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A GENERALIZATION OF THE ZERO-DIVISOR GRAPH FOR MODULES

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Abstract. Let R be a commutative ring with identity and M an R-module. In this paper, we associate a graph to M, say $\Gamma(M)$, such that when $M = R$, $\Gamma(M)$ is exactly the classic zero-divisor graph. Many well-known results by D. F. Anderson and P. S. Livingston, in [5], and by D. F. Anderson and S. B. Mulay, in [6], have been generalized for $\Gamma(M)$ in the present article. We show that $\Gamma(M)$ is connected with $\mathrm{diam}(\Gamma(M)) \leq 3$. We also show that for a reduced module M with $Z(M)^* \neq M \setminus \{0\}$, $\mathrm{gr}(\Gamma(M)) = \infty$ if and only if $\Gamma(M)$ is a star graph. Furthermore, we show that for a finitely generated semisimple R-module M such that its homogeneous components are simple, $x, y \in M \setminus \{0\}$ are adjacent if and only if $xR \bigcap yR = (0)$. Among other things, it is also observed that $\Gamma(M) = \emptyset$ if and only if M is uniform, $\mathrm{ann}(M)$ is a radical ideal, and $Z(M)^* \neq M \setminus \{0\}$, if and only if $\mathrm{ann}(M)$ is prime and $Z(M)^* \neq M \setminus \{0\}$.

1. Introduction

All rings in this paper are commutative with identity and all modules are unitary right modules. Let G be an undirected graph. We say that G is *connected* if there is a path between any two distinct vertices. For distinct vertices x and y in G, the *distance* between x and y, denoted by $d(x, y)$, is the length of a shortest path connecting x and y ($d(x, x) = 0$ and $d(x, y) = \infty$ if no such path exists). The *diameter* of G is

$$\mathrm{diam}(G) = \sup\{d(x, y) \mid \ x \text{ and } y \text{ are vertices of } G\}.$$

A *cycle* of length n in G is a path of the form $x_1 - x_2 - x_3 - \cdots - x_n - x_1$, where $x_i \neq x_j$ when $i \neq j$. We define the *girth* of G, denoted by $\mathrm{gr}(G)$, as the length of a shortest cycle in G, provided G contains a cycle; otherwise, $\mathrm{gr}(G) = \infty$. A graph is *complete* if any two distinct vertices are adjacent. A complete graph with n vertices is denoted by K_n. By a complete subgraph, we mean a subgraph which is complete as a graph. A complete subgraph of G is called a *clique.* The *clique number* of G is $cl(G) = \sup\{|G'| : \ G' \text{ is a complete subgraph of } G\}$. Let

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$K^{m,n}$ denote the complete bipartite graph on two nonempty disjoint sets V_1 and V_2 with $|V_1| = m$ and $|V_2| = n$ (here m and n may be infinite cardinal numbers). A $K^{1,n}$ graph is often called a star graph. In this article, all subgraphs are induced subgraphs, where a subgraph G' of a graph G is an *induced subgraph* of G if two vertices of G' are adjacent in G' if and only if they are adjacent in G. The reader is referred to [7], [19], and [20] for undefined terms and concepts.

In recent decades, the zero-divisor graphs of commutative rings (in this paper, called the *classic zero-divisor graph*) have been extensively studied by many authors and have become a major field of research, see for example [3-15]. Some authors have also extended the graph of zero-divisors to non-commutative rings, see [18] and [2]. In [1], [12], and [13], the graph of zero-divisors for commutative rings has been generalized to the annihilating-ideal graph of commutative rings (two ideals I and J are adjacent if $IJ = (0)$). In [11], the classic zero-divisor graph has been generalized to modules over commutative rings. According to [11], $m, n \in M$ are adjacent if and only if $(mR :_R M)(nR :_R M)M = 0$, which is a direct generalization of the classic zero-divisor graph. In [8] and [9], the authors have associated two different graphs to an R-module M with respect to its first dual, $M^* = \mathrm{Hom}(M, R)$. Though they are not necessarily generalizations of the classic zero-divisor graph, there are some deep interrelations between these two graphs and the classic one. In this article, we introduce a new generalization of the classic zero-divisor graph, which is, at least to the present authors, more natural than the aforementioned generalizations. As any suitable generalization, this one reveals some properties which are sofar untouched in the literature, even for the classic zero-divisor graph (see Proposition 1.7 and Proposition 2.7). In general, proofs of results, are simpler than those proofs given for the counter part results on the classic zero-divisor graph.

Definition 1.1. Let M be an R-module. For every two non-zero elements $x, y \in M$, we say that $x * y = y * x = 0$ provided that

$$x(yR : M) = 0 \ \text{ or } \ y(xR : M) = 0.$$

For an R-module M, by $Z(M)$ we mean the set of all $x \in M$ such that $x * y = 0$ for some non-zero $y \in M$. Put $Z(M)^* = Z(M) \setminus \{0\}$. We associate an undirected graph $\Gamma(M_R)$ to M with vertices $Z(M)^*$ such that for distinct elements $x, y \in Z(M)^*$, the vertices x and y are adjacent provided that $x*y = 0$.

As we observe in the sequel, the graph $\Gamma(M)$ is exactly a generalization of the classic zero-divisor graph. Assume that R is a ring. Then $\Gamma(R_R) = \Gamma(R)$. It is easy to see that for each $x \in R$, $(xR : R) = xR$. Then for all non-zero $x, y \in R$, $xy = 0$ if and only if $x * y = 0$. Along this line, we also have the following proposition.

Proposition 1.2. *For every positive integer number n, $\Gamma(\mathbb{Z}_n) = \Gamma((\mathbb{Z}_n)_{\mathbb{Z}})$.*

Proof. We can show that for each $\bar{x} \in \mathbb{Z}_n$, $(\bar{x}\mathbb{Z}_n : \mathbb{Z}_n) = d\mathbb{Z}$, where $d = (n, x)$. Assume that $\bar{y}(\bar{x}\mathbb{Z} : \mathbb{Z}) = \bar{0}$. There exist $p, q \in \mathbb{Z}$ such that $n = dq$ and $x = dp$. Since $\bar{y}d\mathbb{Z} = \bar{0}$, then n divides yd, and hence $yd = na$ for some $a \in \mathbb{Z}$.

Therefore $nap = ydp = yx$, and hence $n|xy$ (i.e., $\bar{y}\bar{x} = \bar{0}$). Now suppose that $\bar{y}\bar{x} = \bar{0}$ and $d = (n, x)$. There exist $p, q \in \mathbb{Z}$ such that $n = dq$ and $x = dp$. Since $n|xy$, then $yx = na$ for some $a \in \mathbb{Z}$. Thus $na = yx = ydp$, and hence $yp = \frac{n}{d}a$. Since $(p, \frac{n}{d}) = 1$ and $p|\frac{n}{d}a$, then $p|a$, and hence $a = ps$ for some $s \in \mathbb{Z}$. Therefore $ydp = na = nps$, and hence $yd = ns$. Thus $n|yd$; this means that $\bar{y}(\bar{x}\mathbb{Z} : \mathbb{Z}_n) = \bar{y}d\mathbb{Z} = \{\bar{0}\}$. □

The next lemma has a crucial role in this paper.

Lemma 1.3. *Let M be an R-module and m, n two non-zero elements of M.*

(1) *If m and n are adjacent, then $mr * ns = 0$ for every $r, s \in R$ such that $mr \neq 0$ and $ns \neq 0$.*

(2) *If $mR \cap nR = 0$, then m and n are adjacent.*

Proof. (1) Let $m(nR : M) = 0$. It is clear that $mr(nR : M) = 0$ for every $r \in R$. On the other hand, $(nsR : M) \subseteq (nR : M)$ for every $s \in R$. Hence

$$mr(nsR : M) \subseteq mr(nR : M) = 0.$$

(2) Since $m(nR : M) \subseteq nR \cap mR$, consequently $m(nR : M) = 0$, which implies that $m * n = 0$. □

Remark 1.4. The above lemma shows that independent families of submodules, and hence uniform dimension (= Goldie dimension), is a related concept in our discussion. Let M be a module with uniform dimension α (i.e., U.dim $M = \alpha$), where α is an attainable cardinal number (in the sense of Dauns and Fuchs in [15]). Then by Lemma 1.3, we know that $\alpha \leq cl(\Gamma(M)) \leq |Z(M)^*|$. However, this inequality can be strict as we will see in Example 1.12. The converse of Lemma 1.3(2) is not true as we will observe in Example 1.12, where in $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ the elements $(1, 0)$ and $(2, 0)$ are adjacent, but $(1, 0)\mathbb{Z} \cap (2, 0)\mathbb{Z} = \mathbb{Z}_3 \oplus 0$.

The next result is a generalization of [5, Theorem 2.3].

Theorem 1.5. *Let M be an R-module. Then $\Gamma(M)$ is a connected graph with diam$(\Gamma(M)) \leq 3$.*

Proof. Let $m, n \in Z^*(M)$ be distinct vertices. If $m * n = 0$, then $d(m, n) = 1$. Now suppose that $m * n$ is non-zero. Since $m, n \in Z(M)^*$, there exist non-zero elements $x, y \in M$ such that $m * x = 0$ and $n * y = 0$. If either $x * y = 0$ or $x = y$, then $m - x - y - n$ or $m - x - n$ is a path of length less than or equal 3 between n and m. Suppose that $x \neq y$ and $x * y \neq 0$. Then by Lemma 1.3(2), $xR \cap yR \neq 0$, and hence there exists a non-zero element $z = xr = ys \in M \setminus \{m, n\}$. If $m = xr$ or $n = ys$, then by Lemma 1.3(1), we would have $m * n = 0$, which is a contradiction. Again by Lemma 1.3(1), $m - z - n$ is a path of length 2. Therefore diam$(\Gamma(M)) \leq 3$. □

The following theorem is a generalization of [17, Theorem 1.4] (also see [3, page 27] for a brief history of this result). The aforementioned result has also appeared in [5] and [16].

Theorem 1.6. *Let M be an R-module. If $\Gamma(M)$ contains a cycle, then*

$$gr(\Gamma(M)) \leq 4.$$

Proof. Let $x_1 - x_2 - \cdots - x_n$ be a cycle in $\Gamma(M)$. Then one of the following cases holds.
(Case 1) If $x_1R \cap x_3R = 0$, then $x_1 * x_3 = 0$, and hence $x_1 - x_2 - x_3 - x_1$ is cycle.
(Case2) Assume that $x_1R \cap x_3R \neq 0$. There exists a non-zero element $m \in x_1R \cap x_3R$. Then:

(a) If $m = x_2$, then x_2 and x_4 are adjacent because x_4 is adjacent to x_3, and hence by Lemma 1.3(1), it is adjacent to $m = x_2$. Hence $x_2 - x_3 - x_4 - x_2$ is a cycle of length 3.

(b) If $m = x_4$, the same cycle which appeared in part(a) is obtained here.

(c) If $m = x_1$, then x_1 and x_4 are adjacent, and hence $x_1 - x_2 - x_3 - x_4 - x_1$ is a cycle of length 4.

(d) If $m = x_3$, then x_3 and x_n are adjacent, and hence $x_n - x_1 - x_2 - x_3 - x_n$ is a cycle of length 4.

(e) Let $m \in M \setminus \{x_1, x_2, x_3, x_4\}$. Then m is adjacent to both x_4 and x_2. Hence $m - x_2 - x_3 - x_4 - m$ is a cycle of length 4. □

When does $\Gamma(M)$ contain a cycle? The next result gives a partial answer to this question. As we see, it happens when $\Gamma(M)$ contains a path of length 4. In fact, when $\Gamma(M)$ has a path of length 4, then $\text{gr}(\Gamma(M)) \leq 4$.

Proposition 1.7. *Let M be an R-module. If $\Gamma(M)$ contains a path of length 4, then $\Gamma(M)$ contains a cycle.*

Proof. Let $x_1 - x_2 - x_3 - x_4 - x_5$ be a path of length 4. If $x_2R \cap x_4R = 0$, then $x_2 * x_4 = 0$, and hence $x_2 - x_3 - x_4 - x_2$ is cycle. Now assume that $0 \neq z \in x_2R \cap x_4R$. Then one of the following cases holds.
(Case 1) If $z = x_1$, then by Lemma 1.3(1), $x_1 - x_2 - x_3 - x_1$ is a cycle.
(Case 2) If $z = x_2$, then by Lemma 1.3(1), $x_2 - x_3 - x_4 - x_5 - x_2$ is a cycle.
(Case 3) If $z = x_3$, then by Lemma 1.3(1), $x_1 - x_2 - x_3 - x_1$ is a cycle.
(Case 4) If $z = x_4$, then by Lemma 1.3(1), $x_3 - x_4 - x_1 - x_2 - x_3$ is a cycle.
(Case 5) If $z = x_5$, then by Lemma 1.3(1), $x_3 - x_4 - x_5 - x_3$ is a cycle.
(Case 6) If $z \notin \{x_1, x_2, x_3, x_4, x_5\}$, then by Lemma 1.3(1), $x_1 - z - x_3 - x_2 - x_1$ is a cycle. □

The following corollary is also a direct consequence of [16, Lemma 1.5 and Theorem 1.6] or [6, Theorems 2.2-2.5].

Corollary 1.8. *Let R be a ring. If $\Gamma(R)$ contains a path of length 4, then $\Gamma(R)$ contains a cycle.*

Proof. By Proposition 1.7, the verification is immediate. □

It is well-known that R is a domain if and only if the classic zero-divisor graph $\Gamma(R)$ is empty. The following is a natural generalization of this fact.

Theorem 1.9. *Let M be an R-module. Then the following are equivalent.*

1. *$\Gamma(M)$ is the empty graph.*
2. *M is a uniform R-module, $\operatorname{ann}(M)$ is a radical ideal, and $Z(M)^* \neq M \setminus \{0\}$.*
3. *$\operatorname{ann}(M)$ is a prime ideal and $Z(M)^* \neq M \setminus \{0\}$.*

Proof. (1)⇒(2). Let $\Gamma(M) = \emptyset$. Then by Lemma 1.3(2), for all non-zero elements $m, n \in M$, $mR \cap nR$ must be non-zero. This implies that M is a uniform R-module. Now suppose that $a, b \in R$ such that $ab \in \operatorname{ann}(M)$, but neither a nor b belongs to $\operatorname{ann}(M)$. Therefore there exist $m, n \in M$ such that both $ma \neq 0$ and $nb \neq 0$. Hence

$$ma(nbR : M) = am(nbR : M) \subseteq anbR = nabR = 0.$$

Therefore ma and nb belong to $Z(M)^*$. This is a contradiction.

(2)⇒(1). Assume that M is a uniform module with radical annihilator such that $0 \neq m \in Z(M)^*$. There exists $0 \neq n \in Z(M)^*$ such that $m * n = 0$. Since M is uniform, there exists $0 \neq x \in mR \cap nR$. By Lemma 1.3, $x(xR : M) = 0$, and hence $(xR : M) \subseteq \operatorname{ann}(x)$. Now, assume that $r \in (xR : M)$. Then

$$Mr^2 = (Mr)r \subseteq xRr = xrR = 0.$$

Therefore $r^2 \in \operatorname{ann}(M)$, and hence $r \in \operatorname{ann}(M)$ because $\operatorname{ann}(M)$ is a radical ideal. This implies that $(xR : M) \subseteq \operatorname{ann}(M)$. Hence for each non-zero element $y \in M$, $y(xR : M) = 0$. Thus $Z(M)^* = M \setminus \{0\}$. This is a contradiction.

(1)⇒(3). As in the proof of (1)⇒(2).

(3)⇒(1). Suppose that $m \in Z(M)^*$. Then there exists $n \in Z(M)^*$ such that $m * n = 0$. Therefore $(mR : M)(nR : M)M = 0$, hence $(mR : M)(nR : M) \subseteq \operatorname{ann}(M)$. Since $\operatorname{ann}(M)$ is a prime ideal, either $(mR : M) \subseteq \operatorname{ann}(M)$ or $(nR : M) \subseteq \operatorname{ann}(M)$. This implies that $Z(M)^* = M \setminus \{0\}$. This is a contradiction. □

Corollary 1.10. *Let R be a ring. Then $\Gamma(R)$ is the empty graph if and only if R is a domain.*

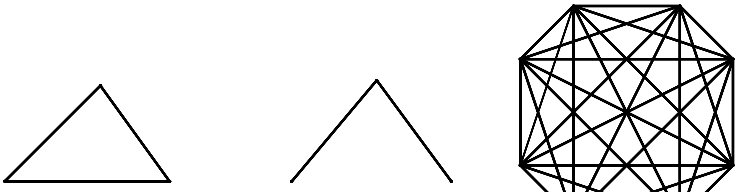
Proof. Since $1 \notin Z^*(R)$, by Theorem 1.9, the proof is clear. □

The next corollary is a consequence of Theorem 1.9. In spite of this, we give an easy and direct proof as well.

Corollary 1.11. *Let S be a simple R-module. Then $\Gamma(M)$ is the empty graph.*

Proof. For every non-zero element $n \in S$, we know that $nR = S$, and hence $(nR : S) = R$. Therefore for all non-zero elements $m, n \in S$, we have $m(nR : S) = mR \neq 0$. Hence $Z(M)^* = \emptyset$. □

Example 1.12. In Figure 1, we give the zero-divisor graph of some $\mathbb{Z}$-modules.



$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ $\mathbb{Z}_6$ $\mathbb{Z}_3 \oplus \mathbb{Z}_3$

Figure 1

2. Complete and bipartite graphs

As we have already observed in the third part of Example 1.12, the zero-divisor graph of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ is a complete graph. It is not difficult to see that this fact holds for $M = \mathbb{Z}_p \oplus \mathbb{Z}_p$ (as a $\mathbb{Z}$-module), where p is any prime number, with $|Z(M)^*| = p^2 - 1$. According to the next result, this is also true for every module which is a direct sum of two isomorphic simple submodules.

Theorem 2.1. *Let S and S' be two isomorphic simple R-modules and $M = S \oplus S'$. Then $Z(M)^* = M \setminus \{0\}$ and $\Gamma(M)$ is a complete graph.*

Proof. Before proving the theorem, we bring the reader's attention to this fact: Let M_1 and M_2 be two isomorphic modules. Then we have $\Gamma(M_1) \cong \Gamma(M_2)$. Using this fact, it is enough to prove our theorem for the case $M = S \oplus S$. For each $0 \neq x \in S$, $((x,0)R : M) = \operatorname{ann}(S)$. Therefore for each $(a,b) \in M$, $(a,b)((x,0)R : M) = 0$, and hence $(x,0)$ is adjacent to each non-zero element $(a,b) \in M$. This argument holds for $(0,x)$, too. Now, suppose that x and y are two non-zero elements of S. It is clear that $\operatorname{ann}(x) = \operatorname{ann}(y) = \operatorname{ann}(S)$ is a maximal ideal of R. Obviously, $((x,y)R : M)$ contains $\operatorname{ann}(S)$. On the other hand, if $1 \in ((x,y)R : M)$, then $(x,0)1 \in (x,y)R$, and hence $(x,0) = (x,y)r$ for some $r \in R$. Thus $yr = 0$ implies that $r \in \operatorname{ann}(y) = \operatorname{ann}(x)$, and hence $x = xr = 0$, a contradiction. Therefore $((x,y)R : M) = \operatorname{ann}(S)$, and hence (x,y) is adjacent to each non-zero element of M. $\square$

Proposition 2.2. *Let R be a commutative ring. Then R is a field if and only if $\Gamma(M)$ is a complete graph for every R-module M.*

Proof. ($\Rightarrow$) Let R be a field. If $\dim(M_R) = 1$, by Lemma 1.11, $\Gamma(M) = \emptyset$, and hence $\Gamma(M)$ is a complete graph. If $\dim(M_R) \geq 2$, then for each $0 \neq m \in M$, $(mR : M) = 0$. Because $0 \neq r \in (mR : M)$ implies that $Mr \subseteq mR$, and hence

$$M \subseteq mr^{-1}R \subseteq mR.$$

This is a contradiction. Thus $n(mR : M) = 0$ for all non-zero elements $m, n \in M$.

($\Leftarrow$) Let N_0 be a maximal ideal of R. Put $M = \frac{R}{N_0} \oplus R$. Then for every $x \in R \setminus N_0$, $(\overline{x}, 0)((0,r)R : \frac{R}{N_0} \oplus R) = 0$ for every non-zero $r \in R$. Hence $(0,r)((0,s)R : \frac{R}{N_0} \oplus R) = 0$ (by completeness) for all distinct non-zero r and s in R. Then for every $0, 1 \neq s \in R$, we have $(0,1)N_0sR = 0$ because $N_0sR \subseteq ((0,s)R : \frac{R}{N_0} \oplus R)$; this implies that for every $0, 1 \neq s$, $N_0s = 0$ and $N_0(1-s) = 0$. This implies that $N_0 = (0)$, i.e., R is a field. $\square$

In Lemma 1.3, we have already observed that if $xR \bigcap yR = (0)$, then x is adjacent to y. In the sequel, we give a partial converse of the aforementioned observation. The reader is reminded that a homogeneous component of a semisimple module is the direct sum of all the simple isomorphic submodules.

Theorem 2.3. *Let M be a finitely generated semisimple R-module such that its homogeneous components are simple. Then $x, y \in M \setminus \{0\}$ are adjacent if and only if $xR \cap yR = 0$.*

Proof. Let $M = \bigoplus_{i \in I} S_i$, where the S_i's are non-isomorphic simple submodules of M. Assume that $x, y \in Z(M)^*$ are adjacent. We must show that $xR \bigcap yR = (0)$. Suppose, to the contrary, $xR \bigcap yR \neq (0)$. Then by hypothesis, there exists $\alpha \in I$ such that $S_\alpha \subseteq xR \bigcap yR$. Since xR, yR are submodules of M, there exist subsets A and B of I such that $M = xR \bigoplus (\oplus_{i \in A} S_i)$ and $M = yR \bigoplus (\oplus_{i \in B} S_i)$ (see [7, Lemma 9.2]). Assume that $x(yR : M) = (0)$. Then

$$(yR : M) = (yR : yR \oplus (\oplus_{i \in B} S_i)) = ann(\oplus_{i \in B} T_i) = \bigcap_{i \in B} ann(S_i)$$

and $xR \cong \oplus_{i \in I \setminus A} S_i$. We conclude that

$$ann(xR) = ann(x) = ann(\oplus_{i \in I \setminus A} S_i) = \bigcap_{i \in I \setminus A} ann(S_i).$$

Since $x(yR : M) = 0$, we have that $(yR : M) \subseteq \text{ann}(x)$, and therefore $\bigcap_{i \in B} ann(S_i) \subseteq ann(\oplus_{i \in I \setminus A} S_i)$. Since for every $i, j \in I$, $ann(S_i)$ and $ann(S_j)$ are coprime, then

$$\bigcap_{i \in B} ann(S_i) = \prod_{i \in B} ann(S_i) \subseteq \bigcap_{i \in I \setminus A} ann(S_i) \subseteq ann(S_r), \ (\forall r \in I \setminus A).$$

Then for every $r \in I \setminus A$, there exists $j_r \in B$ such that $ann(S_{j_r}) \subseteq ann(S_r)$ and hence $ann(S_{j_r}) = ann(S_r)$. Therefore $S_{j_r} \cong S_r$, and hence by hypothesis $S_{j_r} = S_r$. Recall that there exists $\alpha \in I$ such that $S_\alpha \subseteq xR \bigcap yR$. Since $S_\alpha \subseteq xR \cong \oplus_{i \in I \setminus A} S_i$, there exists $i \in I \setminus A$ such that $S_\alpha \cong S_i$, and hence $S_\alpha = S_i$. By the above observation, there exists $j_i \in B$ such that $S_\alpha = S_i = S_{j_i}$. But this implies that

$$S_\alpha \subseteq yR \bigcap (\oplus_{i \in B} S_i) = (0).$$

A contradiction. By Lemma 1.3, the "only if" part is obvious. $\square$

The next corollary gives a partial answer to the question "when is $\Gamma(M)$ a complete bipartite graph?". Here we provide the reader with two different proofs, one uses the above theorem and the other one is direct.

Corollary 2.4. *Let $M = M_1 \oplus M_2$, where M_1 and M_2 are non-isomorphic simple submodules of M. Then $\Gamma(M)$ is a complete bipartite graph.*

Proof. The first Proof. It is not difficult to observe that M satisfies the above theorem's hypothesis. Suppose that $x, y \in Z(M)^*$ are adjacent. By the above theorem, $xR \bigcap yR = (0)$. We have to show that $x \in M_i$ and $y \in M_j$, where $i \neq j$ and $i, j \in \{1, 2\}$. It is clear that xR and yR are non-isomorphic simple submodules of M, and hence $xR = M_i$ and $yR = M_j$, for $i \neq j$.

The second Proof. For every $x \in M_1$ and $y \in M_2$, we have $xR \cap yR = 0$. Hence by Lemma 1.3(2), x and y are adjacent. We observe that no two elements of the M_i's are adjacent; for, if $x, y \in M_1 \setminus \{0\}$ such that $x(yR : M) = 0$, then $xR(yR : M) = M_1(M_1 : M) = 0$. On the other hand, $(M_1 : M) = \operatorname{ann}(M_2)$, and hence $M_1(\operatorname{ann}(M_2)) = 0$, which implies that $\operatorname{ann}(M_2) \subseteq \operatorname{ann}(M_1)$. Since $\operatorname{ann}(M_2)$ is a maximal ideal of R, then $\operatorname{ann}(M_1) = \operatorname{ann}(M_2)$. Therefore

$$M_1 \cong \frac{R}{\operatorname{ann}(M_1)} \cong \frac{R}{\operatorname{ann}(M_2)} \cong M_2,$$

which is a contradiction. In the sequel, we observe that for $0 \neq x \in M_1$ and $0 \neq y \in M_2$, $(x + y)$ is adjacent to no element of M_i, where $i = 1, 2$. For each $z \in M_2$, $(zR : M) = (M_2 : M) = \operatorname{ann}(M_1)$. Therefore $(x + y)(zR : M) = 0$ implies that

$$0 = (x + y)(zR : M) = (x + y)\operatorname{ann}(M_1) = y(\operatorname{ann}(M_1)),$$

and hence $\operatorname{ann}(M_1) \subseteq \operatorname{ann}(y) = \operatorname{ann}(M_2)$. By maximality of $\operatorname{ann}(M_1)$, we have $\operatorname{ann}(M_1) = \operatorname{ann}(M_2)$, a contradiction. If $z((x + y)R : M) = 0$, then by [7, Lemma 9.2] one of the following holds

(Case 1) If $M = (x + y)R \oplus M_2$, then $(x + y)R \cong \frac{M}{M_2} \cong M_1$. On the other hand,

$$(x + y)R \cong \frac{R}{\operatorname{ann}(x + y)} \cong \frac{R}{\operatorname{ann}(x) \cap \operatorname{ann}(y)},$$

which is not a simple R-module because $\operatorname{ann}(x) \cap \operatorname{ann}(y) \subset \operatorname{ann}(x)$ ($\because \operatorname{ann}(x) \cap \operatorname{ann}(y) = \operatorname{ann}(x)$ implies that $\operatorname{ann}(x) \subseteq \operatorname{ann}(y)$. The maximality of $\operatorname{ann}(x)$ and $\operatorname{ann}(y)$ implies that $\operatorname{ann}(x) = \operatorname{ann}(y)$, and hence $M_1 \cong M_2$, a contradiction). Hence

$$\frac{\operatorname{ann}(x)}{\operatorname{ann}(x) \cap \operatorname{ann}(y)} \lneqq \frac{R}{\operatorname{ann}(x) \cap \operatorname{ann}(y)}.$$

This is a contradiction.

(Case 2) If $M = (x + y)R \oplus M_1$, then similarly to case 1, a contradiction may be obtained.

(Case 3) $M = (x + y)R$. Hence $((x + y)R : M) = (M : M) = R$. Therefore $z((x + y)R : M) = 0$ implies that $z = 0$. This is a contradiction.

Similarly, we get a contradiction if we replace M_2 by M_1.
Finally, the case $(x+y)((x'+y')R : M) = 0$ implies that $x((x'+y')R : M) = 0$ and $y((x'+y')R : M) = 0$ which is impossible. ☐

Let $M = \bigoplus_{i=1}^{n} M_i$, where $n \geq 3$ and the homogeneous components are simple. While one expects that, in this case, $\Gamma(M)$ is a complete n-partite graph, one sees that by Theorem 2.3 this is not the case. However, $\Gamma(M)$ contains an n-partite graph.

In [6, Theorem 2.4], it has been proved that for a reduced commutative ring R, $\Gamma(R)$ is nonempty with $\mathrm{gr}(\Gamma(R)) = \infty$ if and only if $\Gamma(R) = K^{1,n}$ for some $n \geq 1$. In the sequel, we generalize this result to $\Gamma(M)$. We need a series of results before proving our main proposition. Recall that a module M is said to be *reduced* if whenever $a \in R$, $m \in M$ satisfy $a^2m = 0$, then $aRm = 0$.

Lemma 2.5. *Let M be a reduced R-module with $Z(M)^* \neq M \setminus \{0\}$. If $\Gamma(M)$ is a bipartite graph with parts V_1 and V_2, then $\overline{V_i} = V_i \bigcup \{0\}$ is a submodule of M for $i = 1, 2$.*

Proof. Let $x_1, x_2 \in \overline{V_1}$ and $r \in R$. We have to show that $x_1 + x_2 \in \overline{V_1}$ and $rx_1 \in \overline{V_1}$. If $rx_1 = 0$, then $rx_1 \in \overline{V_1}$. Now suppose that $rx_1 \neq 0$. By hypothesis, x_1 is adjacent to an element of V_2, say y_1. If $rx_1 = y_1$, then by Lemma 1.3, $y_1(y_1R : M) = 0$. This implies that for every $m \in M$ and $r \in (y_1R : M)$, $mr^2 = 0$. Since M is a reduced R-module, $mr = 0$, which implies that m is adjacent to y_1. This is a contradiction. Therefore $rx_1 \neq y_1$, and by Lemma 1.3, rx_1 is adjacent to y_1. Since $y_1 \in V_2$, we have $rx_1 \in V_1$. If x_1 or x_2 is equal to 0, then $x_1 + x_2 \in \overline{V_1}$. Hence we can suppose that neither x_1 nor x_2 is zero. As $x_1, x_2 \in V_1$, there are $y_1, y_2 \in V_2$ such that x_i is adjacent to y_i for $i = 1, 2$. By Lemma 1.3, $y_1R \bigcap y_2R \neq (0)$. Hence there exists $0 \neq w \in y_1R \bigcap y_2R$. Since

$$x_i(wR : M) \subseteq x_iR \bigcap wR \subseteq \overline{V_1} \bigcap \overline{V_2} = (0)$$

for $i = 1, 2$, $(x_1 + x_2)(wR : M) = (0)$. Now if $x_1 + x_2 = 0$, it belongs to $\overline{V_1}$, and if $x_1 + x_2 \neq 0$, as $w \in V_2$, we have $x_1 + x_2 \in V_1$. Similarly, we may prove that $\overline{V_2}$ is a submodule of M. ☐

Lemma 2.6. *If $m \notin Z(M)^*$, then mR is an essential submodule of M.*

Proof. If mR is not essential, there exists a nonzero submodule K of M such that $mR \bigcap K = (0)$. By Lemma 1.3, m is adjacent to any nonzero element of K, and hence $m \in Z(M)^*$. This is a contradiction. ☐

Proposition 2.7. *Let M be a reduced R-module with $Z(M)^* \neq M \setminus \{0\}$. If $\Gamma(M)$ is a bipartite graph, then the following hold.*

1. *$\Gamma(M)$ is a complete bipartite graph.*
2. *$\mathrm{U.dim} M = 2$.*

Proof. (1) Let $Z(M)^* = V_1 \bigcup V_2$, where $V_1 \bigcap V_2 = \emptyset$ and no two elements of V_i are adjacent. By Lemma 2.5, $\overline{V_1} = V_1 \bigcup \{0\}$ and $\overline{V_2} = V_2 \bigcup \{0\}$ are submodules of M. For every $z \in V_1$ and $y \in V_2$, we have

$$zR \bigcap yR \subseteq \overline{V_1} \bigcap \overline{V_2} = (0).$$

By Lemma 1.3, z and y are adjacent.

(2) Since $Z(\overline{V_1})^*$ and $Z(\overline{V_2})^*$ are empty, by Lemma 2.6, every submodule of $\overline{V_1}$ and also $\overline{V_2}$ is essential. Hence $\overline{V_1}$ and $\overline{V_2}$ are uniform submodules of M. Now we show that $\overline{V_1} \bigoplus \overline{V_2}$ is essential in M. Suppose that K is a submodule such that $K \bigcap \overline{V_1} \bigoplus \overline{V_2} = (0)$ and $0 \neq y \in K$. Then for every $0 \neq z \in \overline{V_1}$ and $0 \neq w \in \overline{V_2}$, we have $zR \bigcap yR = (0) = zR \bigcap wR$. Thus z is adjacent to y and w, i.e., $z \in \overline{V_1} \bigcap \overline{V_2} = (0)$. This is a contradiction. □

Corollary 2.8. *Let M be a reduced R-module with $Z(M)^* \neq M \setminus \{0\}$. Then* $\mathrm{gr}(\Gamma(M)) = \infty$ *if and only if* $\Gamma(M)$ *is a star graph.*

Proof. The "only if" part is obvious. Suppose that $\Gamma(M)$ has no cycles. Then $\Gamma(M)$ is a tree, and hence it is a bipartite graph. Now by Proposition 2.7, $\Gamma(M)$ is a complete bipartite graph. Suppose that V_1 and V_2 are the parts of $\Gamma(M)$. Since $\Gamma(M)$ has no cycles, then either $|V_1| = 1$ or $|V_2| = 1$, which implies that $\Gamma(M)$ is a star graph. □

In [6, Theorem 2.2], it has been proved that for a reduced commutative ring R, $\mathrm{gr}(\Gamma(R)) = 4$ if and only if $\Gamma(R) = K^{m,n}$ with $m, n \geq 2$. Here we state and prove the analog of this result for $\Gamma(M)$. We need an auxiliary lemma before proving our proposition.

In [14], the authors showed that a zero-divisor semigroup graph is bipartite if and only if it contains no triangles. The following lemma is an analog of this result.

Lemma 2.9. *Let M be an R-module. If $\Gamma(M)$ contains a cycle of odd length, then $\Gamma(M)$ contains a triangle.*

Proof. Using induction, we show that for every cycle of odd length $2n+1 \geq 5$, there exists a cycle with length $2k+1$ such that $k < n$. Assume that $x_1 - x_2 - \cdots - x_{2n} - x_{2n+1} - x_1$ is a cycle with odd length $2n+1$. If two distinct non-consecutive x_i and x_j are adjacent, the proof is complete. Otherwise, there exists $0 \neq z \in x_1R \bigcap x_3R = (0)$. By Lemma 1.3, $z \neq x_i$ for all $1 \leq i \leq 2n+1$. Here again z is adjacent to both x_4 and x_{2n+1}; so we have the cycle

$$x_{2n+1} - z - x_4 - x_5 - \cdots - x_{2n+1},$$

which is the desired cycle. □

Proposition 2.10. *Let M be an R-module. If $gr(\Gamma(M)) = 4$, then $\Gamma(M)$ is a bipartite graph with parts V_1 and V_2 such that $|V_1|, |V_2| \geq 2$. The converse is true if M is a reduced module with $Z(M)^* \neq M \setminus \{0\}$.*

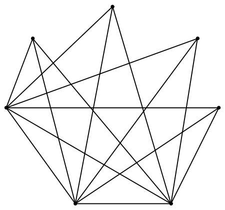


Figure 2

Proof. Let $\mathrm{gr}(\Gamma(M)) = 4$. By the above lemma, we observe that the length of any cycle in $\Gamma(M)$ is even. Since $\Gamma(M)$ has a cycle of length 4, the verification is immediate. The converse follows from Proposition 2.7. □

3. Further notes

In this short section, we are going to explain the relationship between the generalization of the classic zero-divisor graph introduced in [11] (for convenience, we denote it by Γ_b) and the one given in this article. First of all, it is worth mentioning that $\Gamma(M)$ is a subgraph of Γ_b, that is, if $m, n \in Z(M)^*$ are adjacent in $\Gamma(M)$, or equivalently either $n(mR : M) = 0$ or $m(nR : M) = 0$, then $(nR : M)(mR : M)M = 0$. However, the converse is not the case as we observe in the following example.

Example 3.1. Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ as a $\mathbb{Z}$-module. Then the Γ_b is K_7. However, $\Gamma(M)$ is different from K_7 as we observe in Figure 2:

However, when M is a multiplication module (i.e., for every submodule N of M there exists an ideal I of R such that $N = MI$), then $\Gamma(M) = \Gamma_b$. Let $(mR : M)(nR : M)M = 0$. As such, $(nR : M)M = nR$ and $(mR : M)M = mR$; so both $m(nR : M) = 0$ and $n(mR : M) = 0$.

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