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Portfolio Optimization with Groupwise Selection

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ABSTRACT

Portfolio optimization in the presence of estimation error can be stabilized by incorporating norm-constraints; this result was shown by DeMiguel *et al*. (A generalized approach to portfolio optimization: improving performance by constraining portfolio norms, *Management Science*, 5, 798-812, 2009), who reported empirical performance better than numerous competing approaches. We extend the idea of norm-constraints by introducing a powerful enhancement, grouped selection for portfolio optimization. Here, instead of merely penalizing norms of the assets being selected, we penalize groups, where within a group assets are treated alike, but across groups, the penalization may differ. The idea of groupwise selection is grounded in statistics, but to our knowledge, it is novel in the context of portfolio optimization. Novelty aside, the real benefits of groupwise selection are substantiated by experiments; our results show that groupwise asset selection leads to strategies with lower variance, higher Sharpe ratios, and even higher expected returns than the ordinary norm-constrained formulations.

Keywords: Portfolio Optimization, Group-Norm, Asset Class

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1. INTRODUCTION

Modern portfolio theory originated with the seminal work of Markowitz (1951, 1991), who recognized that in an investment portfolio, one should choose assets not individually, but rather by considering how they are related to each other. The resulting 'mean-variance' portfolio selection, which essentially aims to minimize risk (defined in terms of minimizing variance of returns), depends strongly on good estimates of the means and covariances of the asset returns. Typically, these values are estimated by using sample means and covariances, but as strongly stressed by DeMiguel *et al*. (2009), estimation error in these quantities leads to poor out-of-sample performance (see also references therein). To counter this poor performance, DeMiguel *et al.* (2009) introduced the idea solving the traditional minimum-variance problem subject to a norm constraint on the portfolio-weight vector. This constraint was then shown to lead to portfolio strategies that often have higher Sharpe ratios than several competing strategies (DeMiguel *et al*., 2009).

Intuitively, one could attribute the better performance of their norm-constrained portfolios to selection. That is, the norm constraint restricts the choice of assets to which the investment should be allocated. However, which particular assets are chosen depends somewhat arbitrarily on the constraint parameters. So, one may naturally ask whether a more careful asset allocation can lead to even better portfolios?

This question has been addressed at a higher level by considering the notion of asset classes (Maginn *et al*., 2007, Chapter 5). Therein, the authors suggest that one should divide assets into classes which satisfy the following properties: first, assets within an asset class are homogeneous; second, asset classes are mutually exclusive; third, asset classes lead to diversification; forth, the different asset classes should cover a significant fraction of the investor's wealth; and lastly, each asset class should have the capacity to absorb a significant fraction of the investor's wealth.

These properties of asset classes and the good empirical results of the norm-based model motivate us to ask:

Can we combine asset classes and norms to obtain more effective portfolio optimization strategies?

In this paper we provide one answer to this question by presenting several new models based on group norms, which, instead of merely constraining weights of individual assets, constrain the weight associated to a group of assets. Experiments reveal that such groupwise constraints are beneficial— they lead to better portfolio strategies (as indicated by higher Sharpe ratios, lower variance, etc.) than competing approaches. Groupwise selection is a simple, yet powerful generalization to the idea of norm constrained portfolios, especially because it permits use of asset classes, and it opens up the possibility to incorporate expert knowledge for deciding what grouping to use.

Moreover, our mathematical formulation does not exclude overlapping groups; so in case expert knowledge suggests overlapping groups, the model can accommodate this knowledge. Although, as motivated in Maggin *et al.* (2007, Chapter 5), non-overlapping asset classes should be preferred, whereby we focus our attention on non-overlapping groups; we will also briefly discuss examples with overlapping groups.

The remaining part of this paper is organized as follows. In Section 2, we review the previous portfolio optimization strategies. In Section 3, we propose our portfolio optimization strategies. In Section 4, the numerical results are presented and we conclude the paper in Section 5.

2. PORTFOLIO OPTIMIZATION

The goal of portfolio optimization is to maximize its returns with less risk. Single period portfolio optimization using the mean and variance was first suggested by Markowitz (1952).

Markowitz' mean-variance optimization model is a widely used tool for portfolio optimization. It can be formulated in different ways. The typical problem is as follow:

$$\min_w \frac{1}{2} w^T \hat{\Sigma} w, \quad (1)$$

$$\text{subject to } \hat{\mu}^T w \geq \mathrm{R}, \quad (2)$$

$$\mathrm{Aw} = \mathrm{b}, \quad (3)$$

$$\mathrm{Cw} \geq \mathrm{d}, \quad (4)$$

in which $\mathrm{w} \in \mathbb{R}^{\mathrm{N}}$ is the vector of portfolio weights, $\hat{\Sigma} \in \mathbb{R}^{\mathrm{N}\times\mathrm{N}}$ is the estimated covariance matrix, $\mathrm{w}^{\mathrm{T}}\hat{\Sigma}\,\mathrm{w}$ is the variance of the portfolio return, $\hat{\mu}$ is the estimated asset returns.

The other one is a risk-adjusted formulation.

$$\max_w \hat{\mu}^T w - \frac{\lambda}{2} w^T \Sigma w, \quad (5)$$

$$\text{subject to } \mathrm{Aw} = \mathrm{b} \quad (6)$$

$$S_{r_t}\ \mathrm{Cw} \geq \mathrm{d}, \quad (7)$$

where λ is a risk-aversion constant. Recently, some researchers focus on minimum-variance portfolio, because of the estimation error associated with the sample mean. Like previous authors, we too focus on minimum-variance portfolios, noting however that our framework easily applies to other formulations, such as mean-variance.

In the absence of shortsale constraints, the minimum-variance portfolio is the solution to the following problem:

$$\min_w \frac{1}{2} w^T \hat{\Sigma} w, \quad (8)$$

$$\text{subject to } w^T \mathrm{e} = 1, \quad (9)$$

in which $\mathrm{e} \in \mathbb{R}^{\mathrm{N}}$ is the vector of ones.

DeMiguel *et al*. (2009) suggested the additional constraint that the norm of the portfolio-weight vector smaller than a particular value to solve the traditional minimum-variance problem. The p-norm-constraint portfolio is the solution to the problem (8) and (9) subject to the additional constraint on the l_p-norm of the portfolio-weight vector.

$$\|\mathrm{w}\|_{\mathrm{p}} \leq \delta, \quad (10)$$

in which δ is a threshold and it can be calibrated using cross-validation. They showed that the framework nests the shrinkage approaches (Jagannathan and Ma, 2003; Ledoit and Wolf, 2003, 2004) and 1/N portfolio. The norm-constraint portfolio often showed better performance than other portfolios in the literature in terms of out-of-sample Sharpe ratio, although it was accompanied by higher turnover.

We propose more general framework than the norm constrained framework of DeMiguel *et al*. (2009); all of their formulations can be obtained as special cases of our framework.

3. PROPOSED APPROACH: CONSTRAINING GROUP NORM

As mentioned above, Markovitz' mean-variance optimization problem can be formulated with some constraints like shortsale constraint (Jagannathan and Ma, 2003), norm constraint (DeMiguel *et al*., 2009). One can also reduce sector risk by adding the constraint

$$\sum_{i \text{ in sector } k} w_i \leq m_k,$$

where mk is the maximum that can be invested in sector k. However, the more constraints make the objective value degenerates (Cornuejols and Tutuncu, 2007). In this paper, we suggest new portfolio optimization strategies using *group-norms*.

A mixed-norm aka group-norm is usually defined over a set of parameter vectors, where the individual parameters form 'groups' and each group's penalty or contribution may be measured using a different l_p-norm. This notion is described formally in Definition 1–4 below.

3.1 Group-Norms

DEFINITION 1 (Mixed-norm: Vectors).

Let $w \in \mathbb{R}^d$ be partitioned into the set $\{w_t: w_t \in \mathbb{R}^{d_t}, 1 \le t \le n\}$ of (column) vectors. We define the mixed $l_{p,q}$-norm (read as p norm of q norms) ($1 \le p, q \le \infty$) for w as

$$\|w\|_{p,q} = \left\| \left[\|w_1\|_q ; \|w_2\|_q ; \cdots ; \|w_n\|_q \right] \right\|_p. \tag{11}$$

That is, we compute the l_p-norm of the vector of length n formed by computing l_p-norms of the individual vectors w_t ($1 \le t \le n$) (Note: The definition (11) can be further generalized, e.g., if we take the l_{q_t}-norm $\|w_n\|_{q_t}$ of w_t).

This generalization comes up for example, while studying L_p-nested symmetric distributions (Bethge *et al.*, 2009).

DEFINITION 2 (Mixed-norm: Matrices).

For a matrix $\mathbf{W} \in \mathbb{R}^{d \times n}$, we define the mixed-norm to be the l_p-norm of the l_q-norms of the rows; i.e.,

$$\|\boldsymbol{W}\|_{p,q} = \left\| \left[\|w_1\|_q ; \|w_2\|_q ; \cdots ; \|w^d\|_q \right] \right\|_p \tag{12}$$

We define the mixed-norm over rows rather than columns, because usually in multi-task setups one wishes to enforce sparsity across the same feature for multiple tasks, whereby, the same 'row' across all tasks (columns of **W**) is penalized. For example, the $l_{1,\infty}$ norm of $\mathbf{W} \in \mathbb{R}^{d \times n}$ is

$$\|\boldsymbol{W}\|_{1,\infty} = \sum_{i=1}^{d} \|w^i\|_\infty \tag{13}$$

DEFINITION 3 (Group-norm).

Let w be as in Definition 1. We define the group-norm as ($1 \le p \le \infty$)

$$\|w\|_{Gr(p)} = \left\| \left[\|w_1\|_{K_1} ; \|w_2\|_{K_2} ; \cdots ; \|w_n\|_{K_n} \right] \right\|_p, \tag{14}$$

i.e., the l_p-norm of a vector formed by taking Hilbert-Schmidt norms parameterized by the positive-definite matrices $K_t \in S_{++}^{d_t}$ ($1 \le t \le n$). For example, if $K_t \in I_{d_t}$ and p = 1, then (14) becomes

$$\|w\|_{Gr(1)} = \|w\|_{1,2} = \sum_{t=1}^{n} \|w_t\|_2. \tag{15}$$

DEFINITION 4 (Mixed quasi-norms).

In Definitions 1 or 2 we permit any row or subvector (as the case may be) to have norm measured by the l_0-quasi-norm, we obtain a mixed quasi-norm. The most important instances are: $\|\boldsymbol{W}\|_{0,0}$, $\|\boldsymbol{W}\|_{0,p}$, $\|\boldsymbol{W}\|_{p,0}$, $\|\boldsymbol{W}\|_{0,p}$ or $\|\boldsymbol{W}\|_{p,0}$, where $1 \le p \le \infty$.

In this paper, we propose new portfolio optimization strategies using group-norms. The formulation is as follow.

$$\min_w \frac{1}{2} w^T \hat{\Sigma} w, \tag{16}$$

$$\text{subject to } w^T e = 1, \tag{17}$$

$$\|w\|_{p,q} \le \delta, \tag{18}$$

where $\|w\|_{p,q}$ is a group-norm. To apply group-norms, we should partition the stocks into several groups. We use random grouping and k-means clustering algorithm. To implement k-means clustering algorithm, the sample return and variance of assets are used as attributes. We also allow overlapping groups using soft k-means algorithm. Each group is assumed to represent an asset class.

3.2 Interpretation of the Group-Norms

Additional group-norm constraint can be interpreted in various ways. First is introduction of groupwise selection for portfolio optimization. This idea actually goes much further than imagined. Second is explanation of the modeling power of the groupwise selection. For example, in the Black-Litterman model, one of the first steps that is suggested is to divide the assets into various 'asset classes', and Maginn *et al.* (2007) suggest that these asset classes should satisfy:

- Assets within a class should be homogeneous (same l_q-norm used for all elements within a group);
- Assets should be mutually exclusive (non-overlapping variables for groups);
- Asset classes should be diversifying (e.g., use of $l_{1,\infty}$-norm promotes diversity, while being homogeneous within a group respectively asset class);
- Asset class should have capacity to absorb significant fraction of the investor's wealth (again the $l_{1,\infty}$-norm makes sense, because the ∞ portion distributes wealth across the assets within the class; since l_1-favors sparsity, one could use other norms, such as $l_{2,\infty}$ to absorb greater fraction of wealth);
- Asset classes taken together should make up significant

portion of investor's wealth (if we cover all the possible assets this requirement gets addressed automatically; by using l_1-norm, we promote sparsity, so that overly diffuse investments are not made, but rather a few 'groups' or 'asset classes' get selected, and in them the investments are made).

Our model of groupwise selection, however, goes beyond these prescriptions of Miginn *et al*. (2007), and permits arbitrary overlapping groups. That is, the same asset may be part of more than one asset class; even though mathematically we permit this, we do not have a natural interpretation for such grouping, except if one interprets the 'asset classes' as 'hedging' the groups themselves—so that an erroneous asset classification does not have too severe a negative impact on the investment. Last interpretation is mathematical formulation, as well as algorithms for efficiently solving the associated optimization problems.

4. EMPIRICAL RESULTS

For the application, we give an example of the results. We used weekly historical data to estimate the means and covariances of asset returns. The data was taken from Yahoo! Finance (http://finance.yahoo.com). The portfolio is composed of S&P 500 components shares. The S&P 500 is widely regarded as the best representations of the US stock market and a leading indicator of business cycles. It consists of the common stock listed on the NYSE or NASDAQ. The total number of shares is 500 but we use 466 shares. The sample covered the period from August 16, 2004 through August 2, 2010. We estimate sample mean and variance every Monday. An overview of the data is given in Table 1.

To evaluate the performance of the proposed method with N available assets, we compute the out-of-sample variance, Sharpe ratio, and turnover as following:

$$\left(\hat{\sigma}^i\right)^2 = \frac{1}{T-\tau-1}\sum_{t=\tau}^{T-1}\left(w_t^{i^T} r_{t+1} - \hat{\mu}^i\right)^2, \quad (19)$$

$$\text{With } \hat{\mu}^i = \frac{1}{T-\tau}\sum_{t=\tau}^{T-1} w_t^{i^T} r_{t+1}, \quad (20)$$

$$\widehat{SR}^i = \frac{\hat{\mu}^i}{\hat{\sigma}^i}, \quad (21)$$

$$\text{Turnover} = \frac{1}{T-\tau-1}\sum_{t=\tau}^{T-1}\sum_{j=1}^{N}\left(\left|w_{j,t+1}^i - w_{j,t+}^i\right|\right), \quad (22)$$

Table 1. Data set description

Data set	S&P 500 components
No. of shares	466
Time period	16/08/2004 to 02/08/2010
Source	Yahoo! Finance

Table 2. List of benchmark portfolios

Model	Abbreviation
Minimum-variance portfolio with shortsales unconstrained	MINU
1-norm-constrained minimum-variance portfolio	NC1V
2-norm-constrained minimum-variance portfolio	NC2V
Mean-variance portfolio with risk aversion parameter 2	MEAN
1-norm-constrained mean-variance portfolio	NC1M
2-norm-constrained mean-variance portfolio	NC2M

where τ is the length of the estimation window, T is the total number of returns in the dataset, w_t^i is portfolio-weight vectors for each strategy i, and rt denotes the asset returns. In the definition of turnover, $w_{j,t}^i$ is the portfolio weight in asset j at time t for strategy i, $w_{j,t+}^i$ is the portfolio weight before rebalancing.

We use the rolling-window procedure for the comparison. For our empirical test, we use an estimation window of τ = 120, which corresponds to thirty months for weekly data. We compare empirically the out-of-sample performance of group-norms portfolios to other strategies. The portfolios we evaluate are listed in Table 2.

MINU, NC1V, and NC2V are models for minimum-variance portfolio. MEAN, NC1M, and NC2M are models for mean-variance portfolio.

To develop the proposed portfolio in this paper, we should select the number of groups, the value of threshold δ and the orders of group-norm, p and q. We make two, five, and ten groups and use group norm with p = 1, q = 2, 3, ∞. The threshold was calibrated with cross-validation (Efron and Gong, 1983; Campbell *et al*., 1997, Section 12.3.2).

Tables 3 and 4 show the out-of-sample performance of each strategies. Table 3 shows performance of minimum-variance portfolios when the objective function is $\min_w \frac{1}{2} w^T \hat{\Sigma} w.$ The results of mean-variance portfolio are presented in Table 4. The estimation error associated with the sample mean is relatively large so extensive empirical evidence shows that the minimum-variance portfolio often performs better than mean-variance portfolio.

This table reports the weekly out-of-sample mean, Sharpe ratio, variance and turnover for minimum-variance portfolio, $\min_w \frac{1}{2} w^T \hat{\Sigma} w.$

This table reports the weekly out-of-sample mean, Sharpe ratio, variance and turnover for mean-variance portfolio, $\max_w \hat{\mu}^T w - \frac{\lambda}{2} w^T \hat{\Sigma} w.$

From Table 3 and 4 we see that the portfolios with non-overlap groups are usually better than the portfolio with overlap groups. Comparing the benchmark portfolios listed in Table 2, the portfolios developed in this pa-

Table 3. Portfolio performance comparison

Clustering methods		# of groups	p	q	δ	mean	Sharpe ratios	Variances	Turnovers
non-overlap	random	2	1	2	0.7	0.00011	0.01004	0.00012	0.68256
			1	3	0.7	0.00018	0.01672	0.00012	0.70428
			1	∞	0.1	0.00026	0.02291	0.00013	0.75611
		5	1	2	0.7	-0.00021	-0.02101	0.00010	0.51400
			1	3	0.7	0.00029	0.02569	0.00012	0.73130
			1	∞	0.1	-0.00027	-0.02482	0.00012	1.13338
		10	1	2	1	-0.00007	-0.00706	0.00011	0.56017
			1	3	0.7	0.00053	0.04861	0.00012	0.71377
			1	∞	0.1	0.00011	0.01077	0.00010	0.37288
	k-means	2	1	2	0.7	0.00028	0.02628	0.00011	0.66576
			1	3	0.7	0.00018	0.01581	0.00012	0.74035
			1	∞	0.1	0.00009	0.00790	0.00013	0.76474
		5	1	2	0.7	0.00045	0.04081	0.00012	0.65842
			1	3	0.7	0.00037	0.03325	0.00012	0.72649
			1	∞	0.1	0.00095	0.07216	0.00017	1.01678
		10	1	2	1	0.00038	0.03457	0.00012	0.68073
			1	3	0.7	0.00047	0.04326	0.00012	0.76994
			1	∞	0.1	0.00074	0.06147	0.00015	0.80687
overlap	random	2	1	2	2	0.00001	0.00129	0.00012	0.68844
			2	3	2	-0.00009	-0.00828	0.00012	0.70111
			1	∞	0.3	0.00019	0.01705	0.00012	0.69754
		5	1	2	2	0.00007	0.00664	0.00012	0.70323
			5	3	2	-0.00043	-0.03672	0.00014	0.86338
			1	∞	0.3	0.00002	0.00213	0.00013	0.74204
		10	1	2	2	0.00030	0.02812	0.00011	0.69071
			1	3	2	0.00024	0.02266	0.00012	0.87674
			1	∞	0.5	0.00036	0.03216	0.00013	0.74800
	soft k-means	2	1	2	3	0.00008	0.00701	0.00012	0.66751
			1	3	3	-0.00007	-0.00650	0.00013	0.71678
			1	∞	0.5	0.00012	0.01106	0.00012	0.72461
		5	1	2	3	0.00020	0.01806	0.00012	0.69330
			1	3	3	-0.00020	-0.01796	0.00012	0.72799
			1	∞	0.5	0.00013	0.01132	0.00013	0.76152
		10	1	2	3	-0.00018	-0.01655	0.00012	0.73686
			1	3	3	0.00005	0.00426	0.00014	0.83280
			1	∞	0.5	0.00024	0.02291	0.00011	0.78040
Benchmark					δ	mean	Sharpe ratios	Variances	Turnovers
MINU						0.00013	0.01230	0.00012	0.68435
NC1V					1.4	-0.00001	-0.08752	0.00013	0.26938
NC2V					0.2	-0.00001	-0.07766	0.00011	0.42349

per show reasonable results in out-of-sample performance. If we select optimal number of groups, p and q, the performance of the proposed method will be better than the norm-constraint portfolios.

5. CONCLUSION

We provided a general unifying framework for portfolio optimization. This paper contributes to the literature on portfolio optimization strategies. First, our group-norm constraint model is a general form including other constraint portfolio strategies, such as shortsale constraint model (Lintner, 1965; Jagannathan and Ma, 2003), norm-constraint model (DeMiguel *et al.*, 2009), and simple 1/N model. Second, the portfolios developed in this paper often show better mean return and Sharpe ratio than the existing portfolios, although the higher Sharpe ratio is accompanied by higher turnover. Lastly, the proposed model permits use of asset classes naturally. The asset classes lead to diversification of portfolio. This property of asset classes and the good empirical results of the norm-constraint model lead the better performance.

For further research, the strategies of grouping assets

Table 4. Portfolio performance comparison

Clustering methods		# of groups	p	q	δ	mean	Sharpe ratios	Variances	Turnovers
Non overlap	random	2	1	2	0.7	0.00143	0.05703	0.00063	0.84049
			1	3	0.7	0.00194	0.03800	0.00260	2.33944
			1	∞	0.1	-0.00360	-0.07959	0.00205	2.74162
		5	1	2	0.7	0.00049	0.02375	0.00042	0.54908
			1	3	0.7	0.00177	0.05035	0.00123	2.09120
			1	∞	0.1	-0.00221	-0.07225	0.00094	1.55628
		10	1	2	1	-0.00125	-0.06216	0.00040	0.43806
			1	3	0.7	0.00151	0.05558	0.00073	2.88543
			1	∞	0.1	-0.00282	-0.16921	0.00028	1.55390
	k-means	2	1	2	0.7	0.00100	0.03923	0.00065	0.77865
			1	3	0.7	-0.00221	-0.04138	0.00285	2.32636
			1	∞	0.1	-0.00587	-0.10888	0.00291	2.87138
		5	1	2	0.7	0.00071	0.02847	0.00063	0.61404
			1	3	0.7	-0.00390	-0.07374	0.00279	2.23334
			1	∞	0.1	-0.00750	-0.16659	0.00203	3.55052
		10	1	2	1	0.00198	0.07559	0.00068	0.60682
			1	3	0.7	-0.00187	-0.03242	0.00332	1.65515
			1	∞	0.1	0.00178	0.03760	0.00224	1.17426
overlap	random	5	1	2	2	0.00206	0.03551	0.00336	2.35288
			1	3	2	-0.01067	-0.09154	0.01358	5.90414
			1	∞	0.3	-0.00952	-0.08616	0.01220	5.45354
		5	1	2	2	0.00255	0.06483	0.00155	1.70558
			1	3	2	-0.00101	-0.01280	0.00621	5.84109
			1	∞	0.3	-0.00481	-0.09909	0.00236	3.39847
		10	1	2	2	0.00102	0.02777	0.00135	1.36261
			1	3	2	0.00078	0.01489	0.00274	5.26236
			1	∞	0.5	-0.00602	-0.13745	0.00192	4.71586
	soft k-means	5	1	2	3	-0.00229	-0.02816	0.00661	3.54828
			1	3	3	-0.02731	-0.16446	0.02757	9.20783
			1	∞	0.5	-0.02303	-0.13359	0.02971	9.58459
		5	1	2	3	-0.00244	-0.03049	0.00638	3.13377
			1	3	3	-0.01796	-0.12983	0.01913	7.29076
			1	∞	0.5	-0.02812	-0.18545	0.02299	16.71380
		10	1	2	3	0.00997	0.17396	0.00328	1.32832
			1	3	3	-0.00096	-0.01195	0.00646	3.40558
			1	∞	0.5	-0.01147	-0.17059	0.00452	2.97873
Benchmark					δ	mean	Sharpe ratios	Variances	Turnovers
MEAN						-0.27141	-0.18416	2.17215	109.23311
NC1M					1.4	-0.00205	-0.08987	0.00052	0.29046
NC2M					0.2	90.00080	0.03357	0.00057	0.64312

can be analyzed. In this research, we use simple random grouping and k-means clustering algorithm using sample mean and variance without considering expert's knowledge and industrial properties of assets. Especially, as mentioned earlier, it is more difficult to estimate means so the skill to forecast expected returns is needed for better performance. The optimal number of groups should also be studied.

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