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Various Row Invariants on Cohen-Macaulay Rings

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Abstract

We define a numerical invariant $row_j^*(A)$ over Cohen-Macaulay local ring A, which is related to the presenting matrices of the j-th syzygy module (with or without free summands). We show that $row_d(A) = row_{CM}(A)$ and $row_d^*(A) = row_{CM}^*(A)$ for a Cohen-Macaulay local ring A of dimension d.

Key words : Row Invariants, Cohen-Macaulay Local Ring, Maximal Cohen-Macaulay Module, Free Summands

1. Introduction

Throughout this paper, we assume that (A,m) is a Noetherian local ring, and all modules are unitary.

It is proved[1] that there are certain restrictions on the entries of the maps in the minimal free resolutions of finitely generated modules of infinite projective dimension over Noetherian local rings. This fact provides not only a new way to understand some previously known results in commutative ring theory (see for instance Corollary 2.8[1], or Proposition 2.2[1]), but also new interesting invariants of local rings. These invariants have turned out to be quite useful; for example, the Auslander index of A can be described as a column invariant when A is Gorenstein[1], and the multiplicity of A can be also explained by these invariants when A is hypersurface. (For the further background of these invariants, we refer the reader to some papers[1-5].)

This paper particularly deals with invariants $row_d(A)$, $row_{CM}(A)$, $row_d^*(A)$, and $row_{CM}^*(A)$. We will eventually show that all of those are equal to $row(A)$ for a Cohen-Macaulay local ring of dimension d.

We recall that $row(A)$ is defined with the rows of the maps in infinite minimal resolutions: $row(A)$ is the smallest integer $t \geq 1$ such that for each finitely generated A-module M of infinite projective dimension, each row of φ_i contains an element outside m^t for all $i > depth(A)$, where φ_i is the i-th map of a minimal free resolution of M. It was shown[6] that $row(A)$ can be described in terms of the rows of presenting matrices of maximal Cohen-Macaulay modules (not necessarily without free summand) when A is Cohen-Macaulay (not necessarily Gorenstein). In this paper, we define $row_j^*(A)$ as the smallest positive integer $t \geq 1$ such that for each j-th syzygy module (not necessarily without free summand) of infinite projective dimension, each row of its presenting matrix contains an element outside m^t, and show that $row_j^*(A) = row_j(A)$ for each $j \geq d$.

In Section 2, we give the definitions of $row_j(A)$, $row_{CM}(A)$, $row_j^*(A)$, and $row_{CM}^*(A)$ in detail, and study some basic properties of these invariants.

In Section 3, we prove that $row_d^*(A) = row_d(A)$ for a Cohen-Macaulay local ring A of dimension by showing that $row_d(A) = row_{CM}(A)$ and $row_d^*(A) = row_{CM}^*(A)$ which explains $row(A)$ can be described in terms of the rows of presenting matrices of the d-th syzygy modules with or without free summands. We also prove that if A is a Gorenstein local ring of dimension d, then $row(A) = row_j^*(A)$ for all $j \geq d$.

2. Basic Definitions and Facts

Let M be a finitely generated A-module and φ_i denote an i-th map of a minimal free resolution of M: $\cdots \to A^{n_2} \xrightarrow{\varphi_2} A^{n_1} \xrightarrow{\varphi_1} A^{n_0} \to M \to 0$. A map φ_i is

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represented by a matrix of size $n_{i+1} \times n_i$. By a (minimal) presenting matrix of M, we mean the representation matrix of φ_i. Then we note that φ_i is represented by the presenting matrix of the $(i-1)$-st syzygy module, which is the image of φ_i.

Definition 2.1. Let $(A,\boldsymbol{m})$ be a Cohen-Macaulay (non-regular) local ring, and M a (nonfree) maximal Cohen-Macaulay module. We define $row_{CM}(A)$ to be the smallest positive integer c such that each row of the presenting matrix of M contains an element outside $\boldsymbol{m}^c$. We now define $row_{CM}(A) = \sup\{row_{CM}(M) : M : M$ is a maximal Cohen-Macaulay module without free summands$\}$, and $row^*_{CM}(A) = \sup\{row_{CM}(M) : M : M$ is a maximal Cohen-Macaulay module$\}$.

Using the syzygy modules, we also define the following invariants:

Definition 2.2. Let $(A,\boldsymbol{m})$ be a Cohen-Macaulay local ring. For a non negative integer j, we define $row_j(A) = \inf\{ t \geq 1$: for each j-th syzygy module without free summands of infinite projective dimension, each row of its presenting matrix contains an element outside $\boldsymbol{m}^t \}$, and $row^*_j(A) = \inf\{ t \geq 1$: for each j-th syzygy module of infinite projective dimension, each row of its presenting matrix contains an element outside $\boldsymbol{m}^t \}$.

If A is regular, then we define $row_{CM}(A) = 1$, and $row_j(A) = row^*_j(A) = 1$ for each non-negative integer j.

We here remark that if A is not Cohen-Macaulay and the Canonical Element Conjecture holds for A, then $row^*_j(A) = \infty$ for $0 \leq j < \dim A$. We first recall the Canonical Element Conjecture studied[1] for reader's convenience.

Canonical Element Conjecture (Hochster). Let $\boldsymbol{x} = x_1,\cdots,x_n$ be any system of parameters of a Noetherian local ring $(A,\boldsymbol{m})$ of dimension n. Let $\phi_{\bullet}$ be a map of complexes from the Koszul complex $K_{\bullet}(\boldsymbol{x};A)$ to a free resolution $F_{\bullet}$ of the residue field $A/\boldsymbol{m}$ lifting the identity map in degree 0. Then $\phi_n \neq 0$.

It is known[7,8] that this conjecture holds for rings containing a field and also in dimension ≤ 3. P. Roberts also pointed out the following statement, which was proved[1]:

Fact. (P. Roberts) Let $(A,\boldsymbol{m})$ be a local ring of dimension n. Let $\boldsymbol{x} = x_1,\cdots,x_n$ be a system of parameters. Let $(J_{\bullet}, d_{\bullet})$ be a free complex such that $H_0(J_{\bullet}) \cong A/(\boldsymbol{x})$. Let $(K_{\bullet}(n), \delta_{\bullet})$ denote the Koszul complex $K_{\bullet}(\boldsymbol{x};A)$. Let $\lambda_{\bullet}$ be a map of complexes from $(K_{\bullet}(n), \delta_{\bullet})$ to $J_{\bullet}$ which lifts the identity map on $A/(\boldsymbol{x})$. Suppose that the Canonical Element Conjecture holds for all local rings of dimension less than equal to q. Then λ_i splits for all $0 \leq i \leq q$, i.e., we may assume that $J_i = K_{i(n)} \oplus J_i'$, and $d_i = \begin{pmatrix} \delta_i & 0 \\ \alpha & \beta_i \end{pmatrix}$.

Therefore, if $\boldsymbol{x}$ is a system of parameters of a non Cohen-Macaulay ring A, and the Canonical Element Conjecture holds for A, then for each $t \geq 1$, j-th map of the minimal resolution of $A/(\boldsymbol{x}^t)$ contains a row all of whose entries are in $\boldsymbol{m}^t$ for $1 \leq j \leq \dim A$, and hence $row^*_j(A) = \infty$ for $0 \leq j < \dim A$.

The following proposition is obtained immediately from the definitions:

Proposition 2.3. Let $(A,\boldsymbol{m})$ be a Cohen-Macaulay local ring of dimension d. Then

(1) $row^*_j(A) \leq row(A) < \infty$ for all $j \geq d$, and $row^*_{(j+1)}(A) \leq row^*_j(A)$ for all $j \geq 0$.

(2) There is some $j_0 > d$ such that $row^*_j(A) = row(A)$ for all j with $d \leq j < j_0$. In particular, $row(A) = row^*_d(A)$.

Proof. (1) For a fixed $j \geq d$, suppose that $row^*_j(A)$ is finite, and let $t = row^*_j(A)$. Then there is a j-th syzygy module N such that every entry of some row of the presenting matrix of N is contained in $\boldsymbol{m}^{t-1}$. Since the presenting matrix of N is $(j+1)$-th differential map of a minimal resolution of certain module, we know $row(A) > t-1$ by the definition of $row(A)$, and thus $row(A) \geq row^*_j(A)$. If $row^*_j(A) = \infty$, then using the above reasoning, we arrive at $row(A) = \infty$, which is a contradiction.

The second part of (1) follows immediately since a $(j+1)$-th syzygy module is also a j-th syzygy module for any non-negative integer j.

(2) Let $t = row(A)$. Then there is a finitely generated A-module M of infinite projective dimension such that for some $j_0 > d$, every entry of some row of φ_{j_0} in $\boldsymbol{m}^t$,

where $(F_{\bullet}, \varphi_{\bullet})$ is a minimal resolution of M. Since φ_{j_0} is the presenting matrix of a (j_0-1)-th syzygy module, we have $row(A)=t \le row^*_{j_0-1}(A)$. By (1), we also have that for $d \le j < j_0$,

$$row(A) \le row^*_{j_0-1}(A) \le row^*_j(A) \le row(A).$$

Hence the proof is completed. ■

3. Main Results on Row Invariants

It is easy to obtain that $row_j(A)=row^*_j(A)$ for $j > depth(A)$ since the j-th syzygy module has no free summands for $j > depth(A)$ by Fact 3.2. We note that it is not necessarily true that the d-th syzygy module has no free summands for $d = depth(A)$. However, we will show that $row_d(A)=row^*_d(A)$ for $d=depth(A)$ in this section. The techniques used to get this result are quite similar to those in the paper[9]. However, we include the proofs in detail for reader's convenience.

We first see the definition of mapping cone: If $\alpha_{\bullet}$: $(F_{\bullet},\varphi_{\bullet}) \to (G_{\bullet},\psi_{\bullet})$ is a map of complexes, then the mapping cone $(M(\alpha_{\bullet})_{\bullet}, \Psi_{\bullet})$ of $\alpha_{\bullet}$ is a complex such that

$$M(\alpha_{\bullet})_i = G_i \oplus F_{i-1} \text{ and } \Psi_i = \begin{bmatrix} \psi_i & 0 \\ \alpha_{i-1} & -\varphi_{i-1} \end{bmatrix}.$$

Using mapping cones on a M-regular sequence inductively, the following lemma was proved[9].

Lemma 3.1.[9] Let M be a finitely generated A-module, and $(F_{\bullet},\varphi_{\bullet})$ its minimal resolution. Let $x_1,\cdots,x_t$ be a M-regular sequence. The i-th differential map $\psi_i^{x_t}$ of a minimal resolution of $M/(x_1,\cdots,x_t)M$ is of the form

$$\psi_i^{x_t} = \begin{bmatrix} \psi_i^{x_{t-1}} & 0 \\ x_t I & -\psi_{i-1}^{x_{t-1}} \end{bmatrix}$$

where $\psi_i^{x_{t-1}}$ is the i-th differential map of a minimal resolution of $M/(x_1,\cdots,x_t)M$, $\psi_i^{x_1} = \begin{bmatrix} \varphi_i & 0 \\ x_1 I & -\varphi_{i-1} \end{bmatrix}$, and $\psi_i^{x_1} = \begin{bmatrix} \varphi_1 \\ x_1 I \end{bmatrix}$.

The following Fact 2.2 (1), which extends the result in a paper[10], was proved[3]. Fact 2.2 (2) can be proved easily using the Ext characterization of depth and the long exact sequence of Ext.

Fact 3.2. Let (A,m) be a Noetherian local ring, and L, M, N be finitely generated A-modules.

(1) If M has an infinite projective dimension, then $Syz^i(M)$ has no free summands if $i > depth(A) - depth(M)$[3].

(2) If $0 \to L \to M \to N \to 0$ is an exact sequence, then

$$depth(L) \ge \min\{depth(M), depth(N)+1\}.$$

It was proved[9] that a d-th syzygy module without free summands can be obtained from a maximal Cohen-Macaulay module without free summands.

Theorem 3.3.[9] Let (A,m) be a Noetherian local ring and M a finitely generated A-module of positive depth d. Let s be a non-negative integer. Suppose that the j-th syzygy module of M, $Syz^j(M)$, has no A-free summands for all $j \ge s$. Then there exists a nonzero-divisor x of M such that $Syz^{1+j}(M/xM)$ has no A-free summands for all $j \ge s$.

Now we prove our main theorem:

Theorem 3.4. Let (A,m) be a Cohen-Macaulay local ring of dimension d. The values of the invariants in (1) and (2) below are all finite, and we have

(1) $row^*_d(A) = row^*_{CM}(A)$, and

(2) $row_d(A) = row_{CM}(A)$.

Proof. (1) We know that $row^*_d(A) \le row(A) < \infty$ by Proposition 2.3. By the definition of $row^*_d(A)$, there is a d-th syzygy A-module M such that the minimal presenting matrix of M has a row which is contained in $m^{row^*_d(A)-1}$. Since M is a maximal Cohen-Macaulay module by Fact 3.2, one inequality $row^*_{CM}(A) \ge row^*_d(A)$ follows from the definition of $row^*_{CM}(A)$.

For the other inequality, let's assume that $row^*_{CM}(A) = t < \infty$. Then there exists a maximal Cohen-Macaulay module M such that the minimal presenting matrix of M has a row which is contained in m^{t-1}. Let $(F_{\bullet}, \varphi_{\bullet})$ be a minimal resolution of M:

$$(F_{\bullet}, \varphi_{\bullet}) : \cdots \to A^{n_{i+1}} \xrightarrow{\varphi_{i+1}} A^{n_i} \xrightarrow{\varphi_i}$$

$$A^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \cdots \xrightarrow{\varphi_1} A^{n_0} \to 0.$$

Then we know that every entry of some column of φ_1 is contained in m^{t-1}. We choose some M-regular sequences $x_1,\cdots,x_d$ such that $x_1,\cdots,x_d$ are contained in m^{t-1}. Now we consider the d-th syzygy module $Syz^d(M/(x_{1,}\cdots,x_d)M)$. We note that the presenting matrix of $Syz^d(M/(x_{1,}\cdots,x_d)M)$ is the $(d+1)$-th differential map of a minimal resolution of $M/(x_{1,}\cdots,x_d)M$. Lemma 3.1 gives us the information of $(d+1)$-th differential map of a minimal resolution of $M/(x_{1,}\cdots,x_d)M$.

To investigate the behavior of the entries in the maps of a minimal resolution of $M/(x_{1,}\cdots,x_d)M$ and $\psi_i^{x_d}$, we illustrate with some examples of maps: we are particularly interested on the map $\psi_{d+1}^{x_d}$ for $d = depth(A)$:

(i) If $depth(A) = 2$, then

$$\psi_3^{x_2} = \begin{bmatrix} \psi_3^{x_1} & 0 \\ x_2 I & -\psi_2^{x_1} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \varphi_3 & 0 \\ x_1 I & -\varphi_2 \end{bmatrix} & 0 \\ x_2 I & \begin{bmatrix} -\varphi_2 & 0 \\ -x_1 I & \varphi_1 \end{bmatrix} \end{bmatrix}.$$

(ii) If $depth(A) = 3$, then

$$\psi_4^{x_3} = \begin{bmatrix} \psi_4^{x_2} & 0 \\ x_3 I & -\psi_3^{x_2} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \varphi_4 & 0 \\ * & * \end{bmatrix} & 0 \\ x_3 I & \begin{bmatrix} \begin{bmatrix} \varphi_3 & 0 \\ x_1 I & -\varphi_2 \end{bmatrix} & 0 \\ x_2 I & -\begin{bmatrix} -\varphi_2 & 0 \\ -x_1 I & \varphi_1 \end{bmatrix} \end{bmatrix} \end{bmatrix}.$$

(iii) If $depth(A) = d$, then by induction we have

$$\psi_{d+1}^{x_d} = \begin{bmatrix} \begin{bmatrix} \varphi_{d+1} & 0 \\ * & * \end{bmatrix} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ x_d I & \cdots & (-1)^d \begin{bmatrix} -\varphi_2 & 0 \\ -x_1 I & -\varphi_1 \end{bmatrix} \end{bmatrix}.$$

Since each x_1 belongs to m^{t-1} and some row of φ_1 is contained in m^{t-1}, the presenting matrix of $Syz^d(M/(x_{1,}\cdots,x_d)M)$ has a row whose entries are contained in m^{t-1}, and so $row_d^*(A) \geq t = row_{CM}^*(A)$. Hence $row_d^*(A) = row_{CM}^*(A)$.

If $row_{CM}^*(A) = \infty$ and t is any large integer, then using the same argument as above, we can find a d-th syzygy module X such that the minimal presenting matrix of X has a row whose entries are contained in m^{t-1}. Thus $row_d^*(A) \geq t$, i.e., $row_d^*(A)$ is infinite, which is a contradiction, and so $row_{CM}^*(A)$ is finite.

(2) We first note that there exists some M-regular sequence $x_1,\cdots,x_d$ such that $Syz^d(M/(x_{1,}\cdots,x_d)M)$ has no free summands by Theorem 3.3 if M is a maximal Cohen-Macaulay module with no free summands. Secondly, we choose M-regular sequence $x_1,\cdots,x_d$ such that $x_1,\cdots,x_d$ belong to m^k for a large enough k satisfying the conditions in Theorem 3.3 and $k \geq row_{CM}^*(A)$. Then we can complete the proof using the same argument as in (1). ■

Now we have Corollary 3.6 using the following theorem:

Theorem 3.5.[6] Let (A,m) be a Cohen-Macaulay local ring of dimension d. Then

$$row_{CM}(A) = row_{CM}^*(A).$$

Corollary 3.6. Let (A,m) be a Cohen-Macaulay local ring of dimension d. Then

$$row_d^*(A) = row_d(A).$$

Therefore, $row_j(A) = row_j^*(A)$ for $j \geq d = \dim A$.

We close this section by proving some result on a Gorenstein local ring.

Proposition 3.7.[9] Let (A,m) be a Gorenstein local ring and X a maximal Cohen-Macaulay A-module without free summands. Then for any integer $\ell \geq 0$, X is an ℓ-th syzygy.

Using the above proposition, we have the following:

Theorem 3.8. Let (A,m) be a Gorenstein local ring of dimension d. Then $row(A) = row_j^*(A)$ for all $j \geq d$.

Proof. Fix a $j \geq d$. It is enough to show that $row(A)$

$\leq row_j^*(A)$ by Proposition 2.3. Let $t = row(A)$. Then there is a finitely generated A-module M of infinite projective dimension such that for some $q > depth(A)$, every entry of some row of φ_q is in m^{t-1}.

If $q > depth(A)+1$, then since $Im(\varphi_{q-1})$ is a maximal Cohen-Macaulay module without free summands by Fact 3.2, $Im(\varphi_{q-1})$ is the j-th syzygy module by Proposition 3.7. Hence $row_j^*(A) \geq t = row(A)$ by definition.

Suppose that $q = depth(A)+1$. If the $(q-1)$-th syzygy module, $Im(\varphi_{q-1})$, has no free summands, then it is also a j-th syzygy module by Proposition 3.7, and so we have $row_j^*(A) \geq t = row(A)$. Suppose that $Im(\varphi_{q-1}) = N \oplus A$. Then we can show that the presenting matrix of N is obtained from the presenting matrix of $Im(\varphi_{q-1})$ by deleting some column of $Im(\varphi_{q-1})$ (see Theorem 2.2[9]). Thus we may assume that N is a maximal Cohen-Macaulay module without free summands, and so N is again a j-th syzygy module by Proposition 3.7. We note that the submatrix A^*, which is obtained after deleting some column of the presenting matrix of M, is the presenting matrix of N, and every entry of A^* is in m. Since A^* is a submatrix of the presenting matrix of $Im(\varphi_{q-1})$, every entry of some row of A^* is contained in m^{t-1}, and hence $row_j^*(A) \geq t = row(A)$. This completes the proof. ■

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