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Kaifangfa and Translation of Coordinate Axes

開方法과 座標軸의 平行移動

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Since ancient civilization, solving equations has become one of the most important subjects in mathematics and mathematics education. The extractions of square roots and cube roots were first dealt in Jiuzhang Suanshu in the setting of subdivisions. Extending these, Shisuo Kaifangfa and Zengcheng Kaifangfa were introduced in the 11th century and the subsequent development became one of the most important contributions to mathematics in the East Asian mathematics. The translation of coordinate axes plays an important role in school mathematics. Connecting the translation and Kaifangfa, we find strong didactical implications for improving students' understanding the history of Kaifangfa together with the translation itself although the latter is irrelevant to the former's historical development.

Keywords: Shisuo Kaifangfa, Zengcheng Kaifangfa, Translation of coordinate axes, Fanji, Yiji; 釋鎖開方法, 增乘開方法, 座標軸의 平行移動, 飜積, 益積.

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1 Introduction

It is well known that the theory of equations is one of the most important subjects in the history of mathematics. It divides into two parts, namely, constructing equations and solving them [3]. In this paper, our main concern is how to solve polynomial equations so that we will not discuss the former part. As also well known, the first attempt to solve equations was how to extract square roots and cube roots. East Asian mathematics has been developed along with Chinese mathematics. It is fixed in Jiuzhang Suanshu (九章算術) that the field of rational numbers is a basic field for Chinese mathematics and therefore, solving equations in its history is a numerical one. Extractions of square roots and cube roots were also first dealt in Jiu-

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zhang Suanshu, where the extractions were obtained by geometrical subdivisions of squares and cubes. The theory of solving general equations was first discussed in the 11th century by Jia Xian (賈憲) and Liu Yi (劉益) but their contributions were mostly lost and small fragments of its history were quoted in the books published in the 13th century. Indeed, Yang Hui (楊輝) quoted them in his books, Xiangjie Jiuzhang Suanfa (詳解九章算法, 1261) and YangHui Suanfa (1274–1275). Since mathematicians in Song–Yuan era had Tianyuanshu (天元術) to represent and manipulate polynomials, their approaches to solving equations became rather algebraic one apart from geometrical one in Jiuzhang. We discuss briefly the methods of solving equations in Song–Yuan era.

For a polynomial equation $p(x) = 0$, where $p(x) = \sum_{k=0}^{n} a_k x^k$, let α be a guess for the solution, called Chushang (初商) and let X be the remaining part of the solution, called Chishang (次商). Since $x = X + \alpha$, we have the equation for X from $p(x) = 0$:

$$\sum_{k=0}^{n} a_k (X + \alpha)^k = 0.$$

Expanding $a_k(X + \alpha)^k$ for each k, one has an equation

$$\sum_{k=0}^{n} b_k X^k = 0.$$

Repeating the same process as above, one has a numerical solution of the equation $p(x) = 0$. This method is precisely the algebraic version of the method introduced in Jiuzhang and called Shisuo Kaifangfa (釋鎖開方法) by Jia Xian.

Since $X = x - \alpha$, the above equation is

$$\sum_{k=0}^{n} b_k (x - \alpha)^k = 0.$$

For students who have studied the division algorithm for polynomials and synthetic divisions, they can easily find b_k by synthetic divisions. This process to obtain the equation for Chishang is called Zengcheng Kaifangfa (增乘開方法).

Chinese mathematicians in the 11th century have never had the process of divisions of polynomials. Indeed, divisions of polynomials in China were first introduced in Shuli Jingyun (數理精蘊, 1721). Thus Zengcheng Kaifangfa must be introduced without referring to synthetic divisions.

Introducing a concept of synthetic expansions, Hong JeongHa (洪正夏, 1684–?), the greatest Joseon mathematician showed in his book GuIlJib (九一集, 1713 –1724) [2] that the processes of Zengcheng Kaifangfa are independent of synthetic divisions [6].

We now propose that the mathematical relation between the two methods is based on the translation of coordinate axes. It is well known that the translation of coordinate axes is a very important subject in the current school mathematics. Thus most students completely understand the translation of axes. Although it is far from the history of solving equations in Song–Yuan era, our approach to the relation between Shisuo Kaifangfa and Zengcheng Kaifangfa through the translation of coordinate axes can give rise to great benefits for students to understand the numerical method of solving equations in Song–Yuan era and the translation of axes.

Furthermore, we show the mathematical structures of Fanji (飜積) and Yiji (益積) in the process of Zengcheng Kaifangfa through the translation of axes.

The reader may find all the Chinese sources of this paper in Zhongguo Kexue Jishu Dianji Tonghui Shuxuejuan (中國科學技術典籍通彙 數學卷) [1] and hence they will not be numbered as an individual reference.

2 Kaifangfa and translaion of axes

We recall a translation of Cartesian coordinate system i.e., a process of replacing the axes in the system with new axes which have the same direction as and are parallel to the old axes. The new system is determined by shifting the origin. Let (α, β) be the coordinates of the origin of the translated system, and for a point P in the plane, let (x, y) and (X, Y) be the coordinates of P in the original and the translated systems respectively. Then one has the following relations between the coordinates:

$$x = X + \alpha; \quad X = x - \alpha$$
$$y = Y + \beta; \quad Y = y - \beta$$

We now return to Shisuo Kaifangfa and Zengcheng Kaifangfa discussed in the previous section. In the present school mathematics, solutions of a polynomial equation $p(x) = 0$ are geometrically interpreted by the intersection points of the curve $y = p(x)$ and the x-axis, or $y = 0$ in Cartesian coordinate system. For a first guess α, we translate the system by shifting the origin to $(\alpha, 0)$ so the relationship between the two systems is simply

$$x = X + \alpha; \quad X = x - \alpha$$
$$y = Y; \quad Y = y$$

Thus the curve $y = p(x)$ with $p(x) = \sum_{k=0}^{n} a_k x^k$ is represented by

$$Y = p(X + \alpha),$$

i.e., $Y = \sum_{k=0}^{n} a_k (X + \alpha)^k$. We note that the x-axis, $y = 0$ is exactly the same with the X-axis, $Y = 0$. Thus solving the equation $\sum_{k=0}^{n} b_k X^k = 0$ for X in Shisuo Kaifangfa is precisely to find the intersection points of the same curve $y = p(x)$ and the X-axis of the translated system.

Furthermore, shifting the origin of the translated system to the origin of the original system, the equation in Shisuo Kaifangfa becomes

$$\sum_{k=0}^{n} b_k(x-\alpha)^k = 0$$

by the exactly same argument as above. Thus one has the equation in Zengcheng Kaifangfa again by the shift of the origin and synthetic divisions.

In all, Shisuo Kaifangfa and Zengcheng Kaifangfa are simply related by translating the coordinate axes, or shifting the origin to $(\alpha, 0)$ along the x-axis for the same curve $y = p(x)$.

For a real number α, let the affine transformation given by the translation of shifting the origin to $(\alpha, 0)$ along the x-axis be denoted by Γ_α. It is then clear that

$$\Gamma_\beta \circ \Gamma_\alpha = \Gamma_{\beta+\alpha}.$$

The above additive property of the translations implies that the equation for the third part of solution of an equation obtained by the Zengcheng Kaifangfa processes at the Chushang α and then Chishang β is exactly same with the equation at the Chushang $\alpha + \beta$. Thus if one has a sequence $\alpha_1, \alpha_2, \cdots, \alpha_n$ of numerical solutions obtained by successive applications of Zengcheng Kaifangfa to a polynomial equation, then $\sum_{k=0}^{n} \alpha_k$ gives a numerical solution of the equation. This shows the mathematical structure involved in Kaifangfa of the Eastern mathematics.

3 Fanji and Yiji

We now discuss behaviors in the processes of Zengcheng Kaifangfa to extract the nth roots of a positive number a, in other word, to solve the equation $x^n - a = 0$. In the Cartesian coordinate system, the graph $y = x^n - a$ is always strictly increasing on the interval $[0, \rightarrow)$ and hence for any α with $0 < \alpha \leq \sqrt[n]{a}$, $-a < p(\alpha) \leq 0$, where $p(x) = x^n - a$. We note that $p(\alpha)$ is clearly the constant term b_0 of the equation $\sum_{k=0}^{n} b_k X^k = 0$ for Chishang when we take α as the first guess of the solution of the given equation. The graph $Y = \sum_{k=0}^{n} b_k X^k$ in the translated system with the shift of the origin to $(\alpha, 0)$ along the x-axis, is precisely same as the graph of $y = p(x)$ and hence it is also strictly increasing on $[0, \rightarrow)$ in the translated system. Thus once we take a guess for Chishang which is on the interval $(\alpha, \sqrt[n]{a})$ relative to the original system, then the same behavior holds as above. Thus repeating these procedures, one has a good approximation of $\sqrt[n]{a}$.

Clearly every polynomial function $y = p(x)$ with a solution s of $p(x) = 0$ need not be increasing on the interval $[0, s)$. Here we assume that the equation has a positive solution because traditional Chinese and Joseon mathematicians before the 19th century have never paid any attentions to negative solutions. Thus they were puzzled when they found different cases from the extractions of nth roots. In the process of Zengcheng Kaifangfa, one has the case of the change of the signs of constant terms, called Fanji (飜積) in Ceyuan Haijing (測圓海鏡, 1282) of Li Ye (李冶, 1192–1279) or Huangu (換骨) in Shushu Jiuzhang (數書九章, 1247) of Qin Jiushao (秦九韶, 1202–1261), and that of the growth of constant terms in their absolute values with the same signs, called Yiji (益積) in Ceyuan or Toutai (投胎) in Shushu. We must point out that Fanji in Suanxue Qimeng (算學啓蒙, 1299) of Zhu Shijie (朱世傑) means the change of the signs of coefficients. For the detail of Fanji and Yiji, we also refer to [4].

In this paper, we discuss only Fanji and Yiji at the constant terms and define them in terms of modern mathematics.

For a first guess α for a solution of a polynomial equation $p(x) = 0$, the Zengcheng Kaifangfa is called Fanji at α if $p(0)p(\alpha) < 0$.
Further, it is called Yiji at α if $p(0)p(\alpha) > 0$ and $\mid p(0) \mid < \mid p(\alpha) \mid$.

Traditional Chinese and Joseon mathematicians up to the early 20th century did not have the Cartesian coordinate system and hence they could not have any idea on the graph of a function $y = p(x)$ even for polynomial functions. Furthermore, the extractions of roots were not related to equations of the type $p(x) = 0$ but to the type of $q(x) = a$ where $x \mid q(x)$. Thus in the course of solving equations by Zengcheng Kaifangfa, they could hardly accept the change of the signs of constant terms or the growth of the absolute values of constant terms.

But using the modern method of curve sketching, students can easily compare the values of the function $y = p(x)$ at the origins of the original and the translated systems and hence they can figure out Fanji and Yiji at the guess α for the solution in each step of Zengcheng Kaifangfa. Further, they may comprehend that Fanji and Yiji are closely related with the intervals on which the function $p(x)$ is increasing or decreasing as discussed in the above.

We take some examples from YangHui Suanfa and GuIlJib. YangHui Suanfa is the first book where Joseon mathematicians grasp the structure of solving general equations by Zengcheng Kaifangfa and Fanji. In GuIlJib, Hong JeongHa introduced the concept of Yiji independently from Qin Jiushao and Li Ye.

In the second part (卷下) of the book Tainmu Bilei Chengchu Jiefa (田畝比類乘除捷法, 1275) of YangHui Suanfa, Kaifangfa is discussed [5]. As in most of Chinese mathematics books, general quadratic equations are introduced to find sides of a rectangle given with its area and sum or difference of two sides.

Regarding the problems with the difference 12 of two sides and the area 864, two equations $x^2 + 12x - 864 = 0$ and $x^2 - 12x - 864 = 0$ are constructed for the shorter and longer sides x, respectively in Problem 6 and 7. For the former, the graph $y =$

$x^2+12x-864$ is strictly increasing on the interval $[0, \rightarrow)$ and hence Fanji and Yiji do not occur for Chushang which is smaller than the solution $x = 24$. For the latter, the parabola $y = x^2-12x-864$ is decreasing on $[0, 6]$ and increasing on $[6, \rightarrow)$. Thus Yiji may occur at Chushang in the interval $[0, 12]$. But it is customary that the nearest but smaller two digit number 30 than the solution $x = 36$ is taken for Chushang, and therefore, Yang Hui did not introduce Yiji. Instead, Yang Hui explains both Shisuo and Zengcheng Kaifangfa for this problem. He uses the equation of the form $x^2 = 12x + 864$ which is derived by a geometrical reasoning. Thus by the Shishuo Kaifangfa, he has the equation for Chishang by

$$(X + 30)^2 = 864 + 12(30 + X).$$

In the right side, the area (積) 864 is added by 360 so that he called the method Yiji Kaifangshu (益積開方術).

Regarding the problems with the sum 60 and the same area 864, Yang Hui solves two problems for two sides. Clearly, for the both problems, he has the same equation $-x^2+60x = 864$ by a geometrical consideration in Problem 9 and 10. Yang Hui has never constructed equations by Tianyuanshu. The equation has two positive solutions, 24 and 36 but he solves the same equation separately like all the traditional mathematicians. The function $y = -x^2 + 60x - 864$ is strictly increasing on the interval $[0, 30]$ and strictly decreasing on $[30, \rightarrow)$. Thus for the solution $x = 24$, no Fanji occurs and for the solution $x = 36$, Fanji occurs for a Chushang in the interval $[24, 36]$ and hence at the Chushang 30 as explained in Problem 10.

There is only one problem in GuIlJib which involves Yiji. We quote Problem 64 in the Chapter GuGoHoEunMun (句股互隱門) as follows:

> 今有股弦差九尺 只云股除句得七寸半 問各若干

Let the base, height and hypotenuse of the right triangle be a, b and c, respectively. Then the problem is to find three sides with $c - b = 9$ and $\frac{a}{b} = 0.75$. Since $a : b = 3 : 4$, $a : b : c = 3 : 4 : 5$ and hence one has immediately $a = 27, b = 36, c = 45$. Hong JeongHa must have known these solutions but he uses Tianyuanshu to have the equation $0.5625x^2 - 18x - 81 = 0$ for b and shows that Yiji occurs at 30, which is called IkJeokBeob, or Yijifa (益積法). Indeed, the function $y = 0.5625x^2 - 18x - 81$ is decreasing on the interval $[0, 16]$ and increasing on $[16, \rightarrow)$. Thus for any $x \in [0, 32]$, $f(x) \leq f(0)$ and hence for any Chushang in the interval, Yiji occurs. Incidentally, 32 is obtained by $0.5625x^2 - 18x - 81 \leq -81$, equivalently $0.5625x^2 - 18x \leq 0$. Since the natural choice of Chushang is 30, Hong JeongHa gets the case Yiji. Incidentally, Hong JeongHa also quoted Yang's Yiji Kaifang much later in GuIlJib.

One can easily apply the above discussions to negative Chushang or Chushang larger than solutions. We give the following example by the greatest traditional mathematician in Japan, Seki Takakazu (關孝和, ?–1708). Indeed, he makes the most innovative contributions to the theory of equations in East Asian mathematics [9, 10]. He is the first mathematician in East Asia to introduce multiple solutions including negative solutions. His theory appears in Taisei Sankei (大成算經, 1683–1711) compiled by Seki, Takebe Kataakira (建部賢明, 1661–1716) and Takebe Katahiro (建部賢弘, 1664–1739) [10]. It is also presented in a separated booklet, Kaihosanshiki (開方算式). The section Kaiho (開方) in Book 3 of Taisei Sankei, deals with these problems.

In a section Kasho (課商), i.e., the processes of estimating approximations of the solution, he takes a cubic equation $-x^3+22.75x^2-192.1875x+578.640625=0$ with the solution 7.25. This section also indicates Seki's thorough understanding about Zengcheng Kaifangfa.

Although the algorithm in the Kaifangfa involves only simple additions, subtractions and multiplications, it is rather difficult to choose reasonable guesses for the solutions. For the extractions of nth roots, Chushang is almost straightforward. Since $y=x^n-a$ is strictly increasing on the interval $[0,\to)$, one can choose the remaining estimations via $b_1x+b_0=0$ given by the tangent line to the curve at the previous guess, where $\sum_{k=0}^{n} b_kX^k=0$ is the equation for the remaining part of the solution. But for general equations, the above method is almost useless because regarding the intersection points with x-axis, the behaviors of the tangent line vary far from the original curve [7]. In China and Joseon, it is conventional until the 19th century that the next guess is obtained by adding a positive number as in Jiuzhang Suanshu (see the appendix by Yi Zhihan (易之瀚) to Siyuan Yujian Xichao (四元玉鑑細艸, 1835) by Luo Shilin (羅士琳, 1784–1853) [8]). Seki has the guess given by decreasing the previous one first in the section Kasho, although the method cannot completely solve the difficulty of the Kaifangfa.

We now return to the example. Fanji occurs at the Chushang 10 which is larger than 7.25. In fact, Seki obtains this through Chushang 5 and Chishang 5. And Fanji occurs at the Chishang -3. Here again, he has -3 by the two processes of -1 and then -2. At Chushang 10, he has the equation $-x^3-7.25x^2-37.1875x-68.234375=0$ and then he says that the equation cannot have a positive solution because all the coefficients are negative. Thus he chooses negative numbers for Chishang.

4 Conclusions

Curriculums for the current school mathematics are mostly based on the development of Western mathematics so that students and teachers/professors hardly pay any attention to the mathematical contributions achieved in the traditional Eastern mathematics. The theory of equations in Song–Yuan era is a prominent example which should be discussed in school mathematics. We took Shisuo and Zengcheng Kaifangfa, the method of solving equations in the era and investigated them based on the translation of coordinate axes dealt in the current school mathematics to reveal their mathematical structures. Through this discussion, we found strong didactical implications for improving students' understanding the history and the translation of axes although the translation of axes is completely irrelevant to the historical development of Kaifangfa in Song–Yuan era.

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