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A PRIORI L^2 ERROR ANALYSIS FOR AN EXPANDED MIXED FINITE ELEMENT METHOD FOR QUASILINEAR PSEUDO-PARABOLIC EQUATIONS

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Abstract. Based on an expanded mixed finite element method, we consider the semidiscrete approximations of the solution u of the quasilinear pseudo-parabolic equation defined on $\Omega \subset R^d$, $1 \leq d \leq 3$. We construct the semidiscrete approximations of ∇u and $a(u)\nabla u + b(u)\nabla u_t$ as well as u and prove the existence of the semidiscrete approximations. And also we prove the optimal convergence of ∇u and $a(u)\nabla u + b(u)\nabla u_t$ as well as u in L^2 normed space.

1. Introduction

In this paper, we will consider the following equations

$$
\begin{aligned}
&u_t - \nabla \cdot (a(u)\nabla u + b(u)\nabla u_t) = f(u) && \text{in } \Omega \times (0,T], \\
&(a(u)\nabla u + b(u)\nabla u_t) \cdot \boldsymbol{n} = 0 && \text{on } \partial\Omega \times (0,T], \\
&u(x,0) = u_0(x) && \text{on } \Omega,
\end{aligned} \tag{1.1}
$$

where Ω is an open bounded convex domain in R^d, $1 \leq d \leq 3$ with its boundary $\partial\Omega$, $\boldsymbol{n}$ denotes the outward normal vector on $\partial\Omega$ and $a(u)$, $b(u)$ and $f(u)$ are smooth functions. This type of equation which has the mixed derivative term with respect to temporal and spatial variables is called as a pseudo-parabolic equation. It represents physical phenomena arising in the various areas such as in the flow of fluids through fissured materials [3], thermodynamics [7], semiconductor [5] and other applications. For details about the physical significance and various properties of the existence and uniqueness of the solutions of the pseudo-parabolic equations we refer to [3, 5, 6, 7, 9, 11, 24].

Early many authors applied classical Galerkin methods [1, 2, 16, 17, 18] or discontinuous Galerkin methods [22, 23] to construct the semidiscrete or fully

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discrete approximations to the solutions of the various types of the pseudo-parabolic equations. Compared to the classical Galerkin finite element method, the advantage of mixed finite element methods (MFEM) is to compute simultaneously the finite element approximations of the flux and the unknown scalar without requiring the additional regularities of $u(x,t)$. Recently several authors attempted to apply MFEM to the various pseudo-parabolic equations as follows.

For the semilinear pseudo-parabolic equations with $d = 1$, the authors [13] presented a numerical scheme based on the local discontinuous mixed Galerkin method and proved the optimal error estimates of u and u_x. In [12], the authors constructed a numerical scheme to approximate the primal unknown $u(x,t)$ and the unknown flux using a split least-squares characteristic MFEM for linear pseudo-parabolic equations with a convection term and $d = 2$. Shi and Wang [20] adopted a nonconforming Galerkin MFEM for linear pseudo-parabolic equations with $d = 2$ on anisotropic meshes, and proved the error estimates in H^1 normed space. Shi and Zhang [21] introduced a new nonconforming finite element scheme based on a MFEM and an Euler fully discrete method for the linear pseudo-parabolic equation defined on $\Omega \subset R^2$ and analyzed the optimal convergence of the error estimates. Guo [14] introduced split least-squares MFEM procedures for linear pseudo-parabolic equations with $d = 2$ and 3 to construct the approximations of u and the flux variable $-(a(x)\nabla u + b(x)\nabla u_t)$ and proved the optimal order error estimates.

In this paper we will adopt an expanded mixed finite element method to deal with quasilinear pseudo-parabolic equations defined on Ω in R^d, $d = 1, 2$ and 3, with a locally Lipschitz function $f(u)$. The expanded mixed finite expands the classical mixed method by introducing the gradient as a third explicit unknown. Therefore it is suitable to approximate the gradient for the problem with low permeability or small diffusion [8, 15] or with the flux term containing mixed derivative with respect to the spatial variable and the time variable. In this paper based on an expanded mixed method, we construct semidiscrete approximations u_h, $\boldsymbol{\lambda}_h$ and $\boldsymbol{\sigma}_h$ of u, ∇u and $a(u)\nabla u + b(u)\nabla u_t$, respectively and analyze the error estimates of ∇u and $a(u)\nabla u + b(u)\nabla u_t$ as well as u. To approximate ∇u and $a(u)\nabla u + b(u)\nabla u_t$ instead of computing ∇u_h and $a(u_h)\nabla u_h + b(u_h)\nabla u_{ht}$ we construct the semidiscrete approximations of ∇u and $a(u)\nabla u + b(u)\nabla u_t$ directly and we obtain the optimal order of convergence in L^2 normed space.

Our work will be the first trial to adopt an expanded MFEM to quasilinear pseudo-parabolic equations with $d = 1, 2$ and 3 and obtain the optimal error estimates of u, ∇u and $a(u)\nabla u + b(u)\nabla u_t$. This paper is organized as follows. In Section 2 we introduce some notations and also state some necessary assumptions on the data. Next we construct finite element spaces and the weak formulation of (1.1). Then in Section 3 we construct the semidiscrete approximations of u, ∇u and $a(u)\nabla u + b(u)\nabla u_t$ based on an expanded mixed formulation and prove the existence of the semidiscrete approximations. In

Section 4 we prove the convergence of the semidiscrete approximations of u, ∇u and $a(u)\nabla u+b(u)\nabla u_t$. And the optimal error estimates of the semidiscrete approximations in L^2 normed spaces are presented. In Section 5 we describe some conclusions and suggestions. Throughout this paper, the vectors will be denoted by the bold face.

2. Finite element spaces

Now we assume that the following conditions are satisfied:

Condition 1. There exist constants a_*, a^* such that $0 < a_* \le a(u) \le a^*$, and there exist constants b_*, b^* such that $0 < b_* \le b(u) \le b^*$.

Condition 2. $a(u)$ and $b(u)$ are twice differentiable and there exists a constant $K_1 > 0$ such that $|a'(u)| \le K_1$, $|a''(u)| \le K_1$, $|b'(u)| \le K_1$ and $|b''(u)| \le K_1$.

Condition 3. f is locally Lipschitz continuous at u i.e there exist positive constants K_2 and $C(u, K_2)$ such that if $|u(x,t) - v| \le K_2$ then $|f(u(x,t)) - f(v)| \le C(u,K_2)|u(x,t) - v|, \quad \forall (x,t) \in \Omega \times [0,T], \forall v \in R$.

For $1 \le p < \infty$ and s any nonnegative integer, we let $W^{s,p}(\Omega) = \{u \in L^p(\Omega) \mid D^{\boldsymbol{\alpha}} u \in L^p(\Omega),\ |\boldsymbol{\alpha}| \le s\}$ denote the Sobolev space equipped with the norm $\|u\|_{s,p}^p = \left(\sum_{|\boldsymbol{\alpha}|\le s} \|D^{\boldsymbol{\alpha}} u\|_{L^p(\Omega)}^p\right)$ where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ with α_i, $1 \le i \le d$, nonnegative integer, $|\boldsymbol{\alpha}| = \alpha_1+\alpha_2+\cdots+\alpha_d$, and $D^{\boldsymbol{\alpha}} u = \frac{\partial^{|\boldsymbol{\alpha}|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}$. We let $W^{s,\infty}(\Omega) = \{u \in L^\infty(\Omega) \mid D^{\boldsymbol{\alpha}} u \in L^\infty(\Omega),\ |\boldsymbol{\alpha}| \le s\}$ denote the Sobolev space equipped with the norm $\|u\|_{s,\infty} = \max_{0\le|\alpha|\le s} \|D^{\boldsymbol{\alpha}} u\|_\infty$. For our convenience we may skip s if $s = 0$ and we denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$.

If $\boldsymbol{u} = (u_1, u_2, \ldots, u_d)$ is a vector valued function, then we define $\|\boldsymbol{u}\|_{s,p}^p = \sum_{i=1}^d \|u_i\|_{s,p}^p$. And we denote $\boldsymbol{L}^p(\Omega) = (L^p(\Omega))^d$, $\boldsymbol{H}^s(\Omega) = (H^s(\Omega))^d$ and $\boldsymbol{W}^{s,p}(\Omega) = (W^{s,p}(\Omega))^d$. We denote $V = L^2(\Omega)$, $\boldsymbol{\Lambda} = (L^2(\Omega))^d$ and $\boldsymbol{W} = \{\boldsymbol{w} \in \boldsymbol{H}(\text{div} : \Omega) : \boldsymbol{w} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$ where $\boldsymbol{H}(\text{div} : \Omega) = \{\boldsymbol{w} \in (L^2(\Omega))^d : \nabla \cdot \boldsymbol{w} \in L^2(\Omega)\}$, and let $\boldsymbol{H}^s(\text{div} : \Omega) = \{\boldsymbol{w} \in (L^2(\Omega))^d : \text{div } \boldsymbol{w} \in H^s(\Omega)\}$ with a norm $\|\boldsymbol{w}\|_{\boldsymbol{H}^s(\text{div}:\Omega)} = \|\boldsymbol{w}\|_2 + \|\text{div } \boldsymbol{w}\|_{s,2}$. If X is a Sobolev space equipped with a Sobolev norm $\|\cdot\|_X$ and if for each $t \in [0,T]$, $u(x,t)$ is a function which belongs to X, then for $1 \le p < \infty$ we define

$$\|u(x,t)\|_{L^p(0,t_0:X)}^p = \int_0^{t_0} \|u(\cdot,t)\|_X^p dt$$

and for $p = \infty$, $\|u(x,t)\|_{L^\infty(0,t_0:X)} = \text{ess sup}_{0\le t\le t_0} \|u(\cdot,t)\|_X$. To simplify our notation we use $L^p(X)$ and $L^\infty(X)$ instead of $L^p(0,T:X)$ and $L^\infty(0,T:X)$.

Let $\mathcal{E}_h = \{E_1, E_2, \ldots, E_{N_h}\}$ be a regular quasi-uniform subdivision of Ω where E_i is a triangle or a quadrilateral if $d = 2$ and E_i is a 3-simplex or 3-rectangle if $d = 3$. Let h_i be the diameter of E_i and $h = \max_{1\le i\le N_h} h_i$. We assume that there exists a constant $\rho > 0$ such that each E_i contains a ball of radius ρh_i. The quasiuniformity requirement is that there is a constant $\gamma > 0$ such that

$$\frac{h}{h_i} \le \gamma, \quad i = 1, 2, \ldots, N_h.$$

On each element $E \in \mathcal{E}_h$ we define

$$V_h(E) = P_k(E), \quad \boldsymbol{W}_h(E) = (P_k(E))^d \oplus (x_1, x_2, \ldots, x_d)^T P_k(E),$$

where $P_k(E)$ is the space of polynomial of total degree $\le k$ defined on E. Let $V_h \subset V$, $\boldsymbol{\Lambda}_h \subset \boldsymbol{\Lambda}$ and $\boldsymbol{W}_h \subset \boldsymbol{W}$ be finite element spaces defined locally on each element $E \in \mathcal{E}_h$ such that

$$\begin{aligned} V_h &= \{v \in V : v|_E \in V_h(E), \quad \forall E \in \mathcal{E}_h\}, \\ \boldsymbol{\Lambda}_h &= \{\mu \in \boldsymbol{\Lambda} : \mu|_E \in \boldsymbol{W}_h(E), \quad \forall E \in \mathcal{E}_h\}, \\ \boldsymbol{W}_h &= \{w \in \boldsymbol{W} : w|_E \in \boldsymbol{W}_h(E), \quad \forall E \in \mathcal{E}_h\}. \end{aligned}$$

To introduce an expanded mixed weak formulation, we let

$$\boldsymbol{\lambda} = -\nabla u, \tag{2.1}$$

$$\boldsymbol{\sigma} = -(a(u)\nabla u + b(u)\nabla u_t) = a(u)\boldsymbol{\lambda} + b(u)\boldsymbol{\lambda}_t. \tag{2.2}$$

Then $(u, \boldsymbol{\lambda}, \boldsymbol{\sigma})$ is a solution of the following weak formulation of (1.1):

$$(\boldsymbol{\lambda}, \boldsymbol{w}) - (u, \nabla \cdot \boldsymbol{w}) = 0 \qquad \forall\, \boldsymbol{w} \in \boldsymbol{W}, \tag{2.3}$$

$$(a(u)\boldsymbol{\lambda}, \boldsymbol{\mu}) + (b(u)\boldsymbol{\lambda}_t, \boldsymbol{\mu}) - (\boldsymbol{\sigma}, \boldsymbol{\mu}) = 0 \qquad \forall\, \boldsymbol{\mu} \in \boldsymbol{\Lambda}, \tag{2.4}$$

$$(u_t, v) + (\nabla \cdot \boldsymbol{\sigma}, v) = (f(u), v) \qquad \forall\, v \in V. \tag{2.5}$$

3. The existence of an expanded mixed finite element semidiscrete approximation $(u_h, \boldsymbol{\lambda}_h, \boldsymbol{\sigma}_h)$

We shall in this section construct an expanded mixed finite element semidiscrete approximation $(u_h, \boldsymbol{\lambda}_h, \boldsymbol{\sigma}_h)$ and prove the unique existence of the semidiscrete approximation of $(u, \boldsymbol{\lambda}, \boldsymbol{\sigma})$.

Let $\boldsymbol{\Pi}_h : \boldsymbol{W} \to \boldsymbol{W}_h$ denote the Raviart-Thomas projection [4, 9, 17], which satisfies

$$(\nabla \cdot \boldsymbol{w} - \nabla \cdot \boldsymbol{\Pi}_h \boldsymbol{w}, v) = 0, \quad \forall\, v \in V_h, \tag{3.1}$$

$$\|\boldsymbol{w} - \boldsymbol{\Pi}_h \boldsymbol{w}\|_p \le C h^r \|\boldsymbol{w}\|_{r,p}, \quad \forall\, \boldsymbol{w} \in \boldsymbol{H}^r(\Omega) \cap \boldsymbol{W}^{r,p}(\Omega), \ \frac{1}{p} < r \le k+1. \tag{3.2}$$

And we let $P_h : V \to V_h$ be the projection satisfying

$$(v - P_h v, \chi) = 0, \qquad \forall\, \chi \in V_h, \tag{3.3}$$

$$\|u - P_h u\|_p \le C_1 h^r \|u\|_{r,p}, \qquad \forall\, u \in H^r(\Omega) \cap W^{r,p}, \ 0 \le r \le k+1, \tag{3.4}$$

$$\operatorname{div} \boldsymbol{\Pi}_h(\boldsymbol{w}) = P_h \operatorname{div}(\boldsymbol{w}), \qquad \forall \boldsymbol{w} \in \boldsymbol{W}.$$

And also we define $\boldsymbol{R}_h : \boldsymbol{\Lambda} \to \boldsymbol{\Lambda}_h$ be the projection satisfying

$$(\boldsymbol{\lambda} - \boldsymbol{R}_h \boldsymbol{\lambda}, \boldsymbol{\mu}) = 0, \quad \forall\, \boldsymbol{\mu} \in \boldsymbol{\Lambda}_h, \tag{3.5}$$

$$\|\boldsymbol{\lambda} - \boldsymbol{R}_h \boldsymbol{\lambda}\|_p \le C_1 h^r \|\boldsymbol{\lambda}\|_{r,p}, \quad \forall\, \boldsymbol{\lambda} \in \boldsymbol{H}^r(\Omega) \cap \boldsymbol{W}^{r,p}(\Omega), \ 0 \le r \le k+1. \tag{3.6}$$

Then we have the following approximation property

$$\|\mathrm{div}(\boldsymbol{\sigma} - \boldsymbol{\Pi}_h\boldsymbol{\sigma})\|_p \le C_1 h^r \|\mathrm{div}\boldsymbol{\sigma}\|_{r,p}, \quad 0 \le r \le k, \quad \forall \boldsymbol{\sigma} \in \boldsymbol{H}^r(\mathrm{div} : \Omega).$$

Now we can formulate an expanded mixed finite element method to approximate the solution of (1.1) : Find $(u_h, \boldsymbol{\lambda}_h, \boldsymbol{\sigma}_h) \in V_h \times \boldsymbol{\Lambda}_h \times \boldsymbol{W}_h$ such that

$$(\boldsymbol{\lambda}_h, \boldsymbol{w}) - (u_h, \nabla \cdot \boldsymbol{w}) = 0, \qquad \forall \boldsymbol{w} \in \boldsymbol{W}_h, \tag{3.7}$$

$$(a(u_h)\boldsymbol{\lambda}_h, \boldsymbol{\mu}) + (b(u_h)(\boldsymbol{\lambda}_h)_t, \boldsymbol{\mu}) - (\boldsymbol{\sigma}_h, \boldsymbol{\mu}) = 0, \qquad \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}_h, \tag{3.8}$$

$$((u_h)_t, v) + (\nabla \cdot \boldsymbol{\sigma}_h, v) = (f(u_h), v), \qquad \forall v \in V_h, \tag{3.9}$$

where

$$u_h(0) = P_h(u_0(x)), \boldsymbol{\lambda}_h(0) = \boldsymbol{R}_h(\nabla u_0(x)). \tag{3.10}$$

Throughout this paper C denotes a positive generic constant depending on the constants $a_*, a^*, b_*, b^*, K_1, K_2, \Omega$ and the norms of $u, \boldsymbol{\lambda}$ and $\boldsymbol{\sigma}$ declared in the statement in the following lemmas and theorems but independent of h. Any two Cs in different places are not the same.

Lemma 3.1. *The following estimations hold*:

(i) *If* $\tau \in L^4(\Omega)$ *and* $u \in W^{r,4}(\Omega)$, *then*

$$\|(u-\tau)^2\|_2^2 \le C(\|u - P_h u\|_4^4 + \|P_h u - \tau\|_4^4) \le C\{h^{4r}\|u\|_{r,4}^4 + \|P_h u - \tau\|_4^4\}.$$

(ii) *If* $1 < p < \infty$ *and* $u \in W^{r,2p}(\Omega)$, $\tau \in L^p(\Omega)$, *then*

$$\|(u-\tau)^2\|_p \le 2(\|u-P_h u\|_{2p}^2 + \|P_h u-\tau\|_{2p}^2) \le C\{h^{2r}\|u\|_{r,2p}^2 + h^{-\frac{d}{p}}\|P_h u-\tau\|_p^2\}.$$

Proof. The proof of the statement (i) is trivial. Now we prove the statement (ii) in the following.

$$\begin{aligned}
&\|(u-\tau)^2\|_p \\
&\le \|2(u-P_h u)^2 + 2(P_h u - \tau)^2\|_p \le 2\|(u-P_h u)^2\|_p + 2\|(P_h u - \tau)^2\|_p \\
&\le 2\|(u - P_h u)\|_{2p}^2 + 2\|P_h u - \tau\|_{2p}^2 \le C(h^{2r}\|u\|_{r,2p}^2 + h^{2d(\frac{1}{2p}-\frac{1}{p})}\|P_h u - \tau\|_p^2) \\
&\le C(h^{2r}\|u\|_{r,2p}^2 + h^{-\frac{d}{p}}\|P_h u - \tau\|_p^2).
\end{aligned}$$

□

Lemma 3.2. *If* $\tau \in L^{2p}(\Omega)$, $\boldsymbol{\eta} \in \boldsymbol{L}^{2q}(\Omega), u \in W^{r,2p}(\Omega)$ *and* $\boldsymbol{\lambda} \in \boldsymbol{W}^{r,2q}(\Omega)$ *for* $1 < p < \infty$ *and its conjugate* q, *then the following estimations hold*:

(i) $\|(u-\tau)(\boldsymbol{\lambda}-\boldsymbol{\eta})\|^2 \le C\{h^{4r}\|u\|_{r,2p}^4 + \|P_h u - \tau\|_{2p}^4 + h^{4r}\|\boldsymbol{\lambda}\|_{r,2q}^4 + \|\boldsymbol{R}_h\boldsymbol{\lambda} - \boldsymbol{\eta}\|_{2q}^4\}$

(ii) $\|(u-\tau)(\boldsymbol{\lambda}-\boldsymbol{\eta})\|_p^p \le C\|u-\tau\|_{2p}^p\|\boldsymbol{\lambda}-\boldsymbol{\eta}\|_{2p}^p$

for some constant C.

Proof. The proof of the statement (i) can be obtained by the following inequality, (3.4) and (3.6).

$$\begin{aligned}
&\|(u-\tau)(\boldsymbol{\lambda}-\boldsymbol{\eta})\|^2 \\
&\le C\{\|u - P_h u\|_{2p}^4 + \|P_h u - \tau\|_{2p}^4 + \|\boldsymbol{\lambda} - \boldsymbol{R}_h\boldsymbol{\lambda}\|_{2q}^4 + \|\boldsymbol{R}_h\boldsymbol{\lambda} - \boldsymbol{\eta}\|_{2q}^4\}.
\end{aligned}$$

The proof of the statement (ii) is as follow:

$$\|(u-\tau)(\boldsymbol{\lambda}-\boldsymbol{\eta})\|_p^p \le \int (|(u-\tau)||(\boldsymbol{\lambda}-\boldsymbol{\eta})|)^p dx \le C\|u-\tau\|_{2p}^p \|\boldsymbol{\lambda}-\boldsymbol{\eta}\|_{2p}^p. \qquad \square$$

By the Taylor expansion, we have for a function $g(u)$

$$g(p)-g(\rho) = \widetilde{g}_u(p_\rho)(p-\rho) = g'(\rho)(p-\rho) + \widetilde{g}_{uu}(p_\rho)(p-\rho)^2, \tag{3.11}$$

where $\widetilde{g}_u(p_\rho) = \int_0^1 g'(p-t(p-\rho))dt$ and $\widetilde{g}_{uu}(p_\rho) = \int_0^1 (1-t)g''(\rho+t(p-\rho))dt$. By using (3.11) we have

$$\begin{aligned} &a(u)\boldsymbol{\lambda} - a(u_h)\boldsymbol{\lambda}_h \\ &= a(u)(\boldsymbol{\lambda}-\boldsymbol{\lambda}_h) - (a(u)-a(u_h))(\boldsymbol{\lambda}-\boldsymbol{\lambda}_h) + (a(u)-a(u_h))\boldsymbol{\lambda} \\ &= a(u)(\boldsymbol{\lambda}-\boldsymbol{\lambda}_h) - \widetilde{a}_u(u_{u_h})(u-u_h)(\boldsymbol{\lambda}-\boldsymbol{\lambda}_h) \\ &\quad + (a'(u_h)(u-u_h) + \widetilde{a}_{uu}(u_{u_h})(u-u_h)^2)\boldsymbol{\lambda}. \end{aligned} \tag{3.12}$$

Now we let $0 < \varepsilon < 2$ for $d = 1, 2$ and $1 < \varepsilon < 2$ for $d = 3$, $\theta = (4+2\varepsilon)/\varepsilon$ and $\theta' = \theta/(1-\theta)$. We define the spaces $\widetilde{V}_h$, $\widetilde{\boldsymbol{\Lambda}}_h$ and $\widetilde{\boldsymbol{W}}_h$ as follows:

$$\begin{aligned} \widetilde{V}_h &= \{v(x,t)\,|\,v(x,t)\in V_h, \forall t, \|v\|_{L^\infty(L^\theta)} < \infty\}, \\ \widetilde{\boldsymbol{\Lambda}}_h &= \{\boldsymbol{\lambda}(x,t)\,|\,\boldsymbol{\lambda}(x,t)\in \boldsymbol{\Lambda}_h, \forall t, \|\boldsymbol{\lambda}\|_{L^\infty(\boldsymbol{L}^{2+\varepsilon})} + \|\boldsymbol{\lambda}_t\|_{L^\infty(\boldsymbol{L}^{2+\varepsilon})} < \infty\}, \\ \widetilde{\boldsymbol{W}}_h &= \{\boldsymbol{w}(x,t)\,|\,\boldsymbol{w}(x,t)\in \boldsymbol{W}_h, \forall t, \|\boldsymbol{w}\|_{L^\infty(\boldsymbol{L}^2)} < \infty\}. \end{aligned}$$

And we define a function $\boldsymbol{\Phi} : \widetilde{V}_h \times \widetilde{\boldsymbol{\Lambda}}_h \times \widetilde{\boldsymbol{W}}_h \to \widetilde{V}_h \times \widetilde{\boldsymbol{\Lambda}}_h \times \widetilde{\boldsymbol{W}}_h$ by $\boldsymbol{\Phi}((\tau,\boldsymbol{\eta},\boldsymbol{\rho})) = (\overline{\tau},\overline{\boldsymbol{\eta}},\overline{\boldsymbol{\rho}})$, where $(\overline{\tau},\overline{\boldsymbol{\eta}},\overline{\boldsymbol{\rho}})$ is the solution of the following equations:

$$(\boldsymbol{R}_h\boldsymbol{\lambda} - \overline{\boldsymbol{\eta}}, \boldsymbol{w}) - (P_h u - \overline{\tau}, \nabla\cdot\boldsymbol{w}) = 0, \quad \forall\, \boldsymbol{w}\in \boldsymbol{W_h}, \tag{3.13}$$

(3.14)

$$\begin{aligned} &(a(u)(\boldsymbol{R}_h\boldsymbol{\lambda} - \overline{\boldsymbol{\eta}}), \boldsymbol{\mu}) + (b(u)(\boldsymbol{R}_h\boldsymbol{\lambda}_t - \overline{\boldsymbol{\eta}}_t), \boldsymbol{\mu}) \\ &\quad - (\boldsymbol{\Pi}_h\boldsymbol{\sigma} - \overline{\boldsymbol{\rho}}, \boldsymbol{\mu}) + (\boldsymbol{\Gamma_\lambda}(\overline{\tau})(P_h u - \overline{\tau}), \boldsymbol{\mu}) \\ &= (a(u)(\boldsymbol{R}_h\boldsymbol{\lambda} - \boldsymbol{\lambda}), \boldsymbol{\mu}) + (b(u)(\boldsymbol{R}_h\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t), \boldsymbol{\mu}) - (\boldsymbol{\Pi}_h\boldsymbol{\sigma} - \boldsymbol{\sigma}, \boldsymbol{\mu}) \\ &\quad + (\boldsymbol{\Gamma_\lambda}(\tau)(P_h u - u), \boldsymbol{\mu}) - ((\widetilde{a}_{uu}(u_\tau)\boldsymbol{\lambda} + \widetilde{b}_{uu}(u_\tau)\boldsymbol{\lambda}_t)(u-\tau)^2, \boldsymbol{\mu}) \\ &\quad + (\widetilde{a}_u(u_\tau)(u-\tau)(\boldsymbol{\lambda}-\boldsymbol{\eta}), \boldsymbol{\mu}) + (\widetilde{b}_u(u_\tau)(u-\tau)(\boldsymbol{\lambda}_t - \boldsymbol{\eta}_t), \boldsymbol{\mu}), \qquad \forall \boldsymbol{\mu}\in \boldsymbol{\Lambda_h}, \end{aligned}$$

$$\begin{aligned} &(P_h u_t - \overline{\tau}_t, v) + (\nabla\cdot(\boldsymbol{\Pi}_h\boldsymbol{\sigma} - \overline{\boldsymbol{\rho}}), v) + (f(\overline{\tau}) - f(P_h u), v) \\ &= (f(u) - f(P_h u), v), \quad \forall v \in V_h, \end{aligned} \tag{3.15}$$

with $\overline{\tau}(0) = P_h(u_0(x))$, $\overline{\boldsymbol{\eta}}(0) = \boldsymbol{R}_h(\nabla u_0(x))$ and $\boldsymbol{\Gamma_\lambda}(\overline{\tau}) = a'(\overline{\tau})\boldsymbol{\lambda} + b'(\overline{\tau})\boldsymbol{\lambda}_t$. Now we let

$$\begin{aligned} &e_u = P_h u - u, \quad \overline{e}_u = P_h u - \overline{\tau}, \quad e_u^h = P_h u - u_h, \\ &\boldsymbol{e_\lambda} = \boldsymbol{R}_h\boldsymbol{\lambda} - \boldsymbol{\lambda}, \quad \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}} = \boldsymbol{R}_h\boldsymbol{\lambda} - \overline{\boldsymbol{\eta}}, \quad \boldsymbol{e}_{\boldsymbol{\lambda}}^h = \boldsymbol{R}_h\boldsymbol{\lambda} - \boldsymbol{\lambda}_h, \\ &\boldsymbol{e_\sigma} = \boldsymbol{\Pi}_h\boldsymbol{\sigma} - \boldsymbol{\sigma}, \quad \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}} = \boldsymbol{\Pi}_h\boldsymbol{\sigma} - \overline{\boldsymbol{\rho}}, \quad \boldsymbol{e}_{\boldsymbol{\sigma}}^h = \boldsymbol{\Pi}_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h. \end{aligned}$$

Theorem 3.1. *The existence of a semidiscrete solution* $(u_h, \boldsymbol{\lambda}_h, \boldsymbol{\sigma}_h) \in V_h \times \boldsymbol{\Lambda}_h \times \boldsymbol{W}_h$ *of* (3.7)–(3.9) *is equivalent to the existence of a fixed point of* $\boldsymbol{\Phi}$.

Proof. Suppose that there exists a solution $(u_h, \boldsymbol{\lambda}_h, \boldsymbol{\sigma}_h) \in V_h \times \boldsymbol{\Lambda}_h \times \boldsymbol{W}_h$ of (3.7)–(3.9). Now we subtract (3.7) from (2.3), (3.8) from (2.4), and (3.9) from (2.5), respectively and apply (3.12) to get the followings:

$$(\boldsymbol{\lambda} - \boldsymbol{\lambda}_h, \boldsymbol{w}) - (u - u_h, \nabla \cdot \boldsymbol{w}) = 0, \quad \forall\, \boldsymbol{w} \in \boldsymbol{W}_h, \tag{3.16}$$

$$\begin{aligned}
&(a(u)(\boldsymbol{\lambda} - \boldsymbol{\lambda}_h), \boldsymbol{\mu}) + (b(u)(\boldsymbol{\lambda}_t - (\boldsymbol{\lambda}_h)_t), \boldsymbol{\mu}) - (\boldsymbol{\sigma} - \boldsymbol{\sigma_h}, \boldsymbol{\mu}) \\
&\quad + (\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(u_h)(u - u_h), \boldsymbol{\mu}) \\
&= (\widetilde{a}_u(u_{u_h})(u - u_h)(\boldsymbol{\lambda} - \boldsymbol{\lambda_h}), \boldsymbol{\mu}) + (\widetilde{b}_u(u_{u_h})(u - u_h)(\boldsymbol{\lambda}_t - (\boldsymbol{\lambda}_h)_t), \boldsymbol{\mu}) \\
&\quad - ((\widetilde{a}_{uu}(u_{u_h})\boldsymbol{\lambda} + \widetilde{b}_{uu}(u_{u_h})\boldsymbol{\lambda}_t)(u - u_h)^2, \boldsymbol{\mu}), \quad \forall\, \boldsymbol{\mu} \in \boldsymbol{\Lambda}_h,
\end{aligned} \tag{3.17}$$

$$(u_t - (u_h)_t, v) + (\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), v) = (f(u) - f(u_h), v), \quad \forall\, v \in V_h. \tag{3.18}$$

By comparing the above equations with (3.13)-(3.15) and applying (3.5), (3.3) and (3.1), we prove that a semidiscrete solution $(u_h, \boldsymbol{\lambda}_h, \boldsymbol{\sigma}_h)$ is a fixed point of $\boldsymbol{\Phi}$. Reversely now we suppose that there exists a fixed point of $\boldsymbol{\Phi}$, $(\overline{\tau}, \overline{\boldsymbol{\eta}}, \overline{\boldsymbol{\rho}}) \in V_h \times \boldsymbol{\Lambda}_h \times \boldsymbol{W}_h$. From (3.13) we have the following equality

$$\begin{aligned}
(\overline{\boldsymbol{\eta}}, \boldsymbol{w}) - (\overline{\tau}, \nabla \cdot \boldsymbol{w}) &= (\boldsymbol{R}_h \boldsymbol{\lambda}, \boldsymbol{w}) - (P_h u, \nabla \cdot \boldsymbol{w}) \\
&= (\boldsymbol{\lambda}, \boldsymbol{w}) - (u, \nabla \cdot \boldsymbol{w}) = 0, \ \forall\, \boldsymbol{w} \in \boldsymbol{W}_h,
\end{aligned} \tag{3.19}$$

which proves that $(\overline{\boldsymbol{\eta}}, \overline{\tau})$ satisfies (3.7). Then by (3.14) we have

$$\begin{aligned}
&(a(u)\overline{\boldsymbol{\eta}}, \boldsymbol{\mu}) + (b(u)\overline{\boldsymbol{\eta}}_t, \boldsymbol{\mu}) - (\overline{\boldsymbol{\rho}}, \boldsymbol{\mu}) + (\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\overline{\tau})\overline{\tau}, \boldsymbol{\mu}) \\
&= (a(u)\boldsymbol{\lambda}, \boldsymbol{\mu}) + (b(u)\boldsymbol{\lambda}_t, \boldsymbol{\mu}) - (\boldsymbol{\sigma}, \boldsymbol{\mu}) + (\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\overline{\tau})u, \boldsymbol{\mu}) \\
&\quad + ((\widetilde{a}_{uu}(u_{\overline{\tau}})\boldsymbol{\lambda} + \widetilde{b}_{uu}(u_{\overline{\tau}})\boldsymbol{\lambda}_t)(u - \overline{\tau})^2, \boldsymbol{\mu}) - (\widetilde{a}_u(u_{\overline{\tau}})(u - \overline{\tau})(\boldsymbol{\lambda} - \overline{\boldsymbol{\eta}}), \boldsymbol{\mu}) \\
&\quad - (\widetilde{b}_u(u_{\overline{\tau}})(u - \overline{\tau})(\boldsymbol{\lambda}_t - \overline{\boldsymbol{\eta}}_t), \boldsymbol{\mu}), \quad \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}_h.
\end{aligned}$$

By adding $(a(\overline{\tau})\overline{\boldsymbol{\eta}}, \boldsymbol{\mu}) + (b(\overline{\tau})\overline{\boldsymbol{\eta}}_t, \boldsymbol{\mu})$ in both sides of the above equation and applying the definition of $\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\overline{\tau})$, we obtain

$$\begin{aligned}
&(a(\overline{\tau})\overline{\boldsymbol{\eta}}, \boldsymbol{\mu}) + (b(\overline{\tau})\overline{\boldsymbol{\eta}}_t, \boldsymbol{\mu}) - (\overline{\boldsymbol{\rho}}, \boldsymbol{\mu}) \\
&= (a(u)\boldsymbol{\lambda}, \boldsymbol{\mu}) + (b(u)\boldsymbol{\lambda}_t, \boldsymbol{\mu}) - (\boldsymbol{\sigma}, \boldsymbol{\mu}) + (a'(\overline{\tau})\boldsymbol{\lambda} + b'(\overline{\tau})\boldsymbol{\lambda}_t)(u - \overline{\tau}), \boldsymbol{\mu}) \\
&\quad - (\widetilde{a}_u(u_{\overline{\tau}})(u - \overline{\tau})(\boldsymbol{\lambda} - \overline{\boldsymbol{\eta}}), \boldsymbol{\mu}) - (\widetilde{b}_u(u_{\overline{\tau}})(u - \overline{\tau})(\boldsymbol{\lambda}_t - \overline{\boldsymbol{\eta}}_t), \boldsymbol{\mu}) \\
&\quad + ((\widetilde{a}_{uu}(u_{\overline{\tau}})\boldsymbol{\lambda} + \widetilde{b}_{uu}(u_{\overline{\tau}})\boldsymbol{\lambda}_t)(u - \overline{\tau})^2, \boldsymbol{\mu}) \\
&\quad + (a(\overline{\tau})\overline{\boldsymbol{\eta}}, \boldsymbol{\mu}) + (b(\overline{\tau})\overline{\boldsymbol{\eta}}_t, \boldsymbol{\mu}) - (a(u)\overline{\boldsymbol{\eta}}, \boldsymbol{\mu}) - (b(u)\overline{\boldsymbol{\eta}}_t, \boldsymbol{\mu}).
\end{aligned} \tag{3.20}$$

From (3.11) we get the followings

$$\begin{aligned}
&a'(\overline{\tau})\boldsymbol{\lambda}(u - \overline{\tau}) + \widetilde{a}_{uu}(u_{\overline{\tau}})\boldsymbol{\lambda}(u - \overline{\tau})^2 - \widetilde{a}_u(u_{\overline{\tau}})(u - \overline{\tau})\boldsymbol{\lambda} = 0, \\
&b'(\overline{\tau})\boldsymbol{\lambda}_t(u - \overline{\tau}) + \widetilde{b}_{uu}(u_{\overline{\tau}})\boldsymbol{\lambda}_t(u - \overline{\tau})^2 - \widetilde{b}_u(u_{\overline{\tau}})(u - \overline{\tau})\boldsymbol{\lambda}_t = 0, \\
&(a(u) - a(\overline{\tau}))\overline{\boldsymbol{\eta}} + (b(u) - b(\overline{\tau}))\overline{\boldsymbol{\eta}}_t = \widetilde{a}_u(u_{\overline{\tau}})(u - \overline{\tau})\overline{\boldsymbol{\eta}} + \widetilde{b}_u(u_{\overline{\tau}})(u - \overline{\tau})\overline{\boldsymbol{\eta}}_t.
\end{aligned}$$

By applying the above equalities, (2.4) and (3.20) we have

$$(3.21)\quad (a(\overline{\tau})\overline{\boldsymbol{\eta}},\boldsymbol{\mu})+(b(\overline{\tau})\overline{\boldsymbol{\eta}}_t,\boldsymbol{\mu})-(\overline{\boldsymbol{\rho}},\boldsymbol{\mu}) = (a(u)\boldsymbol{\lambda},\boldsymbol{\mu})+(b(u)\boldsymbol{\lambda}_t,\boldsymbol{\mu})-(\boldsymbol{\sigma},\boldsymbol{\mu}) = 0,$$

which shows that $(\overline{\tau},\overline{\boldsymbol{\eta}},\overline{\boldsymbol{\rho}})$ satisfies (3.8). By applying (3.3), (3.1) and (2.5) to (3.15) we have

(3.22)

$$(\overline{\tau}_t, v) + (\nabla\cdot\overline{\boldsymbol{\rho}}, v) = (u_t, v) + (\nabla\cdot\boldsymbol{\sigma}, v) - (f(u)-f(\overline{\tau}), v) = (f(\overline{\tau}), v),$$

which proves that $(\overline{\tau},\overline{\boldsymbol{\rho}})$ satisfies (3.9). Therefore we conclude that the fixed point $(\overline{\tau},\overline{\boldsymbol{\eta}},\overline{\boldsymbol{\rho}}) \in V_h \times \boldsymbol{\Lambda}_h \times \boldsymbol{W}_h$ of $\boldsymbol{\Phi}$ is the solution of (3.7)–(3.9). □

Theorem 3.2. *Suppose that the functions a and b satisfy Conditions* 1 *and* 2*, respectively and f satisfies Condition 3. If $u \in L^2(W^{3,\theta}), u_t \in L^2(W^{3,\theta})$ and $\boldsymbol{\sigma} \in L^2(\boldsymbol{W}^{2,\theta})$, then there exist u_h, $\boldsymbol{\lambda}_h$, and $\boldsymbol{\sigma}_h$ satisfying* (3.7)–(3.9).

Proof. To prove the existence of semidiscrete approximations we will prove that $\boldsymbol{\Phi}$ maps a ball of $\widetilde{V}_h \times \widetilde{\boldsymbol{\Lambda}}_h \times \widetilde{\boldsymbol{W}}_h$ onto itself. We choose a constant δ such that $5Ch^{2-\frac{d}{2+\varepsilon}} < \delta < \frac{1}{5C}h^{\frac{d}{2+\varepsilon}}$. Now we let $(\tau,\boldsymbol{\eta},\boldsymbol{\rho}) \in B_\delta = \{(\tau,\boldsymbol{\eta},\boldsymbol{\rho}) \in \widetilde{V}_h \times \widetilde{\boldsymbol{\Lambda}}_h \times \widetilde{\boldsymbol{W}}_h \mid \|P_h u - \tau\|_{L^\infty(L^\theta)} < \delta,\ \|\boldsymbol{R}_h\boldsymbol{\lambda} - \boldsymbol{\eta}\|_{L^\infty(\boldsymbol{L}^{2+\varepsilon})} + \|\boldsymbol{R}_h\boldsymbol{\lambda}_t - \boldsymbol{\eta}_t\|_{L^\infty(\boldsymbol{L}^{2+\varepsilon})} < \delta,\ \|\boldsymbol{\Pi}_h\boldsymbol{\sigma} - \boldsymbol{\rho}\|_{L^\infty(\boldsymbol{L}^2)} < \delta\}$. If we let $\boldsymbol{\Phi}(\tau,\boldsymbol{\eta},\boldsymbol{\rho}) = (\overline{\tau},\overline{\boldsymbol{\eta}},\overline{\boldsymbol{\rho}})$, then we need to show that

$$\begin{aligned}
&\|P_h u - \overline{\tau}\|_{L^\infty(L^\theta)} < \delta,\\
&\|\boldsymbol{R}_h\boldsymbol{\lambda} - \overline{\boldsymbol{\eta}}\|_{L^\infty(\boldsymbol{L}^{2+\varepsilon})} + \|(\boldsymbol{R}_h\boldsymbol{\lambda})_t - \overline{\boldsymbol{\eta}}_t\|_{L^\infty(\boldsymbol{L}^{2+\varepsilon})} < \delta,\\
&\|\boldsymbol{\Pi}_h\boldsymbol{\sigma} - \overline{\boldsymbol{\rho}}\|_{L^\infty(\boldsymbol{L}^2)} < \delta
\end{aligned}$$

hold for $(\tau,\boldsymbol{\eta},\boldsymbol{\rho}) \in B_\delta$. Now we take $\boldsymbol{\mu} = \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}$ in (3.14) to get

$$\begin{aligned}
(3.23)\quad &(a(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}},\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}) + (b(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t},\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}) - (\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}},\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}) + (\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\overline{\tau})(\overline{e}_u),\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})\\
&= (a(u)\boldsymbol{e}_{\boldsymbol{\lambda}},\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}) + (b(u)\boldsymbol{e}_{\boldsymbol{\lambda}t},\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}) - (\boldsymbol{e}_{\boldsymbol{\sigma}},\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}) + (\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\tau)(e_u),\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})\\
&\quad - ((\widetilde{a}_{uu}(u_\tau)\boldsymbol{\lambda} + \widetilde{b}_{uu}(u_\tau)\boldsymbol{\lambda}_t)(u-\tau)^2,\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}) + (\widetilde{a}_u(u_\tau)(u-\tau)(\boldsymbol{\lambda}-\boldsymbol{\eta}),\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})\\
&\quad + (\widetilde{b}_u(u_\tau)(u-\tau)(\boldsymbol{\lambda}_t-\boldsymbol{\eta}_t),\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}).
\end{aligned}$$

Taking $\boldsymbol{w} = \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}$ in (3.13) we get $(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}},\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}) - (\overline{e}_u, \nabla\cdot\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}) = 0$. From this result and (3.15) with $v = \overline{e}_u$, we have $(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}},\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}) = -((\overline{e}_u)_t,\overline{e}_u) + (f(u)-f(\overline{\tau}),\overline{e}_u)$. By applying this equality to (3.23) we get

(3.24)

$$\begin{aligned}
&(a(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}},\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}) + (b(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t},\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}) + ((\overline{e}_u)_t,\overline{e}_u)\\
=\ & -(\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\overline{\tau})(\overline{e}_u),\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}) + (f(u)-f(\overline{\tau}),\overline{e}_u) + (a(u)\boldsymbol{e}_{\boldsymbol{\lambda}},\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}) + (b(u)\boldsymbol{e}_{\boldsymbol{\lambda}t},\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})\\
&- (\boldsymbol{e}_{\boldsymbol{\sigma}},\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}) + (\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\tau)(e_u),\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}) - ((\widetilde{a}_{uu}(u_\tau)\boldsymbol{\lambda} + \widetilde{b}_{uu}(u_\tau)\boldsymbol{\lambda}_t)(u-\tau)^2,\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})\\
&+ (\widetilde{a}_u(u_\tau)(u-\tau)(\boldsymbol{\lambda}-\boldsymbol{\eta}),\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}) + (\widetilde{b}_u(u_\tau)(u-\tau)(\boldsymbol{\lambda}_t-\boldsymbol{\eta}_t),\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}),\quad \forall\, t \in (0,T].
\end{aligned}$$

To continue our proof, we temporarily assume that

$$(3.25)\qquad \|\overline{e}_u(t)\|_{L^\infty} < K_2/2,\ \|e_u(t)\|_{L^\infty} < K_2/2,\ \forall t \in [0,T]$$

hold for a sufficiently small h. Then we get

$$\int |f(u)-f(\overline{\tau})||\overline{e}_u|dx \leq \int (|f(u)-f(P_hu)|+|f(P_hu)-f(\overline{\tau})|)|\overline{e}_u|dx$$
$$\leq C(K_2)(\|e_u\|^2+\|\overline{e}_u\|^2).$$

By applying this result to (3.24) we have

$$a_*\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|^2+\frac{1}{2}\frac{d}{dt}(b(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}},\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})-\frac{1}{2}\left(\frac{d}{dt}(b(u))\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}},\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\right)+\frac{1}{2}\frac{d}{dt}\|\overline{e}_u\|^2$$
$$\leq C\Big\{\|\overline{e}_u\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|+\|e_u\|^2+\|\overline{e}_u\|^2+\|\boldsymbol{e}_{\boldsymbol{\lambda}}\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|+\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|$$
$$+\|\boldsymbol{e}_{\boldsymbol{\sigma}}\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|+\|e_u\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|+\|(u-\tau)^2\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|+\|(u-\tau)(\boldsymbol{\lambda}-\boldsymbol{\eta})\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|$$
$$+\|(u-\tau)(\boldsymbol{\lambda}_t-\boldsymbol{\eta}_t)\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|\Big\},\quad \forall\, t\in(0,T].$$

From Lemma 3.1, Lemma 3.2, (3.2), (3.4) and (3.6), we have

$$(3.26)\quad 2a_*\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|^2+\frac{d}{dt}\|\sqrt{b(u)}(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})\|^2+\frac{d}{dt}\|\overline{e}_u\|^2$$
$$\leq C\Big\{\|\sqrt{b(u)}(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})\|^2+\|e_u\|^2+\|\overline{e}_u\|^2+\|\boldsymbol{e}_{\boldsymbol{\lambda}}\|^2+\|\boldsymbol{e}_{\boldsymbol{\lambda}t}\|^2+\|\boldsymbol{e}_{\boldsymbol{\sigma}}\|^2+\|e_u\|_4^4$$
$$+\|P_hu-\tau\|_4^4+\|e_u\|_\theta^4+\|P_hu-\tau\|_\theta^4+\|\boldsymbol{e}_{\boldsymbol{\lambda}}\|_{2+\varepsilon}^4+\|\boldsymbol{R}_h\boldsymbol{\lambda}-\boldsymbol{\eta}\|_{2+\varepsilon}^4$$
$$+\|\boldsymbol{e}_{\boldsymbol{\lambda}t}\|_{2+\varepsilon}^4+\|\boldsymbol{R}_h\boldsymbol{\lambda}_t-\boldsymbol{\eta}_t\|_{2+\varepsilon}^4\Big\}.$$

Since $\theta>4$, we have

$$\|P_hu-\tau\|_4\leq C\|P_hu-\tau\|_\theta\leq C\delta,\ \|e_u\|_\theta\leq Ch\|u\|_{1,\theta},$$
$$\|\boldsymbol{e}_{\boldsymbol{\lambda}}\|_{2+\varepsilon}\leq Ch\|\boldsymbol{\lambda}\|_{1,2+\varepsilon}\leq Ch\|u\|_{2,2+\varepsilon}.$$

By applying the above estimations to (3.26), we get

(3.27)

$$2a_*\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|^2+\frac{d}{dt}\big(\|\sqrt{b(u)}(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})\|^2+\|\overline{e}_u\|^2\big)$$
$$\leq C\Big\{\|\sqrt{b(u)}(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})\|^2+h^4\|u\|_{2,2}^2+\|\overline{e}_u\|^2+h^4\|\boldsymbol{\lambda}\|_{2,2}^2+h^4\|\boldsymbol{\lambda}_t\|_{2,2}^2$$
$$+h^4\|\boldsymbol{\sigma}\|_{2,2}^2+h^4\|u\|_{1,4}^4+\delta^4+h^4\|u\|_{1,\theta}^4+Ch^4\|u\|_{2,2+\varepsilon}^4+h^4\|u_t\|_{2,2+\varepsilon}^4\Big\}$$
$$\leq C\Big\{\|\sqrt{b(u)}(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})\|^2+\|\overline{e}_u\|^2+h^4(\|u\|_{3,2}^2+\|u_t\|_{3,2}^2+\|\boldsymbol{\sigma}\|_{2,2}^2+\|u\|_{3,2}^4$$
$$+\|u_t\|_{3,2}^4)+\delta^4\Big\}.$$

Now we integrate both sides of the above inequality with respect to t from 0 to $\widetilde{t}\leq T$ to obtain

$$\|\sqrt{b(u)}(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})(\widetilde{t})\|^2+\|\overline{e}_u(\widetilde{t})\|^2$$
$$\leq C\int_0^{\widetilde{t}}(\|\sqrt{b(u)}(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})(s)\|^2+\|\overline{e}_u(s)\|^2)ds+C(h^4+\delta^4).$$

By applying the Gronwall inequality we get

$$\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|_{L^\infty(\boldsymbol{L}^2)} + \|\overline{e}_u\|_{L^\infty(L^2)} \le C(h^2+\delta^2). \tag{3.28}$$

Now by induction we prove that the hypothesis (3.25) holds. We assume that there exists $\widetilde{t} \in (0,T]$ such that $\|\overline{e}_u(t)\|_{L^\infty} < K_2/2$, $\forall\ 0 \le t < \widetilde{t}$ and $\|\overline{e}_u(\widetilde{t})\|_{L^\infty} \ge K_2/2$. Now we take a sequence $\{t_n\}$ such that $t_n \in [0,\widetilde{t})$, $\lim_{n\to\infty} t_n = \widetilde{t}$ and $\|\overline{e}_u(t_n)\|_{L^\infty} < K_2/2$. By following the procedures of the proof below (3.25) we have $\|\overline{e}_u(t_n)\|_{L^2} \le C(h^2+\delta^2)$. And by the continuity of $\|\cdot\|_{L^2}$ we have $\|\overline{e}_u(\widetilde{t})\| \le C(h^2+\delta^2)$. By applying the inverse inequality and the property of δ, we have

$$\|\overline{e}_u(\widetilde{t})\|_{L^\infty} \le Ch^{-\frac{d}{2}}\|\overline{e}_u(\widetilde{t})\|_{L^2} \le Ch^{-\frac{d}{2}}(h^2+\delta^2) < C(h^{2-\frac{d}{2}}+\delta^2 h^{-\frac{d}{2}}) < K_2/2,$$

which completes the proof of the first inequality of (3.25). By (3.4), the proof of the second inequality of (3.25) is trivial. By taking $\boldsymbol{\mu} = \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}$ in (3.14) we get

$$\begin{aligned}
&(a(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}) + (b(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}, \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}) - (\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}, \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}) + (\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\overline{\tau})(\overline{e}_u), \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}})\\
&= (a(u)\boldsymbol{e}_{\boldsymbol{\lambda}}, \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}) + (b(u)(\boldsymbol{e}_{\boldsymbol{\lambda}t}), \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}) - (\boldsymbol{e}_{\boldsymbol{\sigma}}, \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}) + (\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\tau)(e_u), \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}})\\
&\quad - ((\widetilde{a}_{uu}(u_\tau)\boldsymbol{\lambda} + \widetilde{b}_{uu}(u_\tau)\boldsymbol{\lambda}_t)(u-\tau)^2, \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}) + (\widetilde{a}_u(u_\tau)(u-\tau)(\boldsymbol{\lambda}-\boldsymbol{\eta}), \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}})\\
&\quad + (\widetilde{b}_u(u_\tau)(u-\tau)(\boldsymbol{\lambda}_t - \boldsymbol{\eta}_{\boldsymbol{t}}), \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}),
\end{aligned}$$

which yields that

$$\begin{aligned}
\|\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\|^2 \le C\Big[&\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|^2 + \|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\|^2 + \|\overline{e}_u\|^2 + \|\boldsymbol{e}_{\boldsymbol{\lambda}}\|^2 + \|\boldsymbol{e}_{\boldsymbol{\lambda}t}\|^2 + \|\boldsymbol{e}_{\boldsymbol{\sigma}}\|^2\\
&+ \|e_u\|^2 + \|(u-\tau)^2\|^2 + \|(u-\tau)(\boldsymbol{\lambda}-\boldsymbol{\eta})\|^2 + \|(u-\tau)(\boldsymbol{\lambda}_t - \boldsymbol{\eta}_t)\|^2\Big].
\end{aligned} \tag{3.29}$$

Now we choose $\boldsymbol{\mu} = \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}$ in (3.14). Then we have

$$\begin{aligned}
&(a(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}) + (b(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}, \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}) - (\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}, \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}) + (\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\overline{\tau})(\overline{e}_u), \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t})\\
&= (a(u)\boldsymbol{e}_{\boldsymbol{\lambda}}, \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}) + (b(u)\boldsymbol{e}_{\boldsymbol{\lambda}t}, \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}) - (\boldsymbol{e}_{\boldsymbol{\sigma}}, \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t})\\
&\quad + (\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\tau)(e_u), \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}) - ((\widetilde{a}_{uu}(u_\tau)\boldsymbol{\lambda} + \widetilde{b}_{uu}(u_\tau)\boldsymbol{\lambda}_t)(u-\tau)^2, \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t})\\
&\quad + (\widetilde{a}_u(\tau)(u-\tau)(\boldsymbol{\lambda}-\boldsymbol{\eta}), \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}) + (\widetilde{b}_u(u_\tau)(u-\tau)(\boldsymbol{\lambda}_t - \boldsymbol{\eta}_t), \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}).
\end{aligned} \tag{3.30}$$

Take $\boldsymbol{w} = \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}$ in (3.13) and $v = \nabla\cdot(\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}})$ in (3.15), respectively, to get

$$(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}, \overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}) = (\overline{e}_{ut}, \nabla\cdot(\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}})) = -(\nabla\cdot\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}, \nabla\cdot\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}) + (f(u)-f(\overline{\tau}), \nabla\cdot\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}).$$

By applying the above equality to (3.30), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\sqrt{a(u)}(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})\|^2) - \frac{1}{2}(\frac{d}{dt}(a(u))\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})\\
&\quad + (b(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}, \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}) + (\nabla\cdot\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}, \nabla\cdot\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}) + (\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\overline{\tau})(\overline{e}_u), \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t})\\
&=(f(u)-f(\overline{\tau}), \nabla\cdot\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}) + (a(u)\boldsymbol{e}_{\boldsymbol{\lambda}}, \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}) + (b(u)\boldsymbol{e}_{\boldsymbol{\lambda}t}, \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}) - (\boldsymbol{e}_{\boldsymbol{\sigma}}, \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t})\\
&\quad + (\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\tau)(e_u), \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}) - ((\widetilde{a}_{uu}(u_\tau)\boldsymbol{\lambda} + \widetilde{b}_{uu}(u_\tau)\boldsymbol{\lambda}_t)(u-\tau)^2, \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t})
\end{aligned} \tag{3.31}$$

$$+ (\widetilde{a}_u(\tau)(u-\tau)(\boldsymbol{\lambda}-\boldsymbol{\eta}), \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}) + (\widetilde{b}_u(u_\tau)(u-\tau)(\boldsymbol{\lambda}_t-\boldsymbol{\eta}_t), \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}),$$

which implies that

$$\begin{aligned} & b_*\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\|^2 + \frac{1}{2}\frac{d}{dt}\|\sqrt{a(u)}(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})\|^2 + \|\nabla\cdot\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\|^2 \\ \le & \left(\Big(\frac{1}{2}\frac{d}{dt}a(u)\Big)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\right) + C\Big[\|\overline{e}_u\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\| + \|u-\overline{\tau}\|\|\nabla\cdot\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\| + \|\boldsymbol{e}_{\boldsymbol{\lambda}}\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\| \\ & + \|\boldsymbol{e}_{\boldsymbol{\lambda}t}\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\| + \|\boldsymbol{e}_{\boldsymbol{\sigma}}\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\| + \|e_u\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\| + \|(u-\tau)^2\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\| \\ & + \|(u-\tau)(\boldsymbol{\lambda}-\boldsymbol{\eta})\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\| + \|(u-\tau)(\boldsymbol{\lambda}_t-\boldsymbol{\eta}_t)\|\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\|\Big]. \end{aligned}$$

By applying (3.27) and (3.28) to the above inequality we have

(3.32)

$$\begin{aligned} & b_*\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\|^2 + \frac{d}{dt}\|\sqrt{a(u)}(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})\|^2 + \|\nabla\cdot\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\|^2 \\ \le & C\Big[\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|^2 + \|\overline{e}_u\|^2 + \|e_u\|^2 + \|\boldsymbol{e}_{\boldsymbol{\lambda}}\|^2 + \|\boldsymbol{e}_{\boldsymbol{\lambda}t}\|^2 + \|\boldsymbol{e}_{\boldsymbol{\sigma}}\|^2 + \|e_u\|_4^4 + \|P_hu-\tau\|_4^4 \\ & + \|e_u\|_\theta^4 + \|P_hu-\tau\|_\theta^4 + \|\boldsymbol{e}_{\boldsymbol{\lambda}}\|_{2+\varepsilon}^4 + \|\boldsymbol{R}_h\boldsymbol{\lambda}-\boldsymbol{\eta}\|_{2+\varepsilon}^4 + \|\boldsymbol{e}_{\boldsymbol{\lambda}t}\|_{2+\varepsilon}^4 + \|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\|_{2+\varepsilon}^4\Big] \\ \le & C(h^4+\delta^4). \end{aligned}$$

By simple computation we get

$$\begin{aligned} \frac{d}{dt}\|\sqrt{a(u)}(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}})\|^2 &= \frac{d}{dt}\int a(u)|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}|^2dx = \int\frac{d}{dt}(a(u))|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}|^2dx + (2a(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}) \\ &\ge -\overline{C}\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|^2 - \int\Big(\frac{2}{b^*}(a(u))^2|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}|^2 + \frac{b^*}{2}|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}|^2\Big)dx \end{aligned} \tag{3.33}$$

for some constant $\overline{C} > 0$. Now we apply (3.33) to (3.32) to get

$$\begin{aligned} & b_*\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\|^2 - \overline{C}\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|^2 - \int\Big(\frac{2}{b^*}a(u)^2(|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}|)^2 + \frac{b^*}{2}(|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}|)^2\Big)dx + \|\nabla\cdot(\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}})\|^2 \\ & \le C(h^4+\delta^4), \end{aligned}$$

which by (3.28) implies that

$$\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\|^2 + \|\nabla\cdot\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\|^2 \le C(\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|^2 + h^4 + \delta^4) \le C(h^4+\delta^4). \tag{3.34}$$

By applying (3.28), (3.34) and (3.32) in (3.29), we get $\|\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\| \le C(h^2+\delta^2)$. From the inverse inequality, (3.28) and (3.34) we have

$$\begin{aligned} \|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|_{2+\varepsilon} &\le Ch^{d(\frac{1}{2+\varepsilon}-\frac{1}{2})}\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\| \le Ch^{-\frac{\varepsilon d}{2(2+\varepsilon)}}(h^2+\delta^2), \\ \|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\|_{2+\varepsilon} &\le Ch^{d(\frac{1}{2+\varepsilon}-\frac{1}{2})}\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\| \le Ch^{-\frac{\varepsilon d}{2(2+\varepsilon)}}(h^2+\delta^2). \end{aligned}$$

Since $5Ch^{2-\frac{d}{2+\varepsilon}} < \delta < \frac{1}{5C}h^{\frac{d}{2+\varepsilon}}$ and $d-2<\varepsilon<2$, we have

$$Ch^{-\frac{\varepsilon d}{2(2+\varepsilon)}}(h^2+\delta^2) \le C\big(h^{\frac{(2-\varepsilon)d}{2(2+\varepsilon)}}h^{2-\frac{d}{2+\varepsilon}} + h^{-\frac{\varepsilon d}{2(2+\varepsilon)}}h^{\frac{d}{2+\varepsilon}}\delta\big) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Now we need to prove that $\|\overline{e}_u\|_{L^\infty(L^{\frac{4+2\varepsilon}{\varepsilon}})} < \delta$. Let $r = f(u) - f(\overline{\tau})$ and

$$\boldsymbol{\zeta} = a(u)\boldsymbol{e_\lambda} + b(u)\boldsymbol{e_{\lambda}}_t - \boldsymbol{e_\sigma} + \boldsymbol{\Gamma_\lambda}(\tau)(e_u) - (\widetilde{a}_{uu}(u_\tau)\boldsymbol{\lambda} + \widetilde{b}_{uu}(u_\tau)\boldsymbol{\lambda}_t)(u-\tau)^2$$
$$+ \widetilde{a}_u(u_\tau)(u-\tau)(\boldsymbol{\lambda} - \boldsymbol{\eta}) + \widetilde{b}_u(u_\tau)(u-\tau)(\boldsymbol{\lambda}_t - \boldsymbol{\eta}_t).$$

From (3.13), (3.14) and (3.15) we have

$$(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, \boldsymbol{w}) - (\overline{e}_u, \nabla\cdot\boldsymbol{w}) = 0, \qquad \forall\, \boldsymbol{w} \in \boldsymbol{W}_h, \tag{3.35}$$

$$(a(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, \boldsymbol{\mu}) + (b(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}, \boldsymbol{\mu}) - (\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}, \boldsymbol{\mu}) + (\boldsymbol{\Gamma_\lambda}(\overline{\tau})\overline{e}_u, \boldsymbol{\mu}) = (\boldsymbol{\zeta}, \boldsymbol{\mu}),\ \forall\, \boldsymbol{\mu} \in \boldsymbol{\Lambda}_h, \tag{3.36}$$

$$((\overline{e}_u)_t, v) + (\nabla\cdot\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}, v) = (r, v), \qquad \forall\, v \in V_h. \tag{3.37}$$

For $0 < t < T$ and $\psi(t) \in L^{\theta'}(\Omega)$, we let $\phi(t) \in W^{2,\theta'}(\Omega)$ be the solution of

$$\begin{cases} M^*\phi = \psi \quad \text{in } \Omega, \\ (a(u)\nabla\phi - (b(u)\nabla\phi)_t)\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \\ \phi(T) = 0 \quad \text{on } \partial\Omega, \end{cases}$$

with $M^*\phi = -\nabla\cdot(a(u)\nabla\phi - (b(u)\nabla\phi)_t) - \boldsymbol{\Gamma_\lambda}(u)\cdot\nabla\phi$. Then $\|\phi(t)\|_{2,\theta'} \le C\|\psi(t)\|_{\theta'}$ and $\|\phi_t(t)\|_{2,\theta'} \le C\|\psi(t)\|_{\theta'}$. By the definition of ϕ, (3.35) and (3.36) we get

(3.38)

$$\begin{aligned}
&(\overline{e}_u, \psi) = (\overline{e}_u, M^*\phi) \\
&= (\overline{e}_u, -\nabla\cdot(a(u)\nabla\phi - (b(u)\nabla\phi)_t) - \boldsymbol{\Gamma_\lambda}(u)\cdot\nabla\phi) \\
&= -(\overline{e}_u, \nabla\cdot\boldsymbol{\Pi}_h(a(u)\nabla\phi - (b(u)\nabla\phi)_t)) - (\boldsymbol{\Gamma_\lambda}(u)\overline{e}_u, \nabla\phi) \\
&= -(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, \boldsymbol{\Pi}_h(a(u)\nabla\phi - (b(u)\nabla\phi)_t)) - (\boldsymbol{\Gamma_\lambda}(u)\overline{e}_u, \nabla\phi - \boldsymbol{R}_h\nabla\phi) \\
&\quad - (\boldsymbol{\Gamma_\lambda}(u)\overline{e}_u, \boldsymbol{R}_h\nabla\phi) \\
&= (\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, a(u)\nabla\phi - (b(u)\nabla\phi)_t - \boldsymbol{\Pi}_h(a(u)\nabla\phi - (b(u)\nabla\phi)_t)) \\
&\quad - (\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, a(u)\nabla\phi - (b(u)\nabla\phi)_t) - (\boldsymbol{\Gamma_\lambda}(u)\overline{e}_u, \nabla\phi - \boldsymbol{R}_h\nabla\phi) \\
&\quad - (\boldsymbol{\Gamma_\lambda}(u)\overline{e}_u, \boldsymbol{R}_h\nabla\phi) \\
&= (\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, a(u)\nabla\phi - (b(u)\nabla\phi)_t - \boldsymbol{\Pi}_h(a(u)\nabla\phi - (b(u)\nabla\phi)_t)) \\
&\quad - (a(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, \nabla\phi - \boldsymbol{R}_h\nabla\phi) - (a(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, \boldsymbol{R}_h\nabla\phi) + (\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, (b(u)\nabla\phi)_t) \\
&\quad - (\boldsymbol{\Gamma_\lambda}(u)\overline{e}_u, \nabla\phi - \boldsymbol{R}_h\nabla\phi) - (\boldsymbol{\Gamma_\lambda}(u)\overline{e}_u, \boldsymbol{R}_h\nabla\phi) \\
&= (\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, a(u)\nabla\phi - (b(u)\nabla\phi)_t - \boldsymbol{\Pi}_h(a(u)\nabla\phi - (b(u)\nabla\phi)_t)) \\
&\quad - (a(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, \nabla\phi - \boldsymbol{R}_h\nabla\phi) - (a(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, \boldsymbol{R}_h\nabla\phi) + \frac{d}{dt}(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, b(u)\nabla\phi) \\
&\quad - (b(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}, \nabla\phi - \boldsymbol{R}_h\nabla\phi) - (b(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}, \boldsymbol{R}_h\nabla\phi) - (\boldsymbol{\Gamma_\lambda}(u)\overline{e}_u, \nabla\phi - \boldsymbol{R}_h\nabla\phi) \\
&\quad - (\boldsymbol{\Gamma_\lambda}(u)\overline{e}_u, \boldsymbol{R}_h\nabla\phi) \\
&= (\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, a(u)\nabla\phi - (b(u)\nabla\phi)_t - \boldsymbol{\Pi}_h(a(u)\nabla\phi - (b(u)\nabla\phi)_t))
\end{aligned}$$

$$
\begin{aligned}
&-(a(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, \nabla\phi - \boldsymbol{R}_h\nabla\phi) - (\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}, \boldsymbol{R}_h\nabla\phi) + (\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\overline{\tau})\overline{e}_u, \boldsymbol{R}_h\nabla\phi)\\
&-(\boldsymbol{\zeta}, \boldsymbol{R}_h\nabla\phi) + \frac{d}{dt}(\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, b(u)\nabla\phi) - (b(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}, \nabla\phi - \boldsymbol{R}_h\nabla\phi)\\
&-(\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(u)\overline{e}_u, \nabla\phi - \boldsymbol{R}_h\nabla\phi) - (\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(u)\overline{e}_u, \boldsymbol{R}_h\nabla\phi) := \sum_{i=1}^{9} I_i.
\end{aligned}
$$

I_1 can be estimated as follow:

$$
\begin{aligned}
&I_1\\
&\le Ch\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|_\theta\|a(u)\nabla\phi - (b(u)\nabla\phi)_t\|_{1,\theta'}\\
&\le Ch(\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|_\theta\|a(u)\|_{1,\infty}\|\nabla\phi\|_{1,\theta'} + \|b_u(u)u_t\|_{1,\infty}\|\nabla\phi\|_{1,\theta'} + \|b(u)\|_{1,\infty}\|\nabla\phi_t\|_{1,\theta'})\\
&\le Ch^{1+d(\frac{1}{\theta}-\frac{1}{2})}\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|(\|\phi\|_{2,\theta'} + \|\phi_t\|_{2,\theta'}) \le h^{1+d(\frac{1}{\theta}-\frac{1}{2})}(h^2+\delta^2)\|\psi\|_{\theta'}.
\end{aligned}
$$

By the inverse inequality and (3.6), I_2, I_3, I_5, I_7 and I_8 can be estimated in the following ways:

$$
\begin{aligned}
I_2 &\le \|a(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|_\theta\|\nabla\phi - \boldsymbol{R}_h\nabla\phi\|_{\theta'} \le Ch\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|_\theta\|\nabla\phi\|_{1,\theta'}\\
&\le Ch^{1+d(\frac{1}{\theta}-\frac{1}{2})}\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}\|\|\psi\|_{\theta'} \le h^{1+d(\frac{1}{\theta}-\frac{1}{2})}(h^2+\delta^2)\|\psi\|_{\theta'},\\
I_3 &\le \|\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\|_\theta\|\boldsymbol{R}_h\nabla\phi\|_{\theta'} \le C\|\overline{\boldsymbol{e}}_{\boldsymbol{\sigma}}\|_\theta\|\phi\|_{2,\theta'} \le Ch^{-\frac{d}{2+\varepsilon}}(h^2+\delta^2)\|\psi\|_{\theta'},\\
I_5 &\le C\|\boldsymbol{\zeta}\|_\theta\|\phi\|_{1,\theta'} \le C\|\boldsymbol{\zeta}\|_\theta\|\phi\|_{2,\theta'} \le C\|\boldsymbol{\zeta}\|_\theta\|\psi\|_{\theta'},\\
I_7 &\le Ch^{1+d(\frac{1}{\theta}-\frac{1}{2})}\|\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}t}\|\|\psi\|_{\theta'} \le Ch^{1+d(\frac{1}{\theta}-\frac{1}{2})}(h^2+\delta^2)\|\psi\|_{\theta'},\\
I_8 &\le Ch\|\overline{e}_u\|_\theta\|\psi\|_{\theta'} \le Ch^{1+d(\frac{1}{\theta}-\frac{1}{2})}(h^2+\delta^2)\|\psi\|_{\theta'}.
\end{aligned}
$$

By applying the integration by parts with respect to x to I_6 we have

$$
\begin{aligned}
I_6 &= \frac{d}{dt}(b(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, \nabla\phi) = -\frac{d}{dt}(\nabla\cdot(b(u)\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}), \phi)\\
&= -\frac{d}{dt}(b'(u)\nabla u\cdot\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}} + b(u)\nabla\cdot\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}, \phi).
\end{aligned}
$$

By applying (3.28) and taking into account $\overline{e}_u = P_h u - \overline{\tau}$ we get

$$
\begin{aligned}
I_4 + I_9 &= \Big((\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\overline{\tau}) - \boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(u))\overline{e}_u, \boldsymbol{R}_h\nabla\phi\Big)\\
&= \Big(\{(a'(\overline{\tau}) - a'(u))\boldsymbol{\lambda} + (b'(\overline{\tau}) - b'(u))\boldsymbol{\lambda}_t\}\overline{e}_u, \boldsymbol{R}_h\nabla\phi\Big)\\
&\le C(\|(\overline{\tau} - P_h u)\overline{e}_u\|_\theta + \|(P_h u - u)\overline{e}_u\|_\theta)\|\boldsymbol{R}_h\nabla\phi\|_{\theta'}\\
&\le C(\|\overline{e}_u^2\|_\theta + h\|u\|_{1,\infty}\|\overline{e}_u\|_\theta)\|\nabla\phi\|_{\theta'}\\
&\le C(\|\overline{e}_u\|_{2\theta}^2 + hh^{d(\frac{1}{\theta}-\frac{1}{2})}\|\overline{e}_u\|)\|\psi\|_{\theta'}\\
&\le C(h^{2d(\frac{1}{2\theta}-\frac{1}{2})}(h^2+\delta^2)^2 + h^{1+d(\frac{1}{\theta}-\frac{1}{2})}(h^2+\delta^2))\|\psi\|_{\theta'}.
\end{aligned}
$$

Now we integrate both sides of (3.38) with respect to t from 0 to T and apply the estimations of $I_1 \sim I_9$, $\phi(T) = 0$ and $\overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}(0) = 0$ to obtain

$$\begin{aligned}
&\int_0^T (\overline{e}_u, \psi) dt \\
&\leq C \int_0^T \Big[h^{1+d(\frac{1}{\theta}-\frac{1}{2})}(h^2+\delta^2) + h^{-\frac{d}{2+\varepsilon}}(h^2+\delta^2) + \|\boldsymbol{\zeta}\|_\theta \\
&\quad + h^{-\frac{d(4+\varepsilon)}{2(2+\varepsilon)}}(h^4+\delta^4) \Big] \|\psi\|_{\theta'} dt \\
&\quad + (b'(u_0)\nabla u_0 \cdot \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}(0) + b(u_0)\nabla \cdot \overline{\boldsymbol{e}}_{\boldsymbol{\lambda}}(0))\phi(0) \\
&\leq C \Big[\int_0^T h^{\frac{-d}{2+\varepsilon}}(h^2+\delta^2) + \|\boldsymbol{\zeta}\|_\theta + h^{\frac{-d(4+\varepsilon)}{2(2+\varepsilon)}}(h^4+\delta^4) dt \Big] \|\psi\|_{\theta'}.
\end{aligned}$$

By applying Lemma 3.1 and Lemma 3.2 we have

$$\begin{aligned}
\|\boldsymbol{\zeta}\|_\theta &\leq C(\|\boldsymbol{e}_{\boldsymbol{\lambda}}\|_\theta + \|\boldsymbol{e}_{\boldsymbol{\lambda}t}\|_\theta + \|\boldsymbol{e}_{\boldsymbol{\sigma}}\|_\theta + \|\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}(\tau)(e_u)\|_\theta + \|(u-\tau)^2\|_\theta \\
&\quad + \|(u-\tau)(\boldsymbol{\lambda}-\boldsymbol{\eta})\|_\theta \|(u-\tau)(\boldsymbol{\lambda}_t - \boldsymbol{\eta}_t)\|_\theta) \\
&\leq Ch^2(\|\boldsymbol{\lambda}\|_{2,\theta} + \|\boldsymbol{\lambda}_t\|_{2,\theta} + \|\boldsymbol{\sigma}\|_{2,\theta} + \|u\|_{2,\theta}) \\
&\quad + C\|u-\tau\|_{2\theta}^2 + C\|u-\tau\|_{2\theta}(\|\boldsymbol{\lambda}-\boldsymbol{\eta}\|_{2\theta} + \|\boldsymbol{\lambda}_t - \boldsymbol{\eta}_t\|_{2\theta}) \\
&\leq Ch^2 + C(h^2\|u\|_{1,2\theta}^2 + \|P_h u - \tau\|_{2\theta}^2) \\
&\quad + C(\|u - P_h u\|_{2\theta} + \|P_h u - \tau\|_{2\theta})(\|\boldsymbol{e}_{\boldsymbol{\lambda}}\|_{2\theta} + \|\boldsymbol{R}_h\boldsymbol{\lambda} - \boldsymbol{\eta}\|_{2\theta} \\
&\quad + \|\boldsymbol{R}_h\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t\|_{2\theta} + \|\boldsymbol{R}_h\boldsymbol{\lambda}_t - \boldsymbol{\eta}_t\|_{2\theta}) \\
&\leq Ch^2 + Ch^{2d(\frac{1}{2\theta}-\frac{1}{\theta})}\delta^2 + C(h^2 + h^{d(\frac{1}{2\theta}-\frac{1}{\theta})}\delta)(h^2 + h^{d(\frac{1}{2\theta}-\frac{1}{2+\varepsilon})}\delta) \\
&\leq C(h^2 + h^{2d(-\frac{1}{2\theta})}\delta^2 + h^4 + h^{2+d(\frac{1}{2\theta}-\frac{1}{2+\varepsilon})}\delta + h^{2+d(-\frac{1}{2\theta})}\delta + h^{-\frac{d}{2+\varepsilon}}\delta^2) \\
&\leq C(h^2 + h^{2+d(\frac{1}{2\theta}-\frac{1}{2+\varepsilon})}\delta + h^{-\frac{d}{2+\varepsilon}}\delta^2).
\end{aligned}$$

Therefore, by applying the condition of δ, we have

$$\begin{aligned}
&\int_0^T (\overline{e}_u, \psi) ds \\
&\leq C\Big(\int_0^T h^{-\frac{d}{2+\varepsilon}}(h^2+\delta^2) + h^2 + h^{2+d(\frac{1}{2\theta}-\frac{1}{2+\varepsilon})}\delta + h^{-\frac{d}{2+\varepsilon}}\delta^2 \\
&\quad + h^{\frac{-d(4+\varepsilon)}{2(2+\varepsilon)}}(h^4+\delta^4) dt \Big) \|\psi\|_{\theta'} \\
&\leq C(h^{2-\frac{d}{2+\varepsilon}} + h^{2+d(\frac{1}{2\theta}-\frac{1}{2+\varepsilon})}\delta + h^{-\frac{d}{2+\varepsilon}}\delta^2 + h^{\frac{-d(4+\varepsilon)}{2(2+\varepsilon)}}\delta^4)\|\psi\|_{\theta'} < \delta \|\psi\|_{\theta'}.
\end{aligned}$$

Thus we prove $\|\overline{e}_u\|_{L^\infty(L^\theta)} < \delta$, which completes the proof of the existence of the semidiscrete approximation by Brouwer's fixed point theorem. □

4. The convergence of an expanded mixed finite element semidiscrete approximation $(\boldsymbol{u_h}, \boldsymbol{\lambda_h}, \boldsymbol{\sigma_h})$

Theorem 4.1. *Let $(u_h, \boldsymbol{\lambda}_h, \boldsymbol{\sigma}_h) \in V_h \times \boldsymbol{\Lambda}_h \times \boldsymbol{W}_h$ be the solution of (3.7)-(3.9). If $u \in L^\infty(H^{s+1})$, $u_t \in L^2(H^{s+1})$ and $\boldsymbol{\sigma} \in L^2(\boldsymbol{H}^s)$, then there exists a constant C such that*

$$\begin{aligned}&\|u - u_h\|_{L^\infty(L^2)} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{L^2(\boldsymbol{L}^2)} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{L^\infty(\boldsymbol{L}^2)} \\ \le\, & Ch^\mu(\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^2(H^s)} + \|\boldsymbol{\lambda}\|_{L^\infty(\boldsymbol{H}^s)} + \|\boldsymbol{\lambda}_t\|_{L^2(\boldsymbol{H}^s)} + \|\boldsymbol{\sigma}\|_{L^2(\boldsymbol{H}^s)})\end{aligned}$$

and

$$\begin{aligned}\|\boldsymbol{\sigma} - \boldsymbol{\sigma_h}\|_{L^\infty(\boldsymbol{L}^2)} \le Ch^\mu[&\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|\boldsymbol{\lambda}\|_{L^\infty(\boldsymbol{H}^s)} + \|\boldsymbol{\lambda}_t\|_{L^\infty(\boldsymbol{H}^s)} \\ &+ \|\boldsymbol{\sigma}\|_{L^\infty(\boldsymbol{H}^s)}],\end{aligned}$$

where $\mu = \min(k+1, s)$ and $\mu \ge \frac{d}{2} + 1$.

Proof. By subtracting (3.8) from (2.4) we have

$$\begin{aligned}&((a(u)(\boldsymbol{\lambda} - \boldsymbol{\lambda}_h), \boldsymbol{\mu}) + (b(u)(\boldsymbol{\lambda}_t - (\boldsymbol{\lambda}_h)_t), \boldsymbol{\mu}) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\mu}) \\ &= -\,(a(u) - (a(u_h))\boldsymbol{\lambda}_h, \boldsymbol{\mu}) - (b(u) - (b(u_h))(\boldsymbol{\lambda}_h)_t, \boldsymbol{\mu}), \qquad \forall\, \boldsymbol{\mu} \in \boldsymbol{\Lambda}_h.\end{aligned} \tag{4.1}$$

By applying the Taylor expansion to (4.1) we have

$$\begin{aligned}&(a(u)\boldsymbol{e}^h_{\boldsymbol{\lambda}}, \boldsymbol{\mu}) + (b(u)(\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t, \boldsymbol{\mu}) - (\boldsymbol{e}^h_{\boldsymbol{\sigma}}, \boldsymbol{\mu}) \\ &= (a(u)\boldsymbol{e}_{\boldsymbol{\lambda}}, \boldsymbol{\mu}) + (b(u)(\boldsymbol{e}_{\boldsymbol{\lambda}})_t, \boldsymbol{\mu}) - (\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\mu}) \\ &\quad - (\widetilde{a}_u(u_{u_h})(u - u_h)\boldsymbol{\lambda}_h, \boldsymbol{\mu}) - (\widetilde{b}_u(u_{u_h})(u - u_h)(\boldsymbol{\lambda}_h)_t, \boldsymbol{\mu}).\end{aligned} \tag{4.2}$$

From (3.16) we have

$$(\boldsymbol{e}^h_{\boldsymbol{\lambda}}, \boldsymbol{w}) - (e^h_u, \nabla\cdot\boldsymbol{w}) = 0, \quad \forall\, \boldsymbol{w} \in \boldsymbol{W}_h. \tag{4.3}$$

By (3.18) we have the following

$$((e^h_u)_t, v) + (\nabla\cdot(\boldsymbol{e}^h_{\boldsymbol{\sigma}}), v) = ((e_u)_t, v) + (f(u) - f(u_h), v), \;\; \forall v \in V_h. \tag{4.4}$$

Now by taking $\boldsymbol{\mu} = \boldsymbol{e}^h_{\boldsymbol{\lambda}}$ in (4.2), $\boldsymbol{w} = \boldsymbol{e}^h_{\boldsymbol{\sigma}}$ in (4.3) and $v = e^h_u$ in (4.4) we obtain

$$\begin{aligned}&(a(u)\boldsymbol{e}^h_{\boldsymbol{\lambda}}, \boldsymbol{e}^h_{\boldsymbol{\lambda}}) + (b(u)(\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t, \boldsymbol{e}^h_{\boldsymbol{\lambda}}) + ((e^h_u)_t, e^h_u) \\ &= (a(u)\boldsymbol{e}_{\boldsymbol{\lambda}}, \boldsymbol{e}^h_{\boldsymbol{\lambda}}) + (b(u)(\boldsymbol{e}_{\boldsymbol{\lambda}})_t, \boldsymbol{e}^h_{\boldsymbol{\lambda}}) - (\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{e}^h_{\boldsymbol{\lambda}}) - (\widetilde{a}_u(u_{u_h})(u - u_h)\boldsymbol{\lambda}_h, \boldsymbol{e}^h_{\boldsymbol{\lambda}}) \\ &\quad - (\widetilde{b}_u(u_{u_h})(u - u_h)(\boldsymbol{\lambda}_h)_t, \boldsymbol{e}^h_{\boldsymbol{\lambda}}) + ((e_u)_t, e^h_u) + (f(u) - f(u_h), e^h_u).\end{aligned}$$

Now we temporarily assume that

$$\|e^h_u(t)\|_{L^\infty} < \frac{K_2}{2} \quad \forall t \in [0, T], \tag{4.5}$$

$$\|\boldsymbol{\lambda}_h\|_{L^\infty(\boldsymbol{L}^\infty)} \le C_*, \;\; \|(\boldsymbol{\lambda}_h)_t\|_{L^\infty(\boldsymbol{L}^\infty)} \le C_* \tag{4.6}$$

hold for some constant C_* and a sufficiently small h. Then $\|(u-u_h)(t)\|_{L^\infty} \le \|e_u\|_{L^\infty} + \|e_u^h\|_{L^\infty} \le Ch\|u\|_{1,\infty} + K_2/2 \le K_2$ holds. By the assumptions on the functions $a(u)$ and $b(u)$ in Section 2, we have the following inequality

$$\begin{aligned}
&a_*\|\boldsymbol{e}_{\boldsymbol{\lambda}}^h\|^2 + \frac{1}{2}\frac{d}{dt}(b(u)\boldsymbol{e}_{\boldsymbol{\lambda}}^h, \boldsymbol{e}_{\boldsymbol{\lambda}}^h) - \frac{1}{2}\Big(\Big(\frac{d}{dt}b(u)\Big)\boldsymbol{e}_{\boldsymbol{\lambda}}^h, \boldsymbol{e}_{\boldsymbol{\lambda}}^h\Big) + \frac{1}{2}\frac{d}{dt}\|e_u^h\|^2 \\
&\le a^*\|\boldsymbol{e}_{\boldsymbol{\lambda}}\|\|\boldsymbol{e}_{\boldsymbol{\lambda}}^h\| + b^*\|(\boldsymbol{e}_{\boldsymbol{\lambda}})_t\|\|\boldsymbol{e}_{\boldsymbol{\lambda}}^h\| + \|\boldsymbol{e}_{\boldsymbol{\sigma}}\|\|\boldsymbol{e}_{\boldsymbol{\lambda}}^h\| \\
&\quad + C(K_1)\|\boldsymbol{\lambda}_h\|_{L^\infty}(\|e_u\| + \|e_u^h\|)\|\boldsymbol{e}_{\boldsymbol{\lambda}}^h\| + C(K_1)\|(\boldsymbol{\lambda}_h)_t\|_{L^\infty}(\|e_u\| + \|e_u^h\|)\|\boldsymbol{e}_{\boldsymbol{\lambda}}^h\| \\
&\quad + \|(e_u)_t\|\|e_u^h\| + C(\|e_u\| + \|e_u^h\|)\|e_u^h\|.
\end{aligned}$$

By the assumption (4.6), we have for sufficiently small $\varepsilon > 0$

$$\begin{aligned}
&a_*\|\boldsymbol{e}_{\boldsymbol{\lambda}}^h\|^2 + \frac{1}{2}\frac{d}{dt}(b(u)\boldsymbol{e}_{\boldsymbol{\lambda}}^h, \boldsymbol{e}_{\boldsymbol{\lambda}}^h) - \frac{1}{2}\Big(\Big(\frac{d}{dt}b(u)\Big)\boldsymbol{e}_{\boldsymbol{\lambda}}^h, \boldsymbol{e}_{\boldsymbol{\lambda}}^h\Big) + \frac{1}{2}\frac{d}{dt}\|e_u^h\|^2 \\
&\le C\Big\{\|\boldsymbol{e}_{\boldsymbol{\lambda}}\|^2 + \|(\boldsymbol{e}_{\boldsymbol{\lambda}})_t\|^2 + \|\boldsymbol{e}_{\boldsymbol{\sigma}}\|^2 + \|e_u\|^2 + \|e_u^h\|^2 + \|(e_u)_t\|^2\Big\} + \varepsilon\|\boldsymbol{e}_{\boldsymbol{\lambda}}^h\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
(4.7)\quad &\widetilde{c}\|\boldsymbol{e}_{\boldsymbol{\lambda}}^h\|^2 + \frac{d}{dt}\Big[\Big(\frac{1}{2}b(u)\boldsymbol{e}_{\boldsymbol{\lambda}}^h, \boldsymbol{e}_{\boldsymbol{\lambda}}^h\Big) + \frac{1}{2}\|e_u^h\|^2\Big] \\
&\le C\Big\{\|\boldsymbol{e}_{\boldsymbol{\lambda}}\|^2 + \|(\boldsymbol{e}_{\boldsymbol{\lambda}})_t\|^2 + \|\boldsymbol{e}_{\boldsymbol{\sigma}}\|^2 + \|e_u\|^2 + \|e_u^h\|^2 + \|(e_u)_t\|^2 + \|\boldsymbol{e}_{\boldsymbol{\lambda}}^h\|^2\Big\}
\end{aligned}$$

for some constant $\widetilde{c} > 0$. By integrating both sides of (4.7) with respect to t from 0 to $\widetilde{t}$, we obtain

$$\begin{aligned}
&\widetilde{c}\int_0^{\widetilde{t}}\|\boldsymbol{e}_{\boldsymbol{\lambda}}^h\|^2dt + \frac{1}{2}b(u(t))\|\boldsymbol{e}_{\boldsymbol{\lambda}}^h(\widetilde{t})\|^2 - \frac{1}{2}b(u(0))\|\boldsymbol{e}_{\boldsymbol{\lambda}}^h(0)\|^2 \\
&\quad + \frac{1}{2}\|e_u^h(\widetilde{t})\|^2 - \frac{1}{2}\|e_u^h(0)\|^2 \\
&\le C\Big\{\int_0^{\widetilde{t}}\Big(\|\boldsymbol{e}_{\boldsymbol{\lambda}}\|^2 + \|(\boldsymbol{e}_{\boldsymbol{\lambda}})_t\|^2 + \|e_u^h\|^2 + \|e_u\|^2 + \|(e_u)_t)\|^2 + \|\boldsymbol{e}_{\boldsymbol{\sigma}}\|^2 + \|\boldsymbol{e}_{\boldsymbol{\lambda}}^h\|^2\Big)dt\Big\} \\
&\le Ch^{2\mu}\Big(\|\boldsymbol{\lambda}\|^2_{L^2(\boldsymbol{H}^s)} + \|\boldsymbol{\lambda}_t\|^2_{L^2(\boldsymbol{H}^s)} + \|u\|^2_{L^2(H^s)} + \|u_t\|^2_{L^2(H^s)} + \|\boldsymbol{\sigma}\|^2_{L^2(\boldsymbol{H}^s)}\Big) \\
&\quad + C\int_0^{\widetilde{t}}\Big(\|e_u^h\|^2 + \|\boldsymbol{e}_{\boldsymbol{\lambda}}^h\|^2\Big)dt,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\int_0^{\widetilde{t}}\|\boldsymbol{e}_{\boldsymbol{\lambda}}^h\|^2dt + \|\boldsymbol{e}_{\boldsymbol{\lambda}}^h(\widetilde{t})\|^2 + \|e_u^h(\widetilde{t})\|^2 \\
&\le Ch^{2\mu}\Big(\|\boldsymbol{\lambda}\|^2_{L^2(\boldsymbol{H}^s)} + \|\boldsymbol{\lambda}_t\|^2_{L^2(\boldsymbol{H}^s)} + \|u\|^2_{L^2(H^s)} + \|u_t\|^2_{L^2(H^s)} + \|\boldsymbol{\sigma}\|^2_{L^2(\boldsymbol{H}^s)}\Big) \\
&\quad + C\Big(\|e_u^h(0)\|^2 + \|\boldsymbol{e}_{\boldsymbol{\lambda}}^h(0)\|^2\Big) + C\int_0^{\widetilde{t}}\Big(\|e_u^h\|^2 + \|\boldsymbol{e}_{\boldsymbol{\lambda}}^h\|^2\Big)ds.
\end{aligned}$$

By applying the initial conditions, $u_h(0) = P_h(u_0(x))$, $\boldsymbol{\lambda}_h(0) = \boldsymbol{R}_h(\nabla u_0(x))$ and the Gronwall Lemma, we have

(4.8)
$$\begin{aligned}&\|\boldsymbol{e}^h_{\boldsymbol{\lambda}}(\widetilde{t})\|^2 + \|e^h_u(\widetilde{t})\|^2\\ &\le Ch^{2\mu}\Big(\|\boldsymbol{\lambda}\|^2_{L^2(\boldsymbol{H}^s)} + \|\boldsymbol{\lambda}_t\|^2_{L^2(\boldsymbol{H}^s)} + \|u\|^2_{L^2(H^s)} + \|u_t\|^2_{L^2(H^s)} + \|\boldsymbol{\sigma}\|^2_{L^2(\boldsymbol{H}^s)}\Big)\end{aligned}$$

which by (3.4) and (3.6) implies

$$\begin{aligned}&\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{L^2(\boldsymbol{L^2})} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{L^\infty(\boldsymbol{L}^2)} + \|u - u_h\|_{L^\infty(L^2)}\\ &\le Ch^{\mu}\Big(\|\boldsymbol{\lambda}\|_{L^\infty(\boldsymbol{H}^s)} + \|\boldsymbol{\lambda}_t\|_{L^2(\boldsymbol{H}^s)} + \|u\|_{L^\infty(H^s)} + \|u_t\|_{L^2(H^s)} + \|\boldsymbol{\sigma}\|_{L^2(\boldsymbol{H}^s)}\Big).\end{aligned}$$

Now by induction we will prove that the hypothesis (4.5) holds. Trivially (4.5) holds for $t = 0$. We assume that there exists $t^* \in (0, T]$ such that

$$\|e^h_u(t)\|_{L^\infty} < K_2/2 \quad \forall t \in [0, t^*), \ \|e^h_u(t^*)\|_{L^\infty} \ge K_2/2. \tag{4.9}$$

Now we choose a sequence $\{t_n\}$ such that $t_n \in [0, t^*)$ and $\lim_{n\to\infty} t_n = t^*$. Then obviously we have $\|e^h_u(t_n)\|_{L^\infty} < K_2/2$ and

$$\|(u - u_h)(t_n)\|_{L^\infty} \le \|e_u(t_n)\|_{L^\infty} + \|e^h_u(t_n)\|_{L^\infty} \le h\|u\|_{1,\infty} + K_2/2 \le K_2.$$

Therefore if we follow the procedure after (4.5) we get $\|e^h_u(t^n)\|_{L^2} \le Ch^\mu$. Then by the continuity property of $\|e^h_u(t)\|_{L^2}$ we have $\|e^h_u(t^*)\| \le Ch^\mu$. By applying the inverse property and the condition $\mu \ge d/2 + 1$, we get $\|e^h_u(t^*)\|_{L^\infty} \le Ch^{-\frac{d}{2}}h^\mu < K_2/2$ for a sufficiently small h. It contradicts to (4.9) so that the hypothesis (4.5) holds. Similarly the first inequality of (4.6) can be proved. From (4.3) we have

$$((\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t, \boldsymbol{w}) - ((e^h_u)_t, \nabla\cdot\boldsymbol{w}) = 0, \quad \forall\, \boldsymbol{w} \in \boldsymbol{W}_h. \tag{4.10}$$

Taking $\boldsymbol{\mu} = (\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t$ in (4.2), $\boldsymbol{w} = \boldsymbol{e}^h_{\boldsymbol{\sigma}}$ in (4.10) and $v = \nabla\cdot\boldsymbol{e}^h_{\boldsymbol{\sigma}}$ in (4.4) implies that

$$\begin{aligned}&(a(u)\boldsymbol{e}^h_{\boldsymbol{\lambda}}, (\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t) + (b(u)(\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t, (\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t) + \|\nabla\cdot\boldsymbol{e}^h_{\boldsymbol{\sigma}}\|^2\\ &= (a(u)(\boldsymbol{e}_{\boldsymbol{\lambda}}, (\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t) + (b(u)((\boldsymbol{e}_{\boldsymbol{\lambda}})_t, (\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t) - (\widetilde{a}_u(u_h)(u - u_h)\boldsymbol{\lambda}_h, (\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t)\\ &\quad - (\widetilde{b}_u(u_h)(u - u_h)(\boldsymbol{\lambda}_h)_t, (\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t) + ((e_u)_t, \nabla\cdot(\boldsymbol{e}^h_{\boldsymbol{\sigma}})) + (f(u) - f(u_h), \nabla\cdot(\boldsymbol{e}^h_{\boldsymbol{\sigma}})).\end{aligned}$$

By applying the assumption (4.6) we have the following

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(a(u)\boldsymbol{e}^h_{\boldsymbol{\lambda}}, \boldsymbol{e}^h_{\boldsymbol{\lambda}}\Big) - \frac{1}{2}\Big((\frac{d}{dt}a(u))\boldsymbol{e}^h_{\boldsymbol{\lambda}}, \boldsymbol{e}^h_{\boldsymbol{\lambda}}\Big) + b_*\|(\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t\|^2 + \|\nabla\cdot\boldsymbol{e}^h_{\boldsymbol{\sigma}}\|^2\\ &\le C\Big\{\|\boldsymbol{e}_{\boldsymbol{\lambda}}\|\|(\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t\| + \|(\boldsymbol{e}_{\boldsymbol{\lambda}})_t\|\|(\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t\| + (\|e_u\| + \|e^h_u\|)\|(\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t\|\\ &\quad + \|(e_u)_t\|\|\nabla\cdot\boldsymbol{e}^h_{\boldsymbol{\sigma}}\| + (\|e_u\| + \|e^h_u\|)\|\nabla\cdot\boldsymbol{e}^h_{\boldsymbol{\sigma}}\|\Big\},\end{aligned}$$

which implies that for a sufficiently small $\varepsilon > 0$

$$\frac{1}{2}\frac{d}{dt}\Big(a(u)\boldsymbol{e}^h_{\boldsymbol{\lambda}}, \boldsymbol{e}^h_{\boldsymbol{\lambda}}\Big) + b_*\|(\boldsymbol{e}^h_{\boldsymbol{\lambda}})_t\|^2 + \|\nabla\cdot(\boldsymbol{e}^h_{\boldsymbol{\sigma}})\|^2 \tag{4.11}$$

$$\le C\Big\{\|\boldsymbol{e_\lambda}\|^2+\|(\boldsymbol{e_\lambda})_t\|^2+\|e_u\|^2+\|e_u^h\|^2+\|(e_u)_t\|^2+\|\boldsymbol{e}^h_{\boldsymbol\lambda}\|^2\Big\}$$
$$+\varepsilon\|(\boldsymbol{e}^h_{\boldsymbol\lambda})_t\|^2+\varepsilon\|\nabla\cdot(\boldsymbol{e}^h_{\boldsymbol\sigma})\|^2.$$

Since $-\underline{c}\|\boldsymbol{e}^h_{\boldsymbol\lambda}\|^2-(\frac{b_*}{2}\|(\boldsymbol{e}^h_{\boldsymbol\lambda})_t\|^2+c\|\boldsymbol{e}^h_{\boldsymbol\lambda}\|^2)\le\frac12\frac{d}{dt}(a(u)\boldsymbol{e}^h_{\boldsymbol\lambda},\boldsymbol{e}^h_{\boldsymbol\lambda})$ for some constant $\underline{c}>0$, we have

$$\frac{b_*}{4}\|(\boldsymbol{e}^h_{\boldsymbol\lambda})_t\|^2+\frac12\|\nabla\cdot\boldsymbol{e}^h_{\boldsymbol\sigma}\|^2$$
$$\le C\Big(\|\boldsymbol{e_\lambda}\|^2+\|(\boldsymbol{e_\lambda})_t\|^2+\|e_u\|^2+\|e_u^h\|^2+\|(e_u)_t\|^2+\|\boldsymbol{e}^h_{\boldsymbol\lambda}\|^2\Big)$$
$$\le Ch^{2\mu}\Big[\|\boldsymbol\lambda\|^2_{L^\infty(\boldsymbol{H}^s)}+\|\boldsymbol\lambda_t\|^2_{L^\infty(\boldsymbol{H}^s)}+\|u\|^2_{L^\infty(\boldsymbol{H}^s)}+\|u_t\|^2_{L^\infty(\boldsymbol{H}^s)}+\|\boldsymbol\sigma\|^2_{L^2(\boldsymbol{H}^s)}\Big].$$

Therefore we have the following estimate

$$\|(\boldsymbol{e}^h_{\boldsymbol\lambda})_t(\widetilde{t})\|+\|(\nabla\cdot\boldsymbol{e}^h_{\boldsymbol\sigma})(\widetilde{t})\|$$
$$\le Ch^{\mu}\Big[\|\boldsymbol\lambda\|_{L^\infty(\boldsymbol{H}^s)}+\|\boldsymbol\lambda_t\|_{L^\infty(\boldsymbol{H}^s)}+\|u\|_{L^\infty(H^s)}+\|u_t\|_{L^\infty(H^s)}+\|\boldsymbol\sigma\|_{L^\infty(\boldsymbol{H}^s)}\Big].$$

By the similar method as we performed to prove the hypothesis (4.5) we can show that the second inequality of (4.6) holds.

Now we take $\boldsymbol\mu=\boldsymbol{e}^h_{\boldsymbol\sigma}$ in (4.2) to obtain

$$(a(u)\boldsymbol{e}^h_{\boldsymbol\lambda},\boldsymbol{e}^h_{\boldsymbol\sigma})+(b(u)(\boldsymbol{e}^h_{\boldsymbol\lambda})_t,\boldsymbol{e}^h_{\boldsymbol\sigma})-(\boldsymbol{e}^h_{\boldsymbol\sigma},\boldsymbol{e}^h_{\boldsymbol\sigma})$$
$$=(a(u)\boldsymbol{e_\lambda},\boldsymbol{e}^h_{\boldsymbol\sigma})+(b(u)(\boldsymbol{e_\lambda})_t,\boldsymbol{e}^h_{\boldsymbol\sigma})-(\boldsymbol{e_\sigma},\boldsymbol{e}^h_{\boldsymbol\sigma})-(\widetilde{a}_u(u_{u_h})(u-u_h)\boldsymbol\lambda_h,\boldsymbol{e}^h_{\boldsymbol\sigma})$$
$$-(\widetilde{b}_u(u_{u_h})(u-u_h)(\boldsymbol\lambda_h)_t,\boldsymbol{e}^h_{\boldsymbol\sigma}),$$

which implies that

$$\|\boldsymbol{e}^h_{\boldsymbol\sigma}\|^2\le C(\|\boldsymbol{e}^h_{\boldsymbol\lambda}\|^2+\|\boldsymbol{e_\lambda}\|^2+\|(\boldsymbol{e_\lambda})_t\|^2+\|(\boldsymbol{e}^h_{\boldsymbol\lambda})_t\|^2+\|\boldsymbol{e_\sigma}\|^2+\|u-u_h\|^2)$$
$$+\frac12\|\boldsymbol{e}^h_{\boldsymbol\sigma}\|^2.$$

Therefore we have

$$\|\boldsymbol{e}^h_{\boldsymbol\sigma}\|^2\le Ch^{\mu}(\|\boldsymbol\lambda\|_{L^\infty(\boldsymbol{H}^s)}+\|\boldsymbol\lambda_t\|_{L^\infty(\boldsymbol{H}^s)}+\|u\|_{L^\infty(H^s)}+\|u_t\|_{L^\infty(H^s)}$$
$$+\|\boldsymbol\sigma\|_{L^\infty(\boldsymbol{H}^s)}),$$

which yields that

$$\|(\boldsymbol\sigma-\boldsymbol\sigma_h)\|_{L^\infty(\boldsymbol{L}^2)}\le Ch^{\mu}\Big[\|\boldsymbol\lambda\|_{L^\infty(\boldsymbol{H}^s)}+\|\boldsymbol\lambda_t\|_{L^\infty(\boldsymbol{H}^s)}+\|u\|_{L^\infty(H^s)}$$
$$+\|u_t\|_{L^\infty(H^s)}+\|\boldsymbol\sigma\|_{L^\infty(\boldsymbol{H}^s)}\Big].$$

Thus we prove the optimal convergence of u_h, $\boldsymbol\lambda_h$ and $\boldsymbol\sigma_h$. □

5. Conclusions

In this paper we discussed the quasilinear pseudo-parabolic equations with a locally Lipschitz continuous $f(u)$ which are important in many practical applications as shown in [5, 9, 11]. Applying an expanded mixed finite element method we constructed the approximations of the scalar unknown, its flux and its gradient, directly. We proved the existence of the semidiscrete approximations and derived the optimal order of convergence of the unknown, its flux and its gradient in L^2 normed space.

We also suggest that one can consider the fully discrete implementations using expanded mixed Galerkim methods, fulfill the theoretical analysis of the convergence of the fully discrete approximations and present the numerical result. This is a valuable and challenging topic in the future research.

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