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A CLASSIFICATION OF THE SECOND ORDER PROJECTION METHODS TO SOLVE THE NAVIER-STOKES EQUATIONS

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Abstract. Many projection methods have been progressively constructed to find more accurate and efficient solution of the Navier-Stokes equations. In this paper, we consider most recently constructed projection methods: the pressure correction method, the gauge method, the consistent splitting method, the Gauge-Uzawa method, and the stabilized Gauge-Uzawa method. Each method has different background and theoretical proof. We prove equivalentness of the pressure correction method and the stabilized Gauge-Uzawa method. Also we will obtain that the Gauge-Uzawa method is equivalent to the gauge method and the consistent splitting method. We gather theoretical results of them and conclude that the results are also valid on other equivalent methods.

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1. Introduction

Given an open bounded polyhedral domain Ω in $\mathbb{R}^d$, with $d = 2$ or 3, we consider the time-dependent Navier-Stokes equations of incompressible fluids:

$$
\begin{aligned}
\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mu\triangle\mathbf{u} &= \mathbf{f}, && \text{in } \Omega,\\
\nabla\cdot\mathbf{u} &= 0, && \text{in } \Omega,\\
\mathbf{u}(0,\mathbf{x}) &= \mathbf{u}^0, && \text{in } \Omega,
\end{aligned} \tag{1.1}
$$

with vanishing Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$ and pressure mean-value $\int_\Omega p = 0$. The primitive variables are the (vector) velocity $\mathbf{u}$ and the (scalar) pressure p. The viscosity $\mu = Re^{-1}$ is the reciprocal of the Reynolds number Re. Hereafter, vectors are denoted in boldface.

The original projection method was introduced by Chorin [1] and Temam [13] in the late 60's to decouple the computation of velocity from the pressure, it quickly gained popularity in the computational fluid dynamics community, and over the years, an enormous amount of efforts [2, 4–12, 15] have been devoted to develop more accurate and efficient projection type schemes. So we can say that recently built methods are the most advanced algorithms. In this paper, we consider five projection methods: the pressure correction method [PCM] in [4, 15], the gauge method [GM] in [2, 6], the consistent splitting method [CSM] in [5], the Gauge-Uzawa method [GUM] in [7, 12], and the stabilized Gauge-Uzawa method [SGUM] in [9,10]. These are most recently introduced projection methods. PCM was built in [15] at 1996 and estimated errors for only the Stokes equations via energy estimate in [4] at 2004 and via normal mode analysis in [11] at 2005. GM in [2] was introduced at 2003 and proved stability and estimated errors for only the first order scheme in [6] at 2005. And then the second order GM was introduced and estimated errors in normal mode space for the Stokes equations in [11] at 2005. CSM in [5] had been built in 2003 and it's error is evaluated in normal space in [11] at 2005, but analysis via energy estimate of the scheme is still open problem. Research for the GUM has been started with the back ward Euler time marching algorithm to solve Navier-Stokes equations in [7] and to solve Boussinesq equations in [8], and then it is extended to the second order scheme in [12] at 2007. The stability condition for the method has been proved in [9] at 2011, but the stability condition is too strong to apply to real computational problems. So SGUM has newly been constructed in [10] at 2013.

As we mention above, each method has been developed independently and so has different theoretical result. The goal of this paper is to analyze properties of the five projection methods and then prove the following classification results:

THEOREM 1. *SGUM and PCM are equivalent projection methods.*

THEOREM 2. *GUM, GM and CSM are equivalent projection methods.*

These main theorems lead that theoretical results for a method are also valid for other equivalent algorithms. In order to summarize theoretical results, we use the following notations. One can find more knowledge for these notations in [3, 14]. Let $H^s(\Omega)$ be the Sobolev space with s derivatives in $L^2(\Omega)$, $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$ and $\mathbf{H}^s(\Omega) = (H^s(\Omega))^d$, where $d = 2, 3$. Let $\|\cdot\|_0$ denote the $\mathbf{L}^2(\Omega)$ norm, and $\langle \cdot\, , \cdot \rangle$ the corresponding inner product. Let $\|\cdot\|_s$ denote the norm of $H^s(\Omega)$ for $s \in \mathbb{R}$.

From Theorem 1, we conclude that both SGUM and PCM satisfy Lemmas 2.2, 2.3 and 3.1, below. So it means that both of them are unconditionally stable methods and that error bound of them are, for the Navier-Stokes equations,

$$
\begin{aligned}
&\tau \sum_{n=1}^{N} \left(\left\| \mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1} \right\|_0^2 + \left\| \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}^{n+1} \right\|_0^2 \right) \leq C\tau^4, \\
&\tau \sum_{n=1}^{N} \left(\left\| \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}^{n+1} \right\|_1^2 + \left\| p(t^{n+1}) - p^{n+1} \right\|_0^2 \right) \leq C\tau^2, \\
&\qquad\qquad \|\nabla \cdot \widehat{\mathbf{u}}^{n+1}\|_0 \leq C\tau^{\frac{3}{2}}
\end{aligned} \tag{1.2}
$$

and for the Stokes equations,

$$
\left\| \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}^{n+1} \right\|_1 \leq C\tau^{\frac{3}{2}}.
$$

Additionally, we conclude that Lemma 2.1 for GUM also valid for GM and CSM by Theorem 2.

This paper is organized as follows. We, in §2, reconstruct GUM and SGUM and then state the theoretical results in [9] and [10]. We will derive PCM from SGUM via change variables to prove Theorem 1 in §3. We introduce GM in §4 and then induce GUM and CSM from GM to prove Theorem 2. We finally conclude in §5 with numerical tests for both GUM and SGUM to check accuracy and stability, because we already prove that others are equivalent to one of them.

2. The Gauge-Uzawa and the stabilized Gauge-Uzawa methods

In this section. we summarize GUM and SGUM. We first derive both method briefly and directly from BDF2 time discrete Stokes equations:

$$\frac{3\mathbf{u}^{n+1}-4\mathbf{u}^{n}+\mathbf{u}^{n-1}}{2\tau}+\nabla p^{n+1}-\mu\triangle\mathbf{u}^{n+1}=\mathbf{f}(t^{n+1}). \tag{2.1}$$

GUM hires artificial variables $\widehat{\mathbf{u}}^{n+1}$ and ϕ^{n+1} satisfying

$$\widehat{\mathbf{u}}^{n+1}:=\mathbf{u}^{n+1}-\nabla\left(\phi^{n+1}-2\phi^{n}+\phi^{n-1}\right). \tag{2.2}$$

The main strategy of GUM and SGUM is to compute $\widehat{\mathbf{u}}^{n+1}$ and ϕ^{n+1}, and then calculate $\mathbf{u}^{n+1}$ by addition of the 2 functions. In the view of (2.2), $\widehat{\mathbf{u}}^{n+1}$ and ϕ^{n+1} depend on each other, so the role of $\widehat{\mathbf{u}}^{n+1}$ will be decided automatically, provided that of ϕ^{n+1} is given. We will define ϕ^{n+1} later. If we replace $\widehat{\mathbf{u}}^{n+1}$ in (2.1) with (2.2), then we obtain

$$\begin{aligned}\frac{3\widehat{\mathbf{u}}^{n+1}-4\mathbf{u}^{n}+\mathbf{u}^{n-1}}{2\tau}&+\nabla\left(p^{n+1}+\frac{3\phi^{n+1}-6\phi^{n}+3\phi^{n-1}}{2\tau}\right)\\ &-\mu\triangle\left(\widehat{\mathbf{u}}^{n+1}+\nabla\left(\phi^{n+1}-2\phi^{n}+\phi^{n-1}\right)\right)=\mathbf{f}(t^{n+1}).\end{aligned} \tag{2.3}$$

We now contemplate to define ϕ^{n+1} to split (2.3) with 2 decoupled equations. GUM in [6, 7, 11] impose ϕ^{n+1} as a solution of time discrete heat equation as

$$\frac{3\phi^{n+1}-4\phi^{n}+\phi^{n-1}}{2\tau}-\mu\triangle\phi^{n+1}:=-p^{n+1}. \tag{2.4}$$

Then (2.3) becomes

$$\begin{aligned}&\frac{3\widehat{\mathbf{u}}^{n+1}-4\mathbf{u}^{n}+\mathbf{u}^{n-1}}{2\tau}-\mu\triangle\widehat{\mathbf{u}}^{n+1}\\ &-\nabla\left(\frac{\phi^{n}-\phi^{n-1}}{\tau}-\mu\triangle\left(2\phi^{n}-\phi^{n-1}\right)\right)=\mathbf{f}(t^{n+1}).\end{aligned} \tag{2.5}$$

To deal with the third order term $\nabla\triangle\phi^{n}$, we introduce a new variable $s^{n}:=\triangle\phi^{n}$, then we arrive at GUM by 3 equations (2.2), (2.4) and (2.5) with adding a suitable discretized convection term in (2.5):

ALGORITHM 1 (The Gauge-Uzawa Method). *Set* $\phi^0=s^0=0$ *and then compute* $\mathbf{u}^1$ *and* ϕ^1 *via the first order GUM in [7]. Repeat for* $1\le n\le N=[\frac{T}{\tau}-1]$.

Step 1: *Set* $\mathbf{u}^* = 2\mathbf{u}^n - \mathbf{u}^{n-1}$ *and find* $\widehat{\mathbf{u}}^{n+1}$ *as the solution of*

$$\begin{aligned} &\frac{3\widehat{\mathbf{u}}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\tau} + (\mathbf{u}^* \cdot \nabla)\widehat{\mathbf{u}}^{n+1} - \mu\triangle\widehat{\mathbf{u}}^{n+1} \\ &\qquad -\nabla\left(\frac{\phi^n - \phi^{n-1}}{\tau} - \mu\left(2s^n - s^{n-1}\right)\right) = \mathbf{f}(t^{n+1}). \\ &\qquad \widehat{\mathbf{u}}^{n+1}|_\Gamma = \mathbf{0}. \end{aligned} \tag{2.6}$$

Step 2: *Find* ϕ^{n+1} *as the solution of*

$$\begin{aligned} &-\triangle\phi^{n+1} = -\triangle\left(2\phi^n - \phi^{n-1}\right) + \nabla\cdot\widehat{\mathbf{u}}^{n+1}, \\ &\partial_{\boldsymbol{\nu}}\phi^{n+1}|_\Gamma = 0. \end{aligned} \tag{2.7}$$

Step 3: *Update* $\mathbf{u}^{n+1}$ *and* s^{n+1} *by*

$$\begin{aligned} \mathbf{u}^{n+1} &= \widehat{\mathbf{u}}^{n+1} + \nabla\left(\phi^{n+1} - 2\phi^n + \phi^{n-1}\right) \\ s^{n+1} &= 2s^n - s^{n-1} - \nabla\cdot\widehat{\mathbf{u}}^{n+1}. \end{aligned} \tag{2.8}$$

Step 4: *Update pressure* p^{n+1} *by*

$$p^{n+1} = -\frac{3\phi^{n+1} - 4\phi^n + \phi^{n-1}}{2\tau} + \mu s^{n+1}. \tag{2.9}$$

This GUM performs superior numerical behaviour for accuracy as in §5, but the method requires rather strong stability constraint for τ in [9]:

LEMMA 2.1 (Stability of GUM Algorithm 1). *If* τ *is small enough to satisfy*

$$\tau\mu^2\|\nabla s^n\|_0^2 < M, \quad \forall 1 < n < N,$$

then the a priori bound of GUM Algorithm 1 holds

$$\begin{aligned} &\left\|\mathbf{u}^{N+1}\right\|_0^2 + \left\|2\mathbf{u}^{N+1} - \mathbf{u}^N\right\|_0^2 + 2\left\|\nabla\left(\phi^{N+1} - \phi^N\right)\right\|_0^2 + \tau\mu\sum_{n=1}^N\left\|\nabla\widehat{\mathbf{u}}^{n+1}\right\|_0^2 \\ &+ \sum_{n=1}^N\left(\left\|\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}\right\|_0^2 + 4\left\|\nabla\left(\phi^{n+1} - 2\phi^n + \phi^{n-1}\right)\right\|_0^2\right) \\ &+ 2\tau\mu\left\|s^{N+1} - s^N\right\|_0^2 \le \left\|\mathbf{u}^1\right\|_0^2 + \left\|2\mathbf{u}^1 - \mathbf{u}^0\right\|_0^2 + 2\left\|\nabla\left(\phi^1 - \phi^0\right)\right\|_0^2 \\ &\qquad\qquad + C\frac{\tau}{\mu}\sum_{n=1}^N\left\|\mathbf{f}(t^{n+1})\right\|_{-1}^2 + 2\tau\mu\left\|s^1 - s^0\right\|_0^2 + M. \end{aligned}$$

And the theoretical error estimate for GUM is still open problem.

To make improve stability, the SGUM has been constructed in [10] with replace pressure equation (2.4) to

$$\frac{3\phi^{n+1} - 3\phi^n}{2\tau} - \mu\triangle\left(\phi^{n+1} - \phi^n\right) := -p^{n+1}. \tag{2.10}$$

Then we can rewrite (2.3) by

$$\frac{3\widehat{\mathbf{u}}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\tau} - \nabla\left(\frac{3(\phi^n - \phi^{n-1})}{2\tau} - \mu\triangle\left(\phi^n - \phi^{n-1}\right)\right) \\ -\mu\triangle\widehat{\mathbf{u}}^{n+1} = \mathbf{f}(t^{n+1}). \tag{2.11}$$

Because the functions of ϕ in (2.2), (2.10) and (2.11) are represented by the subtraction of 2 consecutive functions, we use simple notation $\psi^{n+1} := \phi^{n+1} - \phi^n$. Owing to divergence free condition $\nabla\cdot\mathbf{u}^{n+1} = 0$, (2.2) gives

$$\begin{aligned} -\triangle\psi^{n+1} &= -\triangle\left(\phi^{n+1} - \phi^n\right) \\ &= -\triangle\left(\phi^n - \phi^{n-1}\right) + \nabla\cdot\widehat{\mathbf{u}}^{n+1} = -\triangle\psi^n + \nabla\cdot\widehat{\mathbf{u}}^{n+1}. \end{aligned} \tag{2.12}$$

To deal with the third order term $\nabla\triangle\phi^n$, which is a source of trouble due to lack of commutativity of the differential operators at the discrete level, we denote $q^{n+1} := \triangle\psi^{n+1}$. So (2.12) can be rewritten by

$$q^{n+1} = q^n - \nabla\cdot\widehat{\mathbf{u}}^{n+1},$$

which is connected with the Uzawa iteration. If we added up convection term in (2.11) with a suitable approximation $\mathbf{u}^* \approx 2\mathbf{u}^n - \mathbf{u}^{n-1}$, then we arrive at SGUM via gathering above equations.

ALGORITHM 2 (The stabilized Gauge-Uzawa Method). *Compute* $\mathbf{u}^1$ *and* p^1 *via any first order projection method and set* $\psi^1 = \frac{-2\tau}{3}p^1$ *and* $q^1 = 0$. *Repeat for* $1 \le n \le N = [\frac{T}{\tau} - 1]$.

Step 1: *Set* $\mathbf{u}^* = 2\mathbf{u}^n - \mathbf{u}^{n-1}$ *and find* $\widehat{\mathbf{u}}^{n+1}$ *as the solution of*

$$\frac{3\widehat{\mathbf{u}}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\tau} + \nabla p^n + (\mathbf{u}^*\cdot\nabla)\widehat{\mathbf{u}}^{n+1} - \mu\triangle\widehat{\mathbf{u}}^{n+1} = \mathbf{f}(t^{n+1}), \\ \widehat{\mathbf{u}}^{n+1}|_\Gamma = \mathbf{0}. \tag{2.13}$$

Step 2: *Find* ψ^{n+1} *as the solution of*

$$\begin{aligned} -\triangle\psi^{n+1} &= -\triangle\psi^n + \nabla\cdot\widehat{\mathbf{u}}^{n+1}, \\ \partial_{\boldsymbol{\nu}}\psi^{n+1}|_\Gamma &= 0. \end{aligned}$$

Step 3: *Update* $\mathbf{u}^{n+1}$ *and* q^{n+1} *by*

$$\mathbf{u}^{n+1} = \widehat{\mathbf{u}}^{n+1} + \nabla\left(\psi^{n+1} - \psi^{n}\right)$$
$$q^{n+1} = q^{n} - \nabla\cdot\widehat{\mathbf{u}}^{n+1}.$$

Step 4: *Update pressure* p^{n+1} *by*

$$p^{n+1} = -\frac{3\psi^{n+1}}{2\tau} + \mu q^{n+1}. \tag{2.14}$$

We remark that Algorithm 2 consists with (1.1), like GUM. The stability for SGUM Algorithm 2 has been evaluated in [10]:

LEMMA 2.2 (Stability for SGUM Algorithm 2). *The Algorithm 2 is unconditionally stable in the sense that for all* $\tau > 0$ *the following a priori bound holds:*

$$\begin{aligned}
&\left\|\widehat{\mathbf{u}}^{N+1}\right\|_0^2 + \left\|\mathbf{u}^{N+1}\right\|_0^2 + \left\|2\mathbf{u}^{N+1} - \mathbf{u}^{N}\right\|_0^2 + 3\left\|\nabla\psi^{N+1}\right\|_0^2\\
&+ 2\tau\mu\left\|q^{N+1}\right\|_0^2 + \sum_{n=1}^{N}\left(\left\|\delta\delta\mathbf{u}^{N+1}\right\|_0^2 + 3\left\|\nabla\delta\psi^{n+1}\right\|_0^2 + \tau\mu\left\|\nabla\widehat{\mathbf{u}}^{n+1}\right\|_0^2\right)\\
&\le \left\|2\mathbf{u}^{1} - \mathbf{u}^{0}\right\|_0^2 + \left\|\mathbf{u}^{0}\right\|_0^2 + 3\left\|\nabla\psi^{1}\right\|_0^2 + 2\tau\mu\left\|q^{1}\right\|_0^2 + C\frac{\tau}{\mu}\left\|\mathbf{f}(t^{n+1})\right\|_{-1}^2.
\end{aligned}$$

Also errors of SGUM Algorithm 2 has been evaluated in [10] as

LEMMA 2.3 (Error estimates for SGUM Algorithm 2). *The errors of SGUM Algorithm 2 are bounded by*

$$\tau\sum_{n=1}^{N}\left(\left\|\mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1}\right\|_0^2 + \left\|\mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}^{n+1}\right\|_0^2\right) \le C\tau^4,$$
$$\tau\sum_{n=1}^{N}\left(\left\|\mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}^{n+1}\right\|_1^2 + \left\|p(t^{n+1}) - p^{n+1}\right\|_0^2\right) \le C\tau^2,$$
$$\left\|\nabla\cdot\widehat{\mathbf{u}}^{n+1}\right\|_0 \le C\tau^{\frac{3}{2}}.$$

3. The pressure correction projection method and proof of Theorem 1

The pressure correction method has been constructed in [15] and estimated in errors in [4, 11] only for the Stokes equations. In order to prove Theorem 1, we derive the method from SGUM Algorithm 2 with defining a new variable

$$\xi^{n+1} := -\frac{3\left(\psi^{n+1}-\psi^{n}\right)}{2\tau}$$

and we subtract 2 consecutive equations of (2.14) to get

$$p^{n+1} = p^n + \xi^{n+1} - \mu\nabla\cdot\widehat{\mathbf{u}}^{n+1}.$$

Then we arrive at the rotational form of pressure correction projection method in [4, 11, 15]:

ALGORITHM 3 (The pressure correction projection method). *Repeat for* $1 \le n \le N = [\frac{T}{\tau} - 1]$.

Step 1: *Set* $\mathbf{u}^* = 2\mathbf{u}^n - \mathbf{u}^{n-1}$ *and find* $\widehat{\mathbf{u}}^{n+1}$ *as the solution of* (2.13)
Step 2: *Find* ξ^{n+1} *as the solution of*

$$\triangle\xi^{n+1} = \frac{3}{2\tau}\nabla\cdot\widehat{\mathbf{u}}^{n+1},$$
$$\partial_{\boldsymbol{\nu}}\xi^{n+1}|_\Gamma = 0.$$

Step 3: *Update* $\mathbf{u}^{n+1}$ *and* p^{n+1} *by*

$$\mathbf{u}^{n+1} = \widehat{\mathbf{u}}^{n+1} - \frac{2\tau}{3}\nabla\xi^{n+1},$$

$$p^{n+1} = \xi^{n+1} + p^n - \mu\nabla\cdot\widehat{\mathbf{u}}^{n+1}.$$

So we can conclude Theorem 1. The estimate of error of PCM in $\mathbf{L}^2$ space is performed in [4] only for the Stokes equations:

LEMMA 3.1 (The error estimates of the PCM for Stokes equations). *PCM Algorithm 3 for the Stokes equations has error bounds*

$$\tau\sum_{n=1}^{N}\left(\left\|\mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1}\right\|_0^2 + \left\|\mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}^{n+1}\right\|_0^2\right) \le C\tau^4,$$
$$\left\|\mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}^{n+1}\right\|_1 \le C\tau^{\frac{3}{2}}.$$

We note that the stability of PCM Algorithm 3 is unknown, but we conclude that PCM Algorithm 3 is unconditionally stable and holds error bound (1.2).

4. The gauge method and the consistent splitting method

In this section, we will prove Theorem 2. We first introduce GM in §4.1 and then derive GUM Algorithm 1 from GM via change variables in §4.2 to prove the equivalentness of GM and GUM Algorithms 4 and 1. And then we will derive CSM Algorithm 5 from GM Algorithm 4 in §4.3. So we will conclude Theorem 2.

4.1. The gauge method. The gauge formulation consists of rewriting (1.1) in terms of two auxiliary variables, the vector field $\mathbf{a}$ and the scalar field ϕ (gauge variable), which satisfy $\mathbf{u} = \mathbf{a} + \nabla\phi$. Upon replacing this relation into the momentum equation in (1.1), we get

$$\mathbf{a}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\left(\phi_t - \mu\triangle\phi\right) + \nabla p - \mu\triangle\mathbf{a} = \mathbf{f}.$$

Imposing

$$p = -\phi_t + \mu\triangle\phi,$$

we end up with the gauge formulation of (1.1) due to E and Liu [2]:

$$\begin{aligned} \mathbf{a}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\triangle\mathbf{a} = \mathbf{f}, &\quad \text{in } \Omega,\\ -\triangle\phi = \nabla\cdot\mathbf{a}, &\quad \text{in } \Omega,\\ \mathbf{u} = \mathbf{a} + \nabla\phi, &\quad \text{in } \Omega,\\ p = -\phi_t + \mu\triangle\phi, &\quad \text{in } \Omega. \end{aligned}$$

To enforce the boundary condition $\mathbf{u} = \mathbf{0}$, we impose

$$\partial_{\boldsymbol{\nu}}\phi = 0, \qquad \mathbf{a}\cdot\boldsymbol{\nu} = 0, \qquad \mathbf{a}\cdot\boldsymbol{\tau} = -\partial_{\boldsymbol{\tau}}\phi.$$

In order to decouple the calculation of $\mathbf{a}^{n+1}$ and ϕ^{n+1} at time step $n+1$, it is necessary to extrapolate the boundary conditions from the previous time step.

Algorithm 4 (Gauge method). *Set initial values ϕ^0, $s^0 = 0$ and $\mathbf{a}^0 = \mathbf{u}^0 = \mathbf{u}(0,\mathbf{x})$ and then compute $\mathbf{a}^1$, $\mathbf{u}^1$, ϕ^1 and s^1 by using the first order gauge method in [6]. Repeat for $1 \le n \le N$*

Step 1: *Set $\mathbf{u}^* = 2\mathbf{u}^n - \mathbf{u}^{n-1}$ and find $\mathbf{a}^{n+1}$ as the solution of*

$$\begin{aligned} &\frac{3\mathbf{a}^{n+1} - 4\mathbf{a}^n + \mathbf{a}^{n-1}}{2\tau} + (\mathbf{u}^*\cdot\nabla)\mathbf{a}^{n+1}\\ &\quad +(\mathbf{u}^*\cdot\nabla)\nabla\left(2\phi^n - \phi^{n-1}\right) - \mu\triangle\mathbf{a}^{n+1} = \mathbf{f}(t^{n+1}),\\ &\mathbf{a}^{n+1}\cdot\boldsymbol{\nu}|_\Gamma = 0, \quad \mathbf{a}^{n+1}\cdot\boldsymbol{\tau}|_\Gamma = -\,\partial_{\boldsymbol{\tau}}\left(2\phi^n - \phi^{n-1}\right). \end{aligned} \tag{4.1}$$

Step 2: *Find ϕ^{n+1} as the solution of*

$$-\triangle\phi^{n+1} = \nabla\cdot \mathbf{a}^{n+1},$$
$$\partial_{\boldsymbol{\nu}}\phi^{n+1}|_\Gamma = 0.$$

Step 3: *Update* $\mathbf{u}^{n+1}$

$$\mathbf{u}^{n+1} = \mathbf{a}^{n+1} + \nabla\phi^{n+1}. \tag{4.2}$$

One may compute the pressure whenever necessary as

$$p^{n+1} = -\frac{3\phi^{n+1} - 4\phi^{n-1} + \phi^{n-1}}{2\tau} + \mu\triangle\phi^{n+1}. \tag{4.3}$$

The GM has been analyzed only for the first order scheme in [6] and no theoretical proofs for stability and error evaluates, but stability and error bound of the 2nd order GM Algorithm 4 are still open problem.

4.2. The equivalentness of GM and GUM. In order to prove Theorem 2, we first derive GUM from GM Algorithm 4 via using a new variable

$$\widehat{\mathbf{u}}^{n+1} = \mathbf{a}^{n+1} + \nabla\left(2\phi^n - \phi^{n-1}\right). \tag{4.4}$$

If we apply 2 equations (4.2) and (4.4) into Algorithm 4, then we readily get (2.6) in GUM Algorithm 1. In order to derive other equation in GUM, we use (4.4) and $\mathbf{u}^{n+1} = \mathbf{a}^{n+1} + \nabla\phi^{n+1}$ in (4.2) to get

$$\widehat{\mathbf{u}}^{n+1} = \mathbf{u}^{n+1} - \nabla\left(\phi^{n+1} - 2\phi^n + \phi^{n-1}\right). \tag{4.5}$$

Since we have $s^{n+1} = \triangle\phi^{n+1}$, we can readily obtain equations (2.7) and (2.8). Also we already have same pressure formulas (4.3) and (2.9) of GUM and GM Algorithms 1 and 4, respectively. Thus we conclude that GUM and GM are equivalent projection methods.

4.3. The consistent splitting method. In this section, we will derive CSM from GM Algorithm 4 by change variables. We first apply (4.4) in (4.1), in conjunction with (4.3), to get

$$\frac{3\widehat{\mathbf{u}}^{n+1} - 4\widehat{\mathbf{u}}^n + \widehat{\mathbf{u}}^{n-1}}{2\tau} + \nabla\left(2p^n - p^{n-1}\right) - \mu\triangle\widehat{\mathbf{u}}^{n+1} = \mathbf{f}(t^{n+1}). \tag{4.6}$$

And we define a new variable $\psi^{n+1} := \phi^{n+1} - 2\phi^n + \phi^{n-1}$, then (4.5) become

$$\widehat{\mathbf{u}}^{n+1} = \mathbf{u}^{n+1} - \nabla\psi^{n+1}$$

and we obtain, in conjunction with $\nabla \cdot \mathbf{u}^{n+1} = 0$,

$$-\triangle \psi^{n+1} = \nabla \cdot \widehat{\mathbf{u}}^{n+1}. \tag{4.7}$$

From the pressure formula (4.3) and (4.7), we can get

$$\begin{aligned} p^{n+1} - 2p^n + p^{n-1} &= -\frac{3\psi^{n+1} - 4\psi^{n-1} + \psi^{n-1}}{2\tau} + \mu\triangle\psi^{n+1} \\ &= -\frac{3\psi^{n+1} - 4\psi^{n-1} + \psi^{n-1}}{2\tau} - \mu\nabla\cdot\widehat{\mathbf{u}}^{n+1}. \end{aligned} \tag{4.8}$$

If we define a new variable

$$\xi^{n-1} := -\frac{3\psi^{n+1} - 4\psi^{n-1} + \psi^{n-1}}{2\tau}, \tag{4.9}$$

then the pressure equation (4.8) becomes

$$p^{n+1} = 2p^n - p^{n-1} + \xi^{n-1} - \mu\nabla\cdot\widehat{\mathbf{u}}^{n+1}. \tag{4.10}$$

In light of (4.7), taking Laplace operate in both terms (4.9) yields

$$\triangle\xi := \nabla \cdot \left(\frac{3\widehat{\mathbf{u}}^{n+1} - 4\widehat{\mathbf{u}}^{n-1} + \widehat{\mathbf{u}}^{n-1}}{2\tau} \right). \tag{4.11}$$

Finally we arrive at CSM:

Algorithm 5 (The consistent splitting method). *Repeat for* $1 \le n \le N = [\frac{T}{\tau} - 1]$.

Step 1: *Find* $\widehat{\mathbf{u}}^{n+1}$ *as the solution of* (4.6).
Step 2: *Find* ξ^{n+1} *as the solution of* (4.11) *with Neumann boundary condition* $\partial_{\boldsymbol{\nu}}\xi^{n+1}|_\Gamma = 0$.
Step 3: *Update* p^{n+1} *by* (4.10).

So we arrive at Theorem 2. No theoretical proof for CSM Algorithm 5 had been known. Because we now have Theorem 2, we can conclude Lemma 2.1 is valid for CSM Algorithm 5.

5. Numerical tests

In this section, we carried out numerical experiments to compare numerical behaviour of Algorithms 1 and 2, because we proved that others are equivalent to one of them. we perform with a known solution:

$$u = \pi \sin(t)\sin(2\pi y)\sin^2(\pi x),$$
$$v = -\sin(t)\sin(2\pi x)\sin^2(\pi y),$$
$$p = -\sin(t)\cos(\pi x)\sin(\pi y).$$

In this test, we fix $\mu = 1$, and use Taylor-Hood finite element $(\mathcal{P}_2, \mathcal{P}_1)$ on the uniform mesh.

h	1/16	1/32	1/64	1/128	1/256
$\|E\|_0$	0.000374286	5.83689e-05	1.14962e-05	2.65413e-06	6.50019e-07
	Order	2.680869	2.344043	2.114846	2.029685
$\|E\|_{L^\infty}$	0.000835976	0.000166414	3.70314e-05	8.70111e-06	2.11636e-06
	Order	2.328685	2.167956	2.089478	2.039614
$\|E\|_1$	0.0532202	0.0133404	0.00333751	0.000834537	0.000208653
	Order	1.996172	1.998958	1.999724	1.999870
$\|e\|_0$	0.00220608	0.000464423	0.000115444	2.90385e-05	7.28256e-06
	Order	2.247974	2.008246	1.991154	1.995449
$\|e\|_{L^\infty}$	0.0200976	0.00145889	0.00035159	8.72965e-05	2.18391e-05
	Order	3.784080	2.052905	2.009898	1.999010

TABLE 1. Error decay of Algorithm 1.

h	1/16	1/32	1/64	1/128	1/256
$\|E\|_0$	0.000666171	0.000171224	4.67914e-05	1.26051e-05	3.30152e-06
	Order	1.960008	1.871570	1.892236	1.932805
$\|E\|_{L^\infty}$	0.00164411	0.000487813	0.000161503	4.89765e-05	1.39134e-05
	Order	1.752907	1.594767	1.721399	1.815615
$\|E\|_1$	0.053935	0.0136668	0.00347407	0.000886726	0.00022716
	Order	1.980546	1.975977	1.970067	1.964780
$\|e\|_0$	0.00578956	0.00229765	0.000860856	0.000292507	9.15238e-05
	Order	1.333295	1.416315	1.557301	1.676252
$\|e\|_{L^\infty}$	0.061034	0.0380297	0.0183978	0.00794517	0.00322284
	Order	0.682487	1.047593	1.211383	1.301746

TABLE 2. Error decay of Algorithm 2.

Tables 1 and 2 are the error decays of GUM and SGUM Algorithms 1 and 2, respectively. We can conclude that GUM Algorithm 1 is more accurate than SGUM Algorithm 2.

In order to check that SGUM and PCM Algorithms 2 and 3 are unconditionally stable, we compute driven cavity with unstable condition with $\mu = 1/10,000$ under unstable conditions $h = 1/256$ and $\tau = 0.5$. Figure 1 is the numerical result of Algorithm 2 and displays still stable even for high viscosity flow and so we conclude that SGUM and PCM Algorithms 2 and 3 are unconditionally stable.

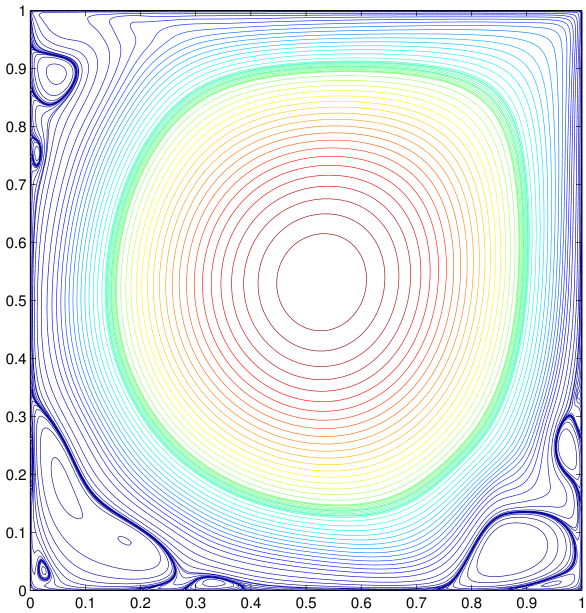

FIGURE 1. Driven cavity for Algorithm 2 with $\mu = 1/10,000$, $h = 1/256$, $\tau = 0.5$

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