrt{\frac{\beta}{\alpha}}w(t_k).$$

Then by using the comparison lemma, we obtain

$$\begin{aligned} w(t) \leq{}& w(t_0)\Big(\prod_{t_0<t_k<t} d_k\sqrt{\frac{\beta}{\alpha}}\Big) \times e^{-\left(\frac{\lambda_{\min}(P)\alpha}{2}\right)(t-t_0)} \\ &+ \frac{2\beta}{\sqrt{\alpha}}\int_{t_0}^{t}\Big(\prod_{s<t_k<t} d_k\sqrt{\frac{\beta}{\alpha}}\Big)e^{-\left(\frac{\lambda_{\min}(P)\alpha}{2}\right)(t-s)}\lambda(s)ds. \end{aligned}$$

This gives the estimate

$$\begin{aligned} \| e(t) \| \leq{}& \| e(t_0) \| \sqrt{\frac{\beta}{\alpha}}\Big(\prod_{t_0<t_k<t} d_k\sqrt{\frac{\beta}{\alpha}}\Big) \times e^{-\left(\frac{\lambda_{\min}(P)\alpha}{2}\right)(t-t_0)} \\ &+ \frac{2\beta}{\alpha}\int_{t_0}^{t}\Big(\prod_{s<t_k<t} d_k\sqrt{\frac{\beta}{\alpha}}\Big)e^{-\left(\frac{\lambda_{\min}(P)\alpha}{2}\right)(t-s)}\lambda(s)ds. \end{aligned}$$

Then

$$\| e(t) \| \leq \| e(t_0) \| \sqrt{\frac{\beta}{\alpha}} \Big(\prod_{k=0}^{+\infty} d_k \sqrt{\frac{\beta}{\alpha}} \Big) \times e^{-\left(\frac{\lambda_{\min}(P)\alpha}{2}\right)(t-t_0)}$$
$$+ \frac{2\beta}{\alpha} \Big(\prod_{s<t_k<t} d_k \sqrt{\frac{\beta}{\alpha}} \Big) \int_{t_0}^{+\infty} \lambda(s) ds.$$

It follows that system (3.5) is uniformly practically exponentially stable with respect to the ball B_ρ with

$$\rho = \frac{2\bar{\lambda}\beta}{\alpha} \Big(\prod_{s<t_k<t} d_k \sqrt{\frac{\beta}{\alpha}} \Big),$$

where $\bar{\lambda} = \int_{t_0}^{+\infty} \lambda(s) ds$. □

Remark 1. Note that the convergence of the series $\sum_{k\geq 0} d_k$ implies the convergence of the infinite product $\prod_{k=0}^{+\infty} \big(d_k \sqrt{\frac{\beta}{\alpha}} \big)$.

3.1. Practical example

Let now consider an other mathematical model which may result from impulsive vaccination affectation and which is very useful to control diseases in ecological problems (see [11]). Such model is described by the following impulsive system

$$\text{(3.6)} \qquad \begin{cases} \dot{x}(t) &= A(t)x(t) + g(t, x(t)) + B(t)u(t), \quad t \neq t_k, \\ x(t_k^+) &= p_k x(t_k), \\ y(t) &= C(t)x(t), \end{cases}$$

where $\forall k \geq 0$, $p_k \in \mathbb{R}^+$.

Corollary 1. *Let assumptions* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_3)$ *hold. In addition, assume that* $\sum_{k\geq 0} p_k < +\infty$ *and* $\lambda(t)$ *in* $(\mathcal{A}_3)$ *is integrable on* $[0, +\infty[$. *Then there exists a practical observer for the impulsive system* (3.6), *of the form*

$$\text{(3.7)} \quad \dot{\hat{x}}(t) = A(t)\hat{x}(t) + g(t, \hat{x}(t)) + B(t)u(t) - L(t)(C(t)\hat{x}(t) - y(t)), \ t \neq t_k,$$
$$\hat{x}(t_k^+) = p_k \hat{x}(t_k),$$

where $L(t)$ *is given by* (3.4).

Proof. The proof is the same as for the above theorem until the following inequality: for $t \neq t_k$,

$$D^+ w(t) \leq -\Big(\frac{\lambda_{\min}(P)\alpha}{2} \Big) w(t) + \frac{2\beta}{\sqrt{\alpha}} \lambda(t).$$

Then for $t = t_k$, we have

$$V(t_k^+, e(t_k^+)) = V(t_k^+, p_k e(t_k) \leq \frac{p_k^2 \beta}{\alpha} V(t_k, e(t_k)).$$

Then we have

$$w(t_k^+) \leq p_k \sqrt{\frac{\beta}{\alpha}} w(t_k).$$

Then by using the comparison lemma, we obtain

$$w(t) \leq w(t_0) \Big(\prod_{t_0 < t_k < t} p_k \sqrt{\frac{\beta}{\alpha}} \Big) \times e^{-\left(\frac{\lambda_{\min}(P)\alpha}{2}\right)(t-t_0)}$$

$$+ \frac{2\beta}{\sqrt{\alpha}} \int_{t_0}^{t} \Big(\prod_{s < t_k < t} p_k \sqrt{\frac{\beta}{\alpha}} \Big) e^{-\left(\frac{\lambda_{\min}(P)\alpha}{2}\right)(t-s)} \lambda(s) ds,$$

which gives

$$\| e(t) \| \leq \| e(t_0) \| \sqrt{\frac{\beta}{\alpha}} \Big(\prod_{k=0}^{+\infty} p_k \sqrt{\frac{\beta}{\alpha}} \Big) \times e^{-\left(\frac{\lambda_{\min}(P)\alpha}{2}\right)(t-t_0)}$$

$$+ \frac{2\bar{\lambda}\beta}{\alpha} \Big(\prod_{s < t_k < t} p_k \sqrt{\frac{\beta}{\alpha}} \Big) \int_{t_0}^{+\infty} \lambda(s) ds,$$

where $\bar{\lambda} = \int_{t_0}^{+\infty} \lambda(s) ds.$ □

Remark 2. In real processes, as well as in predator-prey model with disease in the prey, we release infective prey population and predator population impulsively with constant amounts $p_k > 0$, for integrated pest management [10]. For this natural phenomena, we have the following particular model

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + g(t, x(t)) + B(t)u(t), \quad t \neq t_k, \\ x(t_k^+) &= x(t_k) + p_k, \\ y(t) &= C(t)x(t), \end{cases} \tag{3.8}$$

where $\forall k \geq 0$, $p_k \in \mathbb{R}^+$. For this system, if we consider an observer like previously, it is not difficult to verify that the observer error is reduced to a continuous dynamic.

4. Stabilization

We shall construct now a linear feedback law which makes solutions of the perturbed system (3.2) uniformly exponentially stable toward a new neighborhood of the origin. For this purpose, we need the following assumption:

($\mathcal{A}_4$) System (3.1) is uniformly completely controllable.

Theorem 2. *Let assumptions* ($\mathcal{A}_2$)-($\mathcal{A}_3$) *and* ($\mathcal{A}_4$) *hold. In addition, assume that* $\lambda(t)$ *in* ($\mathcal{A}_3$) *is integrable on* $[0, +\infty[$. *Then there exists a gain matrix* $K(t)$ *given by*

$$\dot{N}(t) = A(t)N(t) + N(t)A^T(t) - N(t)B^T(t)X^{-1}B(t)N(t) + Q, \; t \neq t_k, \tag{4.1}$$

$$N(0) = N_0 = N_0^T > 0, \; X = X^T > 0,$$

$$K(t) = N(t)B^T(t)X^{-1}, \quad \textit{with } Q = Q^T > 0,$$

such that the closed-loop system

$$\begin{aligned} \dot{x}(t) &= \big(A(t) - B(t)K(t)\big)x(t) + g(t, x(t)), \quad t \neq t_k, \\ \Delta x(t_k) &= D_k x(t_k), \end{aligned} \tag{4.2}$$

is globally uniformly practically exponentially stable.

Proof. Let us consider the following Lyapunov function

$$V(t, x) = x^T(t)N^{-1}(t)x(t).$$

On one hand, the Dini derivative of V along the trajectories of (4.2) for $t \neq t_k$ is given by

$$\begin{aligned} D^+V_{(4.2)}(t, x) = {} & x(t)^T\big(- N^{-1}(t)QN^{-1}(t) - B^T(t)X^{-1}B(t)\big)x(t) \\ & + 2g(t, x)^T N^{-1}x(t). \end{aligned}$$

We know that there exist positive constants $\alpha^{'}, \beta^{'}$ and t_0 such that for all $t \geq t_0$: $0 < \alpha^{'} I \leq N^{-1}(t) \leq \beta^{'} I$ (see [4]). Let $\lambda_{\min}(Q) > 0$ be the smallest eigenvalue of the symmetric matrix Q. Then for all $t \geq t_0$, we have

$$-N^{-1}(t)QN^{-1}(t) \leq -\lambda_{\min}(Q)\alpha^{'} N^{-1}(t).$$

This implies that

$$D^+V_{(4.2)}(t) \leq -\lambda_{\min}(Q)\alpha^{'} V(t) + \frac{2\beta^{'}\lambda(t)}{\sqrt{\alpha^{'}}}\sqrt{V(t)}.$$

Let $w(t) = \sqrt{V(t)}$. Then we have the following inequality, if $v(t) \neq 0$:

$$D^+w_{(4.2)}(t) \leq -\Big(\frac{\lambda_{\min}(Q)\alpha^{'}}{2}\Big)w(t) + \frac{\beta^{'}}{\sqrt{\alpha^{'}}}\lambda(t).$$

Now, for $t = t_k$ we have

$$\begin{aligned} V(t_k^+, x(t_k^+)) &= V(t_k^+, x(t_k) + \Delta x(t_k)) \\ &\leq \| I + D_k \|^2 \beta^{'} \| x(t_k) \|^2 \\ &\leq \frac{\beta^{'} d_k^2}{\alpha^{'}} V(t_k, x(t_k)). \end{aligned}$$

Then we obtain

$$w(t_k^+) \leq d_k\sqrt{\frac{\beta^{'}}{\alpha^{'}}}w(t_k).$$

Then by using the comparison lemma, we obtain

$$\begin{aligned} w(t) \leq {} & w(t_0)\Big(\prod_{t_0<t_k<t} d_k\sqrt{\frac{\beta^{'}}{\alpha^{'}}}\Big) \times e^{-\Big(\frac{\lambda_{\min}(Q)\alpha^{'}}{2}\Big)(t-t_0)} \\ & + \frac{\beta^{'}}{\sqrt{\alpha^{'}}}\int_{t_0}^{t}\Big(\prod_{s<t_k<t} d_k\sqrt{\frac{\beta^{'}}{\alpha^{'}}}\Big)e^{-\Big(\frac{\lambda_{\min}(Q)\alpha^{'}}{2}\Big)(t-s)}\lambda(s)ds. \end{aligned}$$

Then

$$\| x(t) \| \leq \| x(t_0) \| \sqrt{\frac{\beta'}{\alpha'}} \Big(\prod_{k=0}^{+\infty} d_k \sqrt{\frac{\beta'}{\alpha'}} \Big) \times e^{-\left(\frac{\lambda_{\min}(Q)\alpha'}{2}\right)(t-t_0)}$$
$$+ \frac{\beta'}{\alpha'} \Big(\prod_{k=0}^{+\infty} d_k \sqrt{\frac{\beta'}{\alpha'}} \Big) \int_{t_0}^{+\infty} \lambda(s) ds,$$

and we can conclude that the closed-loop system is uniformly practically exponentially stable with respect to the ball B_ρ with

$$\rho = \frac{\bar{\lambda}\beta'}{\alpha'} \Big(\prod_{k=0}^{+\infty} d_k \sqrt{\frac{\beta'}{\alpha'}} \Big),$$

where $\bar{\lambda} = \int_{t_0}^{+\infty} \lambda(s) ds$. □

5. Separation principle

Let $K(t)$ be a gain matrix such that system (4.2) is practically exponentially stable and $L(t)$ a gain matrix such that system (3.3) is an observer for (3.2), we consider the following system obtained as the union of (3.3) and (4.2).

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t) + g(t, x(t)) - B(t)K(t)\hat{x}(t), \quad t \neq t_k, \\
\dot{\hat{x}}(t) &= A(t)\hat{x}(t) + g(t, \hat{x}(t)) - B(t)K(t)\hat{x}(t) - L(t)(C(t)\hat{x}(t) - y(t)), \quad t \neq t_k, \\
\Delta x(t_k) &= D_k x(t_k), \\
\Delta \hat{x}(t_k) &= D_k \hat{x}(t_k).
\end{aligned}$$

In order to show that this system is practically exponentially stable, we re-write it by letting $e = \hat{x} - x$:

$$\begin{aligned}
\dot{x}(t) &= \big(A(t) - B(t)K(t)\big)x(t) + g(t, x(t)) - B(t)K(t)e(t), \quad t \neq t_k, \\
\dot{e}(t) &= \big(A(t) - L(t)C(t)\big)e(t) + \big(g(t, e(t) + x(t)) - g(t, x(t))\big), \quad t \neq t_k, \\
\Delta x(t_k) &= D_k x(t_k), \\
\Delta e(t_k) &= D_k e(t_k).
\end{aligned} \tag{5.1}$$

Theorem 3. *Let assumptions* $(\mathcal{A}_1)$-$(\mathcal{A}_2)$-$(\mathcal{A}_3)$ *and* $(\mathcal{A}_4)$ *hold. In addition, assume that* $\lambda(t)$ *in* $(\mathcal{A}_3)$ *is integrable on* $[0, +\infty[$. *Then there exist gain matrices* $K(t)$ *and* $L(t)$ *given by* (4.1) *and* (3.4) *respectively such that system* (5.1) *is globally uniformly practically exponentially stable.*

Proof. Define the Lyapunov function v as follows

$$v(e, x) := e^T(t)M^{-1}(t)e(t) + a x^T(t)N^{-1}(t)x(t) = v_e(t) + a v_x(t),$$

where $a > 0$. Recall that there exist positive constants $\alpha, \alpha', \beta, \beta'$ and t_0 such that $\forall t \geq t_0$, we have

$$0 < \alpha I \leq M^{-1}(t) \leq \beta I$$

$$0 < \alpha' I \leq N^{-1}(t) \leq \beta' I.$$

It is easily to verify that there exist $\lambda_1 > 0, \lambda_2 > 0$ such that

$$0 < \lambda_1 \parallel (e,x) \parallel^2 \leq v(e,x) \leq \lambda_2 \parallel (e,x) \parallel^2 . \tag{5.2}$$

The Dini derivative of v along the trajectories of (5.1) for $t \neq t_k$ is given by

$$\begin{aligned} D^+ v_{(5.1)}(e,x) &= 2e(t)^T M^{-1}(t) \frac{d}{dt}(e(t)) + e(t)^T \frac{d}{dt}(M^{-1}(t))e(t) \\ &\quad + 2ax(t)^T N^{-1}(t) \frac{d}{dt}(x(t)) + ax(t)^T \frac{d}{dt}(N^{-1}(t))x(t) \\ &\leq -e(t)^T M^{-1}(t) P M^{-1}(t) e(t) \\ &\quad + 2\beta \parallel g(t, e(t)+x(t)) - g(t,x(t)) \parallel \parallel e(t) \parallel \\ &\quad - ax(t)^T N^{-1}(t) Q N^{-1}(t) x(t) - 2ax(t)^T N^{-1}(t) B(t) K(t) e(t) \\ &\quad + 2ax(t)^T N^{-1}(t) g(t,x(t)) \\ &\leq -\lambda_{\min}(P)\alpha^2 \parallel e(t) \parallel^2 - a\lambda_{\min}(Q)\alpha'^2 \parallel x(t) \parallel^2 \\ &\quad + 4\beta\lambda(t) \parallel e(t) \parallel + 2a\beta' \lambda(t) \parallel x(t) \parallel \\ &\quad + 2a\beta' \parallel B(t)K(t) \parallel \parallel e(t) \parallel \parallel x(t) \parallel . \end{aligned}$$

We know that $\forall \varepsilon > 0$,

$$2 \parallel e(t) \parallel \parallel x(t) \parallel \leq (\frac{1}{\varepsilon} \parallel e(t) \parallel^2 + \varepsilon \parallel x(t) \parallel^2).$$

Then

$$\begin{aligned} D^+ v_{(5.1)}(e,x) \leq & -\lambda_{\min}(P)\alpha^2 \parallel e(t) \parallel^2 - a\lambda_{\min}(Q)\alpha'^2 \parallel x(t) \parallel^2 \\ & + 4\beta\lambda(t) \parallel e(t) \parallel + 2a\beta' \lambda(t) \parallel x(t) \parallel \\ & + a\beta' \parallel B(t)K(t) \parallel (\frac{1}{\varepsilon} \parallel e(t) \parallel^2 + \varepsilon \parallel x(t) \parallel^2) \\ = & -\Big(\lambda_{\min}(P)\alpha^2 - \frac{a\beta' \parallel B(t)K(t) \parallel}{\varepsilon}\Big) \parallel e(t) \parallel^2 \\ & - a\Big(\lambda_{\min}(Q)\alpha'^2 - \varepsilon\beta' \parallel B(t)K(t) \parallel \Big) \parallel x(t) \parallel^2 \\ & + 4\beta\lambda(t) \parallel e(t) \parallel + 2a\beta' \lambda(t) \parallel x(t) \parallel \\ \leq & -\Big(\lambda_{\min}(P)\alpha^2 - \frac{a\beta' \parallel B(t)K(t) \parallel}{\varepsilon}\Big) \parallel e(t) \parallel^2 \\ & - a\Big(\lambda_{\min}(Q)\alpha'^2 - \varepsilon\beta' \parallel B(t)K(t) \parallel \Big) \parallel x(t) \parallel^2 \\ & + \frac{4\beta}{\sqrt{\alpha}}\lambda(t)\sqrt{v_e(t)} + \frac{2a\beta'}{\sqrt{\alpha'}}\lambda(t)\sqrt{v_x(t)}. \end{aligned}$$

First, we must choose the variable $\varepsilon > 0$ such that

$$\lambda_{\min}(Q)\alpha'^2 - \varepsilon\beta' \parallel B(t)K(t) \parallel > 0.$$

Subsequently we choose the variable $a > 0$ such that

$$\lambda_{\min}(P)\alpha^2 - \frac{a\beta^{'} \parallel B(t)K(t) \parallel}{\varepsilon} > 0.$$

Then by (5.2), it is easy to verify that there exists a constant $\eta > 0$ such that

$$D^+v_{(5.1)}(e,x) \leq -\eta v(e,x) + \Big(\frac{4\beta}{\sqrt{\alpha}} + \frac{2\sqrt{a}\beta^{'}}{\sqrt{\alpha^{'}}}\Big)\lambda(t)\sqrt{v(e,x)}, \quad \forall t \neq t_k. \tag{5.3}$$

Let $w(t) = \sqrt{v(e,x)}$. Then we derive the following inequality

$$D^+w_{(5.1)}(t) \leq -\frac{\eta}{2}w(t) + \Big(\frac{2\beta}{\sqrt{\alpha}} + \frac{\sqrt{a}\beta^{'}}{\sqrt{\alpha^{'}}}\Big)\lambda(t), \quad \forall t \neq t_k. \tag{5.4}$$

For $t = t_k$ we obtain

$$\begin{aligned} v(e(t_k^+), x(t_k^+)) =&\ v(e(t_k) + \Delta e(t_k), x(t_k) + \Delta x(t_k)) \\ &\leq \parallel I + D_k \parallel^2 \frac{\beta}{\alpha} v_e(t_k) + a \parallel I + D_k \parallel^2 \frac{\beta^{'}}{\alpha^{'}} v_x(t_k) \\ &\leq\ \frac{\beta d_k^2}{\alpha} v_e(t_k) + a\frac{\beta^{'} d_k^2}{\alpha^{'}} v_x(t_k) \\ &\leq\ d_k^2\Big(\frac{\beta\alpha^{'} + \alpha\beta^{'}}{\alpha\alpha^{'}}\Big) v(e(t_k), x(t_k)), \end{aligned}$$

which gives

$$w(t_k^+) \leq d_k\sqrt{\frac{\beta\alpha' + \alpha\beta'}{\alpha\alpha'}}w(t_k). \tag{5.5}$$

Then by (5.4), (5.5) and the comparison lemma, we get

$$\begin{aligned} w(t) \leq&\ w(t_0)\Big(\prod_{t_0 < t_k < t} d_k\sqrt{\frac{\beta\alpha' + \alpha\beta'}{\alpha\alpha'}}\Big) \times e^{-\frac{\eta}{2}(t-t_0)} \\ &+ \Big(\frac{2\beta}{\sqrt{\alpha}} + \frac{\sqrt{a}\beta^{'}}{\sqrt{\alpha^{'}}}\Big)\int_{t_0}^{t}\Big(\prod_{s<t_k<t} d_k\sqrt{\frac{\beta\alpha' + \alpha\beta'}{\alpha\alpha'}}\Big) \times e^{-\frac{\eta}{2}(t-s)}\lambda(s)ds \\ \leq&\ w(t_0)\Big(\prod_{k=0}^{+\infty} d_k\sqrt{\frac{\beta\alpha' + \alpha\beta'}{\alpha\alpha'}}\Big) \times e^{-\frac{\eta}{2}(t-t_0)} \\ &+ \Big(\frac{2\beta}{\sqrt{\alpha}} + \frac{\sqrt{a}\beta^{'}}{\sqrt{\alpha^{'}}}\Big)\Big(\prod_{k=0}^{+\infty} d_k\sqrt{\frac{\beta\alpha' + \alpha\beta'}{\alpha\alpha'}}\Big) \times \int_{t_0}^{+\infty} \lambda(s)ds. \end{aligned}$$

By (5.2), it follows that

$$\parallel (e,x) \parallel \leq\ \parallel (e_0, x_0) \parallel \sqrt{\frac{\lambda_2}{\lambda_1}}\Big(\prod_{k=0}^{+\infty} d_k\sqrt{\frac{\beta\alpha' + \alpha\beta'}{\alpha\alpha'}}\Big) \times e^{-\frac{\eta}{2}(t-t_0)}$$

$$+\Big(\frac{2\beta}{\sqrt{\alpha}}+\frac{\sqrt{a}\beta^{'}}{\sqrt{\alpha^{'}}}\Big)\Big(\prod_{k=0}^{+\infty}d_k\sqrt{\frac{\beta\alpha'+\alpha\beta'}{\alpha\alpha^{'}}}\Big)\times\int_{t_0}^{+\infty}\lambda(s)ds.$$

This proves that system (5.1) is uniformly practically exponentially stable with respect to the ball B_ρ with

$$\rho=\bar{\lambda}\Big(\frac{2\beta}{\sqrt{\alpha}}+\frac{\sqrt{a}\beta^{'}}{\sqrt{\alpha^{'}}}\Big)\Big(\prod_{k=0}^{+\infty}d_k\sqrt{\frac{\beta\alpha'+\alpha\beta'}{\alpha\alpha^{'}}}\Big),$$

where $\bar{\lambda}=\int_{t_0}^{+\infty}\lambda(s)ds$. □

6. Conclusion

This paper addresses observer-based control of a class of linear impulsive systems with non-vanishing perturbation. In particular, we present an observer construction and a state feedback construction that deliver practical regulation of the observer estimation error response and closed-loop state response, respectively, in a uniform-ultimate-boundedness sense with exponential convergence to an ultimate bound.

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