J. Chosun Natural Sci.
Vol. 7, No. 4 (2014) pp. 273 − 277
http://dx.doi.org/10.13160/ricns.2014.7.4.273

A Priori Boundary Estimations for an Elliptic Operator

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Abstract

In this article, we consider a singular and a degenerate elliptic operators in a divergence form. The singularities exist on a part of boundary, and comparable to the logarithmic distance function or its inverse. If we assume that the operator can be treated in a pointwise sense than distribution sense, with this operator we obtain a priori Harnack continuity near the boundary. In the proof we transform the singular elliptic operator to uniformly bounded elliptic operator with unbounded first order terms. We study this type of estimations considering a De Giorgi conjecture. In his conjecture, he proposed a certain ellipticity condition to guarantee a continuity of a solution.

Key words : Partial Differential Equations, Singular Elliptic, Degenerate Elliptic, De Giorgi Conjecture

1. Introduction

We consider the second order, linear, elliptic partial differential equation with divergence structure:

(D) $\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_ju(x)) = 0,\ n \geq 2, x \in \Omega \subset R^n.$

Here, $a_{ij}(x)$ satisfies the following elliptic condition:

(E) $\lambda(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_j\xi_j \leq \Lambda(x)|\xi|^2$

for some measurable, finite, positive function

$\lambda(x), \Lambda(x)$, for all $\xi = (\xi_1, \xi_2, .., \xi_n) \in R^n$, and almost every $x \in \Omega$. Originally, the derivative D_i, D_j is understood in a weak sense. Namely, the equation (D) means

(W) $-\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x)D_ju(x)D_i\phi(x)dx = 0$

for all $\phi(x) \in C_0^\infty(\Omega)$, where $C_0^\infty(\Omega)$ is a set of all smooth functions with compact support in a given domain Ω. Thus, we seek a solution u in some Sobolev spaces, which guarantee that the integral of (W) is finite.

In the case of $a_{ij} = \delta_{ij}$ the elliptic operator reduce to the well known Laplace operator. The equation (D) is called uniformly elliptic if $\Lambda(x)/\lambda(x)$ is essentially bounded, and strictly elliptic if $\lambda(x) \geq \lambda_0$ for some positive constant λ_0. We call uniformly bounded elliptic if $\lambda(x)^{-1}$ and $\Lambda(x)$ is essentially bounded.

On the other hand, if $\lambda(x)^{-1}$ is unbounded, it is called degenerate, and $\Lambda(x)$ is unbounded, it is called singular.

When the equation is uniformly and strictly elliptic, there are many established theories and results, for example, maximum principle, a regularity of solution, the existence of a solution, representation of solutions, etc.

Among them, we are interested in a regularity of a solution, especially continuity and discontinuity of a solution for the singular or degenerate elliptic operator.

Here, we briefly explain some classical works regarding a regularity theory for strictly uniformly elliptic case.

In the planar case, Morrey proved that a weak solution is eventually Holder continuous[1,2].

For a higher dimensions (R^n, $n \geq 3$), in the late 1950's, De Giorgi[3] and Nash[4] obtained Holder continuity for elliptic and parabolic case, independently.

A bit later, Moser proved the Harnack inequality, which easily leads to Holder continuity[5,6].

For the degenerate or singular case, to find optimal conditions for $\lambda(x)$ and $\Lambda(x)$ to guarantee the continuity of a solution is completely unsettled.

In this regard, De Giorgi gave a talk in Italy regarding the continuity of a solution, and proposed some con-

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(Received : December 7, 2014, Revised : December 15, 2014, Accepted : December 25, 2014)

jectures[7], which are enlisted below. The first one is about the singular case in higher dimensions.

Conjecture 1.1 Let $n \geq 3$. Suppose that a_{ij} satisfies (E) with $\lambda(x) = 1$ and $\Lambda(x)$ satisfying

(C1) $\int_{\Omega} \exp(\Lambda(x))dx < \infty$.

Then all weak solutions of (D) are continuous in Ω.

The second one is concerned about the degenerate case in higher dimensions.

Conjecture 1.2 Let $n \geq 3$. Suppose that a_{ij} satisfies (E) with $\Lambda(x) = 1$ and $\lambda(x)$ satisfying

(C2) $\int_{\Omega} \exp(\lambda(x)^{-1})dx < \infty$.

Then all weak solutions of (D) are continuous in Ω.

The third one concerns the singular and degenerate case in higher dimensions.

Conjecture 1.3 Let $n \geq 3$. Suppose that a_{ij} satisfies (E) with $\Lambda(x) = \lambda(x)^{-1}$ satisfying

$$\int_{\Omega} \exp(\Lambda(x)^2)dx < \infty.$$

Then all weak solutions of (D) are continuous in Ω. The fourth one concerns the degenerate case in planar case, n=2.

Conjecture 1.4 Let n=2. Suppose that a_{ij} satisfies (E) with $\Lambda(x) = 1$ and with $\lambda(x)$ satisfying

$$\int_{\Omega} \exp(\sqrt{\lambda(x)^{-1}})dx < \infty.$$

Then all weak solutions of (D) are continuous in Ω.

Conjectures 1-3 still remains open. In this direction, the best known result is due to Trudinger[8]. Regarding Conjecture 1.4, Onninen and Zhong[9] proved that all weak solution are continuous under the assumption that

$$\int_{\Omega} \exp(\alpha\sqrt{\lambda(x)^{-1}})dx < \infty \text{ for some } \alpha > 1.$$

In his talk, De Giorgi also conjectured that the previous conditions are optimal. For example, in Conjecture 1.1, one can not replace (C1) by

$$\int_{\Omega} \exp(\alpha\Lambda(x)^{1-\delta})dx < \infty$$

for some $\delta > 0$ and any $\alpha > 0$.

He gave a hint how one construct a counter example. Following his idea, Zhong[10] construct some discontinuous solutions which illustrate that Conjectures 1.1, 1.2, 1.4 are optimal. For more details, one may refer to Zhong' paper[10].

In this paper we treat some special case of a singular and a degenerate elliptic operators in a n-dimensional domain, $B_2 - B_1$. The singularities exist on a certain part of boundary, and comparable to the logarithmic distance function or its inverse. We assume that the divergence operator can be treated in a pointwise sense than distribution sense. Namely we impose the next assumption:

Assumption 1.5 We assume that the coefficients a_{ij} are smooth enough such that the solution of operator L is smooth.

The assumption is a qualitative purpose. Namely, our a priori estimations do hold independent of any norm of derivatives.

With this operator we obtain a priori Harnack continuity near the boundary. In the proof we transform the singular and degenerate elliptic operator to uniformly bounded elliptic operator with unbounded first order terms.

Theorem 1.6 For the singular case we consider $a_{ij} = -\ln(|x|-1)a'_{ij}$ for L, and a_{ij} satisfies Assumption 1.1, for some $\nu \in (0,1]$,

$$\nu|\xi|^2 \leq a'_{ij}\xi_i\xi_j \leq \nu^{-1}|\xi|^2 \text{ for any } \xi \in R^n.$$

For the degenerate case, we let $a_{ij} = \frac{1}{-\ln(|x|-1)}a'_{ij}$ for L.

And u be a solution of the following:

$$\begin{cases} Lu(x) = 0, & x \in B_2 - B_1, \\ u(x) = 0, & x \in \partial B_1, \\ u(x) = g(x), & x \in \partial B_2, \end{cases}$$

for a given bounded function g. Then the solution u is Holder continuous at any point on ∂B_1.

The singular and degenerate case is treated in Proposition 3.1 and 3.2, respectively.

2. Preliminaries

We state the comparison principle without a proof.

Lemma 2.1 Let *u* and *v* satisfy the following

$$\begin{cases} Lu \geq Lv, x \in \Omega, \\ u \leq v, \quad x \in \partial\Omega. \end{cases}$$

Then $u \leq v$ in Ω.

Note that this comparison principle immediately implies the uniqueness of a solution provided that a solution exists.

The following a priori estimation is proved as a Corollary 3.3 by Cho and Safonov[11].

Lemma 2.2 Let $r_0 = const > 0$, an open subset $\Omega' \subseteq \Omega$, and $u \in W(\Omega')$ be such that

$u > 0, Lu \leq 0$ in $\Omega' \cap \{d < r_0\}$;
$u = 0$ on $(\partial\Omega') \cap \{d < r_0\}$,

where $d = d(x) := dist(x, \partial\Omega)$, and W is a solution space. Set $u \equiv 0$ on $\Omega - \Omega'$. Then,

$$\omega(\rho) := \sup_{\Omega \cap \{d < \rho\}} u \leq \left(\frac{4\rho}{r_0}\right)^{\gamma_1} \omega(r_0)$$

for all $\rho \in (0, r_0]$, where $\gamma_1 = \gamma_1(n, \nu, \theta_0) \in (0,1]$ is the constant depending on the given quantities.

In fact, the previous lemma is proved for a uniformly bounded elliptic operator L without lower order terms. But the results[11] is improved and do hold even for a parabolic operator with singular lower order terms. For details one may refers the reference[12]. We state the following simplified form to our purpose:

Lemma 2.3 Let $r_0 = const > 0$, an open subset $\Omega' \subseteq \Omega$, and $u \in W(\Omega')$ be such that

$u > 0, Lu \leq 0$ in $\Omega' \cap \{d < r_0\}$;
$u = 0$ on $(\partial\Omega') \cap \{d < r_0\}$,

where $d = d(x) := dist(x, \partial\Omega)$, and W is a solution space. Set $u \equiv 0$ on $\Omega - \Omega'$. Then

$$\omega(\rho) := \sup_{\Omega \cap \{d < \rho\}} u \leq \left(\frac{4\rho}{r_0}\right)^{\gamma_1} \omega(r_0)$$

for all $\rho \in (0, r_0]$, where $\gamma_1 = \gamma_1(n, \nu, \theta_0) \in (0,1]$ is the constant depending on the given quantities.

Here L means the following elliptic operator:

$$\sum_{i,j=1}^{n} D_i(a'_{ij}(x) D_j u) + b_i D_i u,$$

a'_{ij} is uniformly bounded elliptic with a constant ν, $|b_i| = o(d^{-1}), d = d(X)$, namely, there exists a non-decreasing positive valued function γ on R such that $\gamma(d) \to 0$ as $d \to 0$ and $|b_i| \leq d^{-1}\gamma(d)$.

Following the proof of Theorem 3.5[11], we can easily derive the following local version of it:

Lemma 2.4 Let $\Omega := B_2 - B_1$ be a bounded open set in R^n and $\gamma_1 = \gamma_1(n, \nu, \theta_0) \in (0,1]$ be the constant in Lemma 2.3. Let a uniformly bounded elliptic operator L, and functions

$u \in W(\Omega), f \in L^{\infty}_{loc}(\Omega)$ be such that $Lu = f$ in Ω,
$u = 0$ on ∂B_1.

Then for any constant $\gamma \in (0, \gamma_1)$,

$$\sup_{B_{5/4} - B_1} d^{-\gamma}|u| \leq NF,$$

where $F := \sup_{\Omega} d^{2-\gamma} f_+$, $d = d(x) := dist(x, \partial B_1)$, and $N = N(n, \nu, \theta_0, \gamma) > 0$.

In particular, the above lemma implies the local Holder continuity on ∂B_1 from the next inequality:

$|u| \leq NFd^{\gamma}$ in $B_{5/4} - B_1$.

3. Results and Discussion

For the singular elliptic operator we have a local upper bound and Holder continuity on the boundary.

Proposition 3.1 Let $a_{ij} = -\ln(|x| - 1) a'_{ij}$ for L, and a_{ij} satisfies Assumption 1.1, for some $\nu \in (0,1]$,

$\nu|\xi|^2 \leq a'_{ij}\xi_i\xi_j \leq \nu^{-1}|\xi|^2$ for any $\xi \in R^n$.

Furthermore *u* be a solution of the following:

$$\begin{cases} Lu(x) = 0, & x \in B_2 - B_1, \\ u(x) = 0, & x \in \partial B_1, \\ u(x) = g(x), & x \in \partial B_2, \end{cases}$$

for a given positive bounded function *g*. Then the solution *u* is Holder continuous at any point on ∂B_1.

proof. For a given operator *L*, define *L'* by

$$L' = D_i(a'_{ij}D_ju) + \frac{1}{-\ln(|x|-1)}D_i(-\ln(|x|-1))a'_{ij}D_ju$$

Note that

$$D_i(\ln(|x|-1)) = (|x|-1)^{-1}\frac{x_i}{|x|}.$$

Thus

$$\left|\frac{1}{-\ln(|x|-1)}D_i(-\ln(|x|-1))\right| \le \frac{-1}{(|x|-1)\ln(|x|-1)}$$
$$= o(\frac{1}{d}).$$

By Assumption 1.5, note that $Lv = -1$ if and only if

$$L'v = \frac{1}{\ln(|x|-1)}.$$

Let v solve the following boundary value problem:

$$\begin{cases} L'v(x) = \frac{1}{\ln(|x|-1|)}, & x \in B_2 - B_1, \\ v(x) = 0, & x \in \partial B_1, \\ v(x) = |g(x)|, & x \in \partial B_2. \end{cases}$$

By simple comparison principle, we get $u \le v$ in $B_2 - B_1$. Considering $-u$, we have $|u| \le v$.

By Lemma 2.4, we also have u(x) converges to 0 as x goes to x_0 for any $x_0 \in \partial B_1$, and u is Holder continuous at any boundary point of ∂B_1.

□

Note that this result related to Conjecture 1.1, and

$$\int_{B_2-B_1} \exp(-\ln(|x|-1))dx = \infty,$$

but

$$\int_{B_2-B_1} \exp(-\alpha\ln(|x|-1))dx < \infty \text{ for } 0 < \alpha < 1.$$

This means the result is critical in a sense of De Giorgi conjecture.

Remark 3.2 We may try with $a_{ij} = (|x|-1)^{2-\beta}a'_{ij}$ for L, and uniformly bounded a_{ij}, which satisfies Assumption 1.5.

And β be any positive constant such that $0 < \beta \le \gamma_1$, where γ_1 is the constant from Lemma 2.3. But, then during the proof, for a given operator L, we need to define L' by

$$L' = D_i(a'_{ij}D_ju) + \frac{1}{(|x|-1)^{2-\beta}}D_i((|x|-1)^{2-\beta})a'_{ij}D_ju$$

Note that

$$D_i((|x|-1)^{2-\beta}) = (2-\beta)(|x|-1)^{1-\beta}\frac{x_i}{|x|}.$$

Thus we have,

$$\left|\frac{1}{(|x|-1)^{2-\beta}}D_i((|x|-1)^{2-\beta})\right| \le (2-\beta)\frac{1}{(|x|-1)}.$$

Note that this does not bounded by $o(\frac{1}{d})$.

Similarly, for the degenerate case, we obtain the following:

Proposition 3.3 Let $a_{ij} = \frac{1}{-\ln(|x|-1)}a'_{ij}$ for L, and a_{ij} satisfies Assumption 1.1, for some $\nu \in (0,1]$,

$\nu|\xi|^2 \le a'_{ij}\xi_i\xi_j \le \nu^{-1}|\xi|^2$ for any $\xi \in R^n$.

Furthermore u be a solution of the following:

$$\begin{cases} Lu(x) = 0, & x \in B_2 - B_1, \\ u(x) = 0, & x \in \partial B_1, \\ u(x) = g(x), & x \in \partial B_2, \end{cases}$$

for a given positive bounded function g. Then the solution u is Holder continuous at any point on ∂B_1.

proof. We simply follow the previous proof. For the L', we put

$$L' = D_i(a'_{ij}D_ju) - \ln(|x|-1)D_i(\frac{1}{-\ln(|x|-1)})a'_{ij}D_ju$$

Then

$$D_i(\frac{1}{-\ln(|x|-1)}) = \frac{1}{(\ln(|x|-1))^2}\frac{1}{(|x|-1)}\frac{x_2}{|x|},$$
$$\left|-\ln(|x|-1)D_i(\frac{1}{-\ln(|x|-1)})\right| \le \frac{-1}{(\ln(|x|-1))(|x|-1)}$$
$$= o(\frac{1}{(|x|-1)}) = o(\frac{1}{d}).$$

The remaining part is similar to the previous one. We omitted. □

4. Conclusion

In this paper, we treat the special type of a singular and a degenerate elliptic operators in a model domain $B_2 - B_1$, and obtain a priori Holder continuity on the boundary. For this purpose, we assume that the coefficients are smooth enough, so that we treat the divergence operator in a pointwise sense.

In future works, we have plans to extend the results to a more general regular domain and with a more general condition for the coefficients a_{ij}. This result is a priori estimations in its nature, which need to be improved to one without a existence assumption.

Acknowledgements

This work is supported by Gwangju National University of Education Research Grant 2014.

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