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HIGHER ORDER DISCONTINUOUS GALERKIN FINITE ELEMENT METHODS FOR NONLINEAR PARABOLIC PROBLEMS

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ABSTRACT. In this paper, we consider discontinuous Galerkin finite element methods with interior penalty term to approximate the solution of nonlinear parabolic problems with mixed boundary conditions. We construct the finite element spaces of the piecewise polynomials on which we define fully discrete discontinuous Galerkin approximations using the Crank-Nicolson method. To analyze the error estimates, we construct an appropriate projection which allows us to obtain the optimal order of a priori $\ell^\infty(L^2)$ error estimates of discontinuous Galerkin approximations in both spatial and temporal directions.

1. INTRODUCTION

Discontinuous Galerkin finite element methods(DGM) have recently received a lot of interest. The advantage of DGM is the flexible decomposition of the spatial domain and the construction of the spaces of finite elements consisting of different order of polynomials without continuity requirement. Since the classical DGM was first introduced by Nitsche [8] as a method which enforced the Dirichlet boundary conditions weakly, various types of DGMs are applied to solve time dependent problems as well as elliptic problems. And the DGM was applied to solve the interface problems in [5, 6]. In [3], the authors introduced the local DGM for time-dependent convection-diffusion systems and analyzed the convergence of the approximations.

In [10], Rievière and Wheeler formulated and analyzed a family of discontinuous methods to approximate the solution of the transport problem with nonlinear reaction. They constructed

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semidiscrete approximations which converge optimally in h and suboptimally in p for the energy norm and suboptimally for the L^2 norm. They also constructed fully discrete approximations and proved the optimal convergence in the temporal direction. To solve reactive transport problems, Sun and Wheeler in [14] analyzed three discontinuous Galerkin methods, which were symmetric interior penalty Galerkin method, nonsymmetric interior penalty Galerkin method, and incomplete interior penalty Galerkin method. They obtained error estimates in $L^2(H^1)$ which are optimal in h and nearly optimal in p and they developed a parabolic lift-technique for SIPG which leads to h-optimal and nearly p-optimal error estimates in $L^2(L^2)$ and negative norms.

Rievière and Wheeler [13] formulated semidiscrete and a family of time-discrete locally conservative discontinuous Galerkin procedures for approximations to nonlinear parabolic equations and obtained the optimal spatial rates in H^1 and time truncation errors in L^2. In [9], the authors constructed discontinuous Galerkin semidiscrete approximations of the nonlinear parabolic differential problems and proved the optimal order of convergence in L^2 normed space. Furthermore the authors in [7] applied the hp-version discontinuous Galerkin finite element method with interior penalty to semilinear parabolic problems with locally Lipschitz continuous nonlinearity and analyzed the error bound of the spatially semidiscrete hp-DGM.

Rievière and Shaw [11] developed the discontinuous Galerkin finite element approximation of a nonlinear model of non-fickian diffusion in viscoelastic polymers and proved optimal orders of convergence. In [12], the authors considered dynamic linear solid viscoelasticity problems, defined a fully discrete approximation based on a spatially discontinuous Galerkin finite element method and provided an a priori error estimate. We may refer other references [4, 15] concerning DGM applied to time-dependent problems, for example, the Camassa-Holm equation or the Keller-Segel chemitoxis model.

In this paper, we approximate the solution of nonlinear parabolic problems using a discontinuous Galerkin method with interior penalties for the spatial discretization and Crank-Nicolson method for the time stepping. The main object of this paper is to obtain the optimal $\ell^\infty(L^2)$ error estimates in both spatial and temporal directions by adopting an appropriate elliptic-type projection. The rest of this paper is organized as follows: In section 2, we introduce our problem and some notations. In section 3, we construct appropriate finite element spaces, define an elliptic-type projection, and prove its approximation properties. In section 4, by applying the Crank-Nicilson method, we construct discontinuous Galerkin fully discrete approximations which yield optimal order convergence in the temporal direction as well as the spatial direction.

2. MODEL PROBLEM AND NOTATIONS

In this paper, we consider the following nonlinear parabolic equation:

$$\begin{aligned} &u_t - \nabla \cdot \{a(x,u)\nabla u\} = f(x,t,u), && \text{in } \Omega \times (0,T], \\ &u = g_D, && \text{on } \partial\Omega_D \times (0,T], \\ &(a(x,u)\nabla u) \cdot n = g_N, && \text{on } \partial\Omega_N \times (0,T], \\ &u(x,0) = u_0(x), && \text{in } \Omega, \end{aligned} \tag{2.1}$$

where Ω denotes an open convex polygonal domain in $\mathbb{R}^d$, $d = 1, 2, 3$ with its boundary $\partial\Omega$, $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N, \partial\Omega_D \cap \partial\Omega_N = \phi$, T is a given positive real number, n denotes the unit outward normal vector to $\partial\Omega$, and $u_0(x)$ and $f(x,t,u)$ are given functions. We assume that $u_0(x) \in H^s(\Omega)$ and f satisfies the locally Lipschitz continuous condition in u and assume that there exist positive constants a_* and a^* such that $a_* \leq a(x,u) \leq a^*$ for all (x,u), and a_u, a_{uu}, and a_{uuu} are bounded.

Let $\Omega_h = \{K_i\}_{i=1}^{N_h}$ be a regular quasi-uniform subdivision of Ω where K_i is an interval if $d = 1$, K_i is a triangle or a quadrilateral if $d = 2$, and K_i is a 3-simplex or parallelogram if $d = 3$. Let $h_j = \text{diam}(K_j)$ and $h = \max\limits_{1\leq j\leq N_h} h_j$. The regular subdivision requires that there exists a constant $\rho > 0$ such that each K_j contains a ball of radius ρh_j. The quasi-uniformity requires the existence of a constant $\gamma > 0$ such that

$$h/h_j \leq \gamma \ \text{ for } \ j = 1, 2, \cdots, N_h.$$

If $d = 2$(or 3), then we denote the set of the edges (resp., faces for $d = 3$) of $K_i,\ 1 \leq i \leq N_h$ by $\{e_1, e_2, \cdots, e_{M_h}\}$ where e_k has positive $d-1$ dimensional Lebesque measure, $e_k \subset \Omega$ for $1 \leq k \leq P_h$, $e_k \subset \partial\Omega_D$ for $P_h + 1 \leq k \leq L_h$ and $e_k \subset \partial\Omega_N$ for $L_h + 1 \leq k \leq M_h$. With each edge (or face) $e_k = \partial K_i \cap \partial K_j$ and $i < j$, we associate a unit normal vector n_k to E_i. For $k \geq P_h + 1$, n_k is taken to be the unit outward normal vector to $\partial\Omega$.

For an $s \geq 0,\ 1 \leq p \leq \infty$, and a domain $K \subset \mathbb{R}^d$, we denote by $W^{s,p}(K)$ the Sobolev space of order s equipped with the usual Sobolev norm $\|\cdot\|_{W^{s,p}(K)}$. As usual we simply use the notation $H^s(K)$ instead of $W^{s,2}(K)$, $\|\cdot\|_s$ instead of $\|\cdot\|_{W^{s,2}(\Omega)}$, and $\|\cdot\|_K$ instead of $\|\cdot\|_{L^p(K)}$ if $p = 2$. And we also define the usual norm and seminorm on $H^s(K)$ denoted by $\|\cdot\|_{s,K}$ and $|\cdot|_{s,K}$, respectively. We denote $(\cdot,\cdot)$ for the usual inner product of two functions.

Now for an $s \geq 0$ and a given subdivision Ω_h, let

$$H^s(\Omega_h) = \{v \in L^2(\Omega) \mid v|_{K_i} \in H^s(K_i),\ \ i = 1, 2, \cdots, N_h\}.$$

For a $v \in H^s(\Omega_h)$ with $s > \dfrac{1}{2}$, we define the average function $\{v\}$ and the jump function $[v]$ such that

$$\begin{aligned} \{v\} &= \frac{1}{2}(v|_{K_i})|_{e_k} + \frac{1}{2}(v|_{K_j})|_{e_k},\ \ \forall x \in e_k,\ 1 \leq k \leq P_h, \\ [v] &= (v|_{K_i})|_{e_k} - (v|_{K_j})|_{e_k},\ \ \forall x \in e_k,\ 1 \leq k \leq P_h, \end{aligned} \tag{2.2}$$

where $e_k = \partial K_i \cap \partial K_j$ with $i < j$. If $e_k \in \partial\Omega \cap K_i$, then

$$\{v\} = v|_{K_i}, \quad \forall x \in e_k,\ P_h + 1 \le k \le M_h,$$
$$[v] = v|_{K_i}, \quad \forall x \in e_k,\ P_h + 1 \le k \le M_h.$$

We define the following broken norms on $H^s(\Omega_h)$

$$\begin{aligned} |||v|||_0^2 &= \sum_{i=1}^{N_h} \|v\|_{0,K_i}^2, \\ |||v|||_1^2 &= \sum_{i=1}^{N_h} \|v\|_{1,K_i}^2 + \sum_{k=1}^{L_h} h\|\{\nabla v \cdot n_k\}\|_{e_k}^2 + J^\sigma(v,v), \end{aligned} \tag{2.3}$$

where

$$J^\sigma(v,w) = \sum_{k=1}^{L_h} \sigma h^{-1} \int_{e_k} [v][w]ds \tag{2.4}$$

is an interior penalty term and σ is a positive constant.

3. Approximation properties and an auxiliary projection

For a positive integer r, we construct the following finite element spaces

$$D_r(\Omega_h) = \{v \in L^2(\Omega) \mid \quad v|_{K_i} \in P_r(K_i),\ i = 1, 2, \cdots, N_h\} \tag{3.1}$$

where $P_r(K_i)$ denotes the set of polynomials of total degree less than or equal to r on K_i.

Now we state the following approximation properties and trace inequalities whose proofs are provided in [1,2]. Hereafter C denotes a positive generic constant depending on u, Ω, γ and ρ but independent of h and Δt defined in Section 4 and any two Cs in different places don't need to be equal.

Lemma 3.1. *Let $K_j \in \Omega_h$ and $v \in H^s(K_j)$. Then there exist a positive constant C depending on s, γ, and ρ but independent of v, r and h and a sequence $\{z_r^h\}_{r\ge1} \in P_r(K_j)$ such that for any $0 \le q \le s$ and $1 \le p \le \infty$*

$$\begin{aligned} &\|v - z_r^h\|_{W^{q,p}(K_j)} \le Ch_j^{\mu-q}\|v\|_{W^{s,p}(K_j)}, \quad s \ge 0, \\ &\|v - z_r^h\|_{e_j} \le Ch_j^{\mu-\frac{1}{2}}\|v\|_{s,K_j}, \quad s > \frac{1}{2}, \\ &\|v - z_r^h\|_{1,e_j} \le Ch_j^{\mu-\frac{3}{2}}\|v\|_{s,K_j}, \quad s > \frac{3}{2}, \end{aligned} \tag{3.2}$$

where $\mu = \min(r+1, s)$ and e_j is an edge or a face of K_j.

Lemma 3.2. *For each $K_j \in \Omega_h$, there exists a positive constant C depending only on γ and ρ such that the following trace inequalities hold:*

$$
\begin{aligned}
\|v\|_{e_j}^2 &\le C\left(\frac{1}{h_j}|v|_{0,K_j}^2 + h_j|v|_{1,K_j}^2\right), \ \ \forall v \in H^1(K_j), \\
\left\|\frac{\partial v}{\partial n_j}\right\|_{e_j}^2 &\le C\left(\frac{1}{h_j}|v|_{1,K_j}^2 + h_j|v|_{2,K_j}^2\right), \ \ \forall v \in H^2(K_j),
\end{aligned}
\tag{3.3}
$$

where e_j is an edge or a face of K_j and n_j is the unit outward normal vector to K_j.

Now we define the bilinear mapping $A(u : \cdot, \cdot)$ on $H^s(\Omega_h) \times H^s(\Omega_h)$ as follows:

$$
\begin{aligned}
A(u : v, w) = (a(x,u)\nabla v, \nabla w) &- \sum_{k=1}^{L_h} \int_{e_k} \{a(x,u)\nabla v \cdot n_k\}[w]dx \\
&- \sum_{k=1}^{P_h} \int_{e_k} \{a(x,u)\nabla w \cdot n_k\}[v]dx + J^\sigma(v,w).
\end{aligned}
\tag{3.4}
$$

Then the weak formulation of the problem (2.1) is given as follows:

$$
\begin{aligned}
&(u_t, v) + A(u : u, v) = (f(x,t,u)), v) + l(v), \ \forall \, v \in H^s(\Omega_h), \\
&l(v) = \sum_{k=L_h+1}^{M_h} (g_N, [v])_{e_k} + \sum_{k=P_h+1}^{L_h} (g_D, \sigma h^{-1}[v])_{e_k}, \ \forall \, v \in H^s(\Omega_h).
\end{aligned}
\tag{3.5}
$$

For a given $\lambda > 0$, we define the bilinear form $A_\lambda(u : \cdot, \cdot)$ on $H^s(\Omega_h) \times H^s(\Omega_h)$ as follows:

$$
A_\lambda(u : v, w) = A(u : v, w) + \lambda(v, w). \tag{3.6}
$$

Lemma 3.3. *For a given $\lambda > 0$, there exists a constant $C > 0$, independent of u, such that*

$$
|A_\lambda(u : v, w)| \le C|||v|||_1 |||w|||_1, \ \forall v, w \in H^s(\Omega_h). \tag{3.7}
$$

Proof. Let $v, w \in H^s(\Omega_h)$. Then we have

$$\begin{aligned}
|A_\lambda(u : v, w)| \leq & \sum_{i=1}^{N_h}(a(x,u)\nabla v, \nabla w) + \sum_{k=1}^{L_h}\int_{e_k}|\{a(x,u)\nabla v\cdot n_k\}[w]|dx \\
& + \sum_{k=1}^{P_h}\int_{e_k}|\{a(x,u)\nabla w\cdot n_k\}[v]|dx + J^\sigma(v,w) + \lambda(v,w) \\
\leq & a^*\sum_{i=1}^{N_h}\|\nabla v\|_{K_i}\|\nabla w\|_{K_i} \\
& + a^*\Big(\sum_{k=1}^{L_h}\frac{\sigma}{h}\|[w]\|_{e_k}^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{L_h}\frac{h}{\sigma}\|\{\nabla v\cdot n_k\}\|_{e_k}^2\Big)^{\frac{1}{2}} \\
& + a^*\Big(\sum_{k=1}^{P_h}\frac{h}{\sigma}\|\{\nabla w\cdot n_k\}\|_{e_k}^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{P_h}\frac{\sigma}{h}\|[v]\|_{e_k}^2\Big)^{\frac{1}{2}} \\
& + \Big(\sum_{k=1}^{L_h}\sigma h^{-1}\|[v]\|_{e_k}^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{L_h}\sigma h^{-1}\|[w]\|_{e_k}^2\Big)^{\frac{1}{2}} \\
& + \lambda(v,w) \\
\leq & C|\!|\!|v|\!|\!|_1|\!|\!|w|\!|\!|_1.
\end{aligned}$$

This completes the proof. □

Lemma 3.4. *For a given $\lambda > 0$, there exists a constant $\tilde{c} > 0$, independent of u, such that*

$$A_\lambda(u : v, v) \geq \tilde{c}|\!|\!|v|\!|\!|_1^2,\ \forall v \in D_r(\Omega_h). \tag{3.8}$$

Proof. Let $v \in D_r(\Omega_h)$. Then we have

$$\begin{aligned}
A_\lambda(u : v, v) = & \sum_{i=1}^{N_h}(a(x,u)\nabla v, \nabla v)_{K_i} - \sum_{k=1}^{L_h}\int_{e_k}\{a(x,u)\nabla v\cdot n_k\}[v]dx \\
& - \sum_{k=1}^{P_h}\int_{e_k}\{a(x,u)\nabla v\cdot n_k\}[v]dx + J^\sigma(v,v) + \lambda(v,v) \\
\geq & a_*\sum_{i=1}^{N_h}\|\nabla v\|_{K_i}^2 - 2\sum_{k=1}^{L_h}(\frac{a^*}{\mathcal{E}}h\|\{\nabla v\cdot n_k\}\|_{e_k}^2 + \mathcal{E}h^{-1}\|[v]\|_{e_k}^2) \\
& + \sum_{k=1}^{L_h}\sigma h^{-1}\|[v]\|_{e_k}^2 + \lambda\|v\|^2.
\end{aligned}$$

By Lemma 3.2, the following estimation can be obtained

$$\begin{aligned} A_\lambda(u:v,v) \geq& (a_* - \frac{a^*C}{\mathcal{E}}) \sum_{i=1}^{N_h} \|\nabla v\|_{K_i}^2 + \sum_{k=1}^{L_h} (\sigma - 2\mathcal{E}) h^{-1} \|[v]\|_{e_k}^2 + \lambda \|v\|^2 \\ \geq& C\left(\sum_{i=1}^{N_h} \|\nabla v\|_{K_i}^2 + \sum_{k=1}^{L_h} h \|\{\nabla v \cdot n_k\}\|_{e_k}^2 \right) + C \sum_{k=1}^{L_h} h^{-1} \|[v]\|_{e_k}^2 + \lambda \|v\|^2 \\ \geq& \tilde{c} \|\!|v|\!\|_1^2, \end{aligned}$$

for sufficiently large $\mathcal{E}$ and $\sigma > 2\mathcal{E}$. This completes the proof. □

Now for a given u, we construct a projection $\widetilde{u} \in D_r(\Omega_h)$ satisfying

$$A_\lambda(u : u - \widetilde{u}, v) = 0, \quad \forall v \in D_r(\Omega_h), \tag{3.9}$$

Then, by Lemma 3.3 and Lemma 3.4, $\widetilde{u}$ is obviously well-defined. We denote

$$\eta(x,t) = u(x,t) - \tilde{u}(x,t), \;\; \theta(x,t) = \hat{u}(x,t) - \tilde{u}(x,t),$$

where $\hat{u}(x,t)$ is the approximation of $u(x,t)$ satisfying the approximation properties of Lemma 3.1.

Lemma 3.5. *If $\mu \geq \frac{d}{2} + 1$ and $u \in H^s(\Omega)$, then there exist constants $C > 0$ and $C^* > 0$ such that*

$$\|\!|\theta|\!\|_1 \leq C h^{\mu-1} \|u\|_s, \;\; \|\theta\|_{L^\infty} \leq \frac{C^*}{3},$$

where $\mu = \min(r+1, s)$.

Proof. By Lemma 3.4, we have

$$\begin{aligned} \tilde{c} \|\!|\theta|\!\|_1^2 \leq A_\lambda(u : \theta, \theta) &= A_\lambda(u : \theta - \eta, \theta) \\ &= A_\lambda(u : \hat{u} - u, \theta) \leq C \|\!|\hat{u} - u|\!\|_1 \|\!|\theta|\!\|_1 \end{aligned}$$

and hence $\|\!|\theta|\!\|_1 \leq C \|\!|\hat{u} - u|\!\|_1$. By Lemma 3.1 we get

$$\begin{aligned} \|\!|\theta|\!\|_1 \leq& C (\sum_{i=1}^{N_h} \|\hat{u} - u\|_{1,K_i}^2 + \sum_{k=1}^{L_h} h \|\{\nabla(\hat{u} - u) \cdot n_k\}\|_{e_k}^2 + \sum_{k=1}^{L_h} h^{-1} \|[\hat{u} - u]^2\|_{e_k})^{\frac{1}{2}} \\ \leq& C (h^{\mu-1} + h^{\frac{1}{2}} h^{\mu - \frac{3}{2}} + h^{-\frac{1}{2}} h^{\mu - \frac{1}{2}}) \|u\|_s \leq C h^{\mu-1} \|u\|_s. \end{aligned}$$

If $\mu \geq \frac{d}{2} + 1$, then by the inverse inequality

$$\|\theta\|_{L^\infty} \leq C h^{-\frac{d}{2}} h^{\mu-1} \|u\|_s \leq \frac{C^*}{3}$$

for some constant C^*. This completes the proof. □

Now we state the following approximation results for η and η_t whose proofs can be found in [9].

Theorem 3.1. *If $u \in H^s(\Omega)$ and $u_t \in H^s(\Omega)$, then there exists a constant C, independent of h, such that*
(i) $\|\eta\| + h|\!|\!|\eta|\!|\!|_1 \leq Ch^\mu \|u\|_s$;
(ii) $\|\eta_t\| + h|\!|\!|\eta_t|\!|\!|_1 \leq Ch^\mu(\|u\|_s + \|u_t\|_s)$,
where $\mu = \min(r+1, s)$.

And by following the ideas in the proofs of Theorem 3.1, we obtain the following results for η_{tt} and η_{ttt}.

Theorem 3.2. *If $u \in H^s(\Omega)$, $u_t \in H^s(\Omega)$, $u_{tt} \in H^s(\Omega)$ and $u_{ttt} \in H^s(\Omega)$, then there exists a constant C, independent of h, such that*
(i) $\|\eta_{tt}\| + h|\!|\!|\eta_{tt}|\!|\!|_1 \leq Ch^\mu\{\|u\|_s + \|u_t\|_s + \|u_{tt}\|_s\}$;
(ii) $\|\eta_{ttt}\| + h|\!|\!|\eta_{ttt}|\!|\!|_1 \leq Ch^\mu\{\|u\|_s + \|u_t\|_s + \|u_{tt}\|_s + \|u_{ttt}\|_s\}$,
where $\mu = \min(r+1, s)$.

4. The optimal $\ell^\infty(L^2)$ error estimates of fully discrete approximations

Now using Crank-Nicolson method, we construct the fully discrete discontinuous Galerkin approximations for nonlinear parabolic problems and prove the optimal convergence in L^2 normed space. For a positive integer N, let $\Delta t = \dfrac{T}{N}$, $t^j = j(\Delta t)$ for $j = 0, 1, \cdots, N$, and $t^{j+\frac{1}{2}} = \dfrac{1}{2}(t^j + t^{j+1})$ for $j = 0, 1, \cdots, N-1$. For a function $g(x,t)$ defined on $\Omega \times [0,T]$, let $g^j = g(t^j) = g(x, t^j)$ for $j = 0, 1, \cdots, N$ and $\partial_t g^j = \dfrac{g^{j+1} - g^j}{\Delta t}$ and $g^{j+\frac{1}{2}} = \dfrac{1}{2}(g^j + g^{j+1})$ for $j = 0, 1, \cdots, N-1$.

Then the extrapolated Crank-Nicolson discontinuous Galerkin approximation $\{U^j\}_{j=0}^N \subset D_r(\Omega_h)$ is defined as follows: for $j = 1, 2, \cdots, N-1$

$$(\partial_t U^j, v) + A(EU^j : U^{j+\frac{1}{2}}, v) = (f(x, t^{j+\frac{1}{2}}, EU^j), v) + l(v), \ \ \forall v \in D_r(\Omega_h) \tag{4.1}$$

and

$$\begin{aligned} &(\partial_t U^0, v) + A(U_0 : U^{\frac{1}{2}}, v) = \left(f(x, t^{\frac{1}{2}}, U^{\frac{1}{2}}), v\right) + l(v), \ \ \forall v \in D_r(\Omega_h), \\ &U^0(x) = \widetilde{u}(x, 0) = \widetilde{u}_0(x), \end{aligned} \tag{4.2}$$

where $EU^j = \frac{3}{2}U^j - \frac{1}{2}U^{j-1}$. To prove the optimal convergence of $u^j - U^j$ in L^2 normed space, we denote

$$\xi^j = \widetilde{u}^j - U^j, \ j = 0, \ 1, \ \cdots, \ N.$$

By simple computations and the applications of Theorem 3.2, we obtain the following Lemma 4.1 and Lemma 4.2.

Lemma 4.1. *If $u \in L^\infty(H^s(\Omega)), u_t \in L^\infty(H^s(\Omega)), u_{tt} \in L^\infty(H^s(\Omega))$ and $u_{ttt} \in L^\infty(H^s(\Omega))$ and if*

$$\rho^{j+\frac{1}{2}} = \frac{\partial_t \widetilde{u}^j - \widetilde{u}_t(t^{j+\frac{1}{2}})}{\Delta t},$$

then there exists a constant C, independent of h and Δt, such that

$$\|\rho^{j+\frac{1}{2}}\|_0 \le C\Delta t(\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(H^s)} + \|u_{ttt}\|_{L^\infty(H^s)});$$
$$\|\rho^{j+\frac{1}{2}}\|_1 \le C\Delta t(\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(H^s)} + \|u_{ttt}\|_{L^\infty(H^s)}).$$

Lemma 4.2. *If $u \in L^\infty(H^s(\Omega)), u_t \in L^\infty(H^s(\Omega))$ and $u_{tt} \in L^\infty(H^s(\Omega))$ and if $r^{j+\frac{1}{2}} = \widetilde{u}(t^{j+\frac{1}{2}}) - \widetilde{u}^{j+\frac{1}{2}}$, then there exists a constant C, independent of h and Δt, such that*

$$\|r^{j+\frac{1}{2}}\|_0 \le C(\Delta t)^2(\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(H^s)});$$
$$\|r^{j+\frac{1}{2}}\|_1 \le C(\Delta t)^2(\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(H^s)}).$$

Lemma 4.3. *If $u \in L^\infty(H^s(\Omega)), u_t \in L^\infty(H^s(\Omega))$ and $u_{tt} \in L^\infty(H^s(\Omega))$ and if $\mu > \frac{d}{2}+1$ and $\varphi^{j+\frac{1}{2}} = \tilde{u}(t^{j+\frac{1}{2}}) - E\tilde{u}(t^j)$, then the following statements hold:*
(i) $\|\varphi^{j+\frac{1}{2}}\| \le C(\Delta t)^2$;
(ii) $\|\nabla \tilde{u}^j\|_\infty$ *is bounded.*

Proof. By the simple calculation, we obtain

$$\begin{aligned}\|\varphi^{j+\frac{1}{2}}\| =& \|\tilde{u}(t^{j+\frac{1}{2}}) - \frac{3}{2}\tilde{u}(t^j) + \frac{1}{2}\tilde{u}(t^{j-1})\| \\ \le& \|\tilde{u}(t^j) + \frac{1}{2}\Delta t\tilde{u}_t(t^j) - \frac{3}{2}\tilde{u}(t^j) + \frac{1}{2}(\tilde{u}(t^j) - \Delta t\tilde{u}_t(t^j))\| + C(\Delta t)^2 \\ \le& C(\Delta t)^2.\end{aligned}$$

Therefore the bound in (i) holds. And we get the statement (ii) in the following way

$$\begin{aligned}\|\nabla\tilde{u}^j\|_\infty &\le \|\nabla\tilde{u}^j - \nabla u^j\|_\infty + \|\nabla u^j\|_\infty \\ &\le \|\nabla\tilde{u}^j - \nabla\hat{u}^j\|_\infty + \|\nabla\hat{u}^j - \nabla u^j\|_\infty + \|\nabla u^j\|_\infty \\ &\le Ch^{-\frac{d}{2}}\|\nabla\theta^j\| + h^{\mu-1-\frac{d}{2}}\|u\| + \|\nabla u^j\|_\infty \\ &\le Ch^{\mu-1-\frac{d}{2}}\|u\| + \|\nabla u^j\|_\infty \le C\end{aligned}$$

if $\mu > \frac{d}{2}+1$. This completes the proof. □

Theorem 4.1. *For $0 < \lambda < 1$, if $u \in L^\infty(H^s(\Omega)), u_t \in L^\infty(H^s(\Omega)), u_{tt} \in L^\infty(H^s(\Omega))$ and $u_{ttt} \in L^\infty(H^s(\Omega))$, then there exists a constant $C > 0$, independent on h and Δt, such that*

$$\|\xi^1\|_0 \le C(h^\mu + (\Delta t)^2),$$
$$\|e^1\|_0 \le C(h^\mu + (\Delta t)^2),$$

where $\mu = \min(r+1, s)$ and $\mu \ge \frac{d}{2}+1$.

Proof. The proof of Theorem 4.1 is very similar to one of Theorem 4.2 which will be given later, in detail. So we will skip the proof of Theorem 4.1. This completes the proof. □

Theorem 4.2. *Suppose that the assumptions of Lemma 4.1 and 4.2 hold and that*

$$|f(x,t,u)-f(x,t,\alpha)| \le C(C^*,u)|u-\alpha|,$$
$$|a(x,u)-a(x,\alpha)| \le C(C^*,u)|u-\alpha|$$

for $|u-\alpha| < C^*$. *Then for* $0<\lambda<1$, *there exists a constant* $C>0$, *independent on* h *and* Δt, *such that for* $j = 0, 1, \cdots, N$

$$\|u(t^j)-U^j\|_0 \le C(h^\mu + (\Delta t)^2). \tag{4.3}$$

hold where $\mu = \min(r+1, s)$ *and* $\mu \ge \frac{d}{2}+1$.

Proof. To prove this theorem, we will prove, by mathematical induction, that

$$\|\xi^n\|_0 \le C(h^\mu + (\Delta t)^2), \qquad n = 0, 1, \cdots, N.$$

By (4.2) and Theorem 4.1, $\|\xi^0\|_0 \le Ch^\mu$ and $\|\xi^1\|_0 \le C(h^\mu + (\Delta t)^2)$, respectively. Now we suppose that for all j, $0 \le j \le N-1$ we have

$$\|\xi^j\|_0 \le C(h^\mu + (\Delta t)^2). \tag{4.4}$$

From (4.1) and (3.5), we have

$$\begin{aligned} &(u_t(t^{j+\frac12}) - \partial_t U^j, v) + A_\lambda(u(t^{j+\frac12}) : u(t^{j+\frac12}), v) - A_\lambda(EU^j : U^{j+\frac12}, v) \\ &= (f(x, t^{j+\frac12}, u(t^{j+\frac12})) - f(x, t^{j+\frac12}, EU^j), v) + \lambda(u(t^{j+\frac12}) - U^{j+\frac12}, v). \end{aligned} \tag{4.5}$$

By the notations of η and ξ, we get

$$u_t(t^{j+\frac12}) - \partial_t U^j = u_t(t^{j+\frac12}) - \partial_t \widetilde{u}^j + \partial_t \widetilde{u}^j - \partial_t U^j = \eta_t(t^{j+\frac12}) - \Delta t \rho^{j+\frac12} + \partial_t \xi^j. \tag{4.6}$$

From the definition of η and ξ, we obtain

$$\begin{aligned} &A_\lambda(u(t^{j+\frac12}) : u(t^{j+\frac12}), v) - A_\lambda(EU^j : U^{j+\frac12}, v) \\ &= A_\lambda(EU^j : \xi^{j+\frac12}, v) + A_\lambda(u(t^{j+\frac12}) : \eta(t^{j+\frac12}), v) \\ &\quad + A_\lambda(u(t^{j+\frac12}) : \widetilde{u}(t^{j+\frac12}) - \widetilde{u}^{j+\frac12}, v) \\ &\quad + A_\lambda(u(t^{j+\frac12}) : \tilde{u}^{j+\frac12}, v) - A_\lambda(EU^j : \tilde{u}^{j+\frac12}, v). \end{aligned} \tag{4.7}$$

Substituting (4.6) and (4.7) in (4.5) and choosing $v = \xi^{j+\frac12}$ in (4.5), we have

$$\begin{aligned} &(\partial_t \xi^j, \xi^{j+\frac12}) + A_\lambda(EU^j : \xi^{j+\frac12}, \xi^{j+\frac12}) \\ &= -(\eta_t(t^{j+\frac12}) - \Delta t \rho^{j+\frac12}, \xi^{j+\frac12}) - A_\lambda(u(t^{j+\frac12}) : \eta(t^{j+\frac12}), \xi^{j+\frac12}) \\ &\quad - A_\lambda(u(t^{j+\frac12}) : r^{j+\frac12}, \xi^{j+\frac12}) - A_\lambda(u(t^{j+\frac12}) : \tilde{u}^{j+\frac12}, \xi^{j+\frac12}) \\ &\quad + A_\lambda(EU^j : \tilde{u}^{j+\frac12}, \xi^{j+\frac12}) + \lambda(u(t_{j+\frac12}) - U^{j+\frac12}, \xi^{j+\frac12}) \\ &\quad + (f(x, t^{j+\frac12}, u(t^{j+\frac12})) - f(x, t^{j+\frac12}, EU^j), \xi^{j+\frac12}). \end{aligned} \tag{4.8}$$

Notice that

$$(\partial_t \xi^j, \xi^{j+\frac{1}{2}}) = \frac{1}{2\Delta t}(|||\xi^{j+1}|||_0^2 - |||\xi^j|||_0^2). \tag{4.9}$$

Applying (4.9) in (4.8) and using Lemma 3.4, we obtain

$$\begin{aligned}
&\frac{1}{2\Delta t}(|||\xi^{j+1}|||_0^2 - |||\xi^j|||_0^2) + \tilde{c}|||\xi^{j+\frac{1}{2}}|||_1^2 \\
\leq\ & -(\eta_t(t^{j+\frac{1}{2}}) - \Delta t \rho^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}}) \\
& + \lambda(u(t^{j+\frac{1}{2}}) - U^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}}) \\
& - A_\lambda(u(t^{j+\frac{1}{2}}) : r^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}}) \\
& - \left(A_\lambda(u(t^{j+\frac{1}{2}}) : \tilde{u}^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}}) - A_\lambda(EU^j : \tilde{u}^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}})\right) \\
& + \left(f(x, t^{j+\frac{1}{2}}, u^{j+\frac{1}{2}}) - f(x, t^{j+\frac{1}{2}}, EU^j), \xi^{j+\frac{1}{2}}\right) \\
=\ & \sum_{i=1}^{5} I_i.
\end{aligned} \tag{4.10}$$

By applying Lemma 4.1 there exists a constant $C > 0$ such that

$$\begin{aligned}
|I_1| &\leq (|||\eta_t(t^{j+\frac{1}{2}})|||_0 + |||\Delta t \rho^{j+\frac{1}{2}}|||_0)|||\xi^{j+\frac{1}{2}}|||_0 \\
&\leq C(|||\eta_t(t^{j+\frac{1}{2}})|||_0^2 + (\Delta t)^2 |||\rho^{j+\frac{1}{2}}|||_0^2 + |||\xi^{j+1}|||_0^2 + |||\xi^j|||_0^2) \\
&\leq C(h^{2\mu} + (\Delta t)^4) + |||\xi^{j+1}|||_0^2 + |||\xi^j|||_0^2).
\end{aligned}$$

For sufficiently small $\epsilon > 0$ we obtain the following estimates of I_2 and I_3:

$$\begin{aligned}
|I_2| &\leq \lambda(|||\eta(t^{j+\frac{1}{2}})|||_0 + |||r^{j+\frac{1}{2}}|||_0 + |||\xi^{j+\frac{1}{2}}|||_0)|||\xi^{j+\frac{1}{2}}|||_0 \\
&\leq C(h^{2\mu} + (\Delta t)^4 + |||\xi^j|||_0^2 + |||\xi^{j+1}|||_0^2),
\end{aligned}$$

$$|I_3| \leq C|||r^{j+\frac{1}{2}}|||_1 |||\xi^{j+\frac{1}{2}}|||_1 \leq C(\Delta t)^4 + \varepsilon |||\xi^{j+\frac{1}{2}}|||_1^2.$$

Now to calculate the bound for I_4, we split it into 3 terms as follows:

$$\begin{aligned}
I_4 =& \left((a(x, u(t^{j+\frac{1}{2}})) - a(x, EU^j))\nabla \tilde{u}^{j+\frac{1}{2}}, \nabla \xi^{j+\frac{1}{2}}\right) \\
&- \sum_{k=1}^{L_h} \int_{e_k} \{(a(x, u(t^{j+\frac{1}{2}})) - a(x, EU^j))\nabla \tilde{u}^{j+\frac{1}{2}} \cdot n_k\}[\xi^{j+\frac{1}{2}}] \\
&- \sum_{k=1}^{P_h} \int_{e_k} \{(a(x, u(t^{j+\frac{1}{2}})) - a(x, EU^j))\nabla \xi^{j+\frac{1}{2}} \cdot n_k\}[\tilde{u}^{j+\frac{1}{2}}] = \sum_{i=1}^{3} I_{4i}.
\end{aligned}$$

Note that by Taylor's expansion, Lemma 3.1, Lemma 3.5 and the assumption (4.4), we get

$$\begin{aligned}\|u(t^{j+\frac{1}{2}})-EU^j\|_{L^\infty}\leq&\|u(t^{j+\frac{1}{2}})-Eu^j\|_{L^\infty}+\|Eu^j-E\hat{u}^j\|_{L^\infty}\\&+\|E\hat{u}^j-E\tilde{u}^j\|_{L^\infty}+\|E\tilde{u}^j-EU^j\|_{L^\infty}\\\leq&C((\Delta t)^2+h^\mu)+\frac{C^*}{3}+\frac{C^*}{3}\leq C^*\end{aligned}\tag{4.11}$$

for sufficiently small h and Δt. By (4.11) we obtain

$$|a(x,u(t^{j+\frac{1}{2}}))-a(x,EU^j)|\leq C(C^*)|u(t^{j+\frac{1}{2}})-EU^j|$$

and by Lemma 4.3

$$\begin{aligned}I_{41}&=\left|\left((a(x,u(t^{j+\frac{1}{2}}))-a(x,EU^j))\nabla\tilde{u}^{j+\frac{1}{2}},\nabla\xi^{j+\frac{1}{2}}\right)\right|\\&\leq C\|\nabla\tilde{u}^{j+\frac{1}{2}}\|_{L^\infty}\left(\|\eta(t^{j+\frac{1}{2}})\|+\Delta t^2+|\!|\!|\xi^j|\!|\!|_0+|\!|\!|\xi^{j-1}|\!|\!|_0\right)|\!|\!|\nabla\xi^{j+\frac{1}{2}}|\!|\!|_0\\&\leq C(h^{2\mu}+(\Delta t)^4+|\!|\!|\xi^j|\!|\!|_0^2+|\!|\!|\xi^{j-1}|\!|\!|_0^2)+\epsilon|\!|\!|\xi^{j+\frac{1}{2}}|\!|\!|_1^2.\end{aligned}$$

Similarly there exists a constant $C>0$ such that

$$\begin{aligned}|I_{42}|\leq&\sum_{k=1}^{L_h}C\|\nabla\tilde{u}^{j+\frac{1}{2}}\|_{L^\infty(e_k)}(\|\eta(t^{j+\frac{1}{2}})\|_{e_k}+\|\varphi^{j+\frac{1}{2}}\|_{e_k}+\|\xi^j\|_{e_k}+\|\xi^{j-1}\|_{e_k})\|[\xi^{j+\frac{1}{2}}]\|_{e_k}\\\leq&C\sum_{i=1}^{N_h}\|\nabla\tilde{u}^{j+\frac{1}{2}}\|_{L^\infty(K_i)}(h^{-\frac{1}{2}}\|\eta(t^{j+\frac{1}{2}})\|_{K_i}+h^{\frac{1}{2}}\|\nabla\eta(t^{j+\frac{1}{2}})\|_{K_i}+h^{-\frac{1}{2}}\|\varphi^{j+\frac{1}{2}}\|_{K_i}\\&+h^{-\frac{1}{2}}\|\xi^j\|_{K_i}+h^{-\frac{1}{2}}\|\xi^{j-1}\|_{K_i})h^{\frac{1}{2}}|\!|\!|\xi^{j+\frac{1}{2}}|\!|\!|_1\\\leq&C(|\!|\!|\eta(t^{j+\frac{1}{2}})|\!|\!|_0^2+h^2|\!|\!|\nabla\eta(t^{j+\frac{1}{2}})|\!|\!|_0^2+(\Delta t)^4+|\!|\!|\xi^j|\!|\!|_0^2+|\!|\!|\xi^{j-1}|\!|\!|_0^2)+\epsilon|\!|\!|\xi^{j+\frac{1}{2}}|\!|\!|_1^2\\\leq&C(h^{2\mu}+(\Delta t)^4+|\!|\!|\xi^j|\!|\!|_0^2+|\!|\!|\xi^{j-1}|\!|\!|_0^2)+\epsilon|\!|\!|\xi^{j+\frac{1}{2}}|\!|\!|_1^2.\end{aligned}$$

Since $[u]=0$ on $e_k\in P_h$, we have

$$\begin{aligned}|I_{43}|\leq&C\sum_{k=1}^{P_h}\|\nabla\xi^{j+\frac{1}{2}}\|_{L^\infty(e_k)}(\|\eta(t^{j+\frac{1}{2}})\|_{e_k}+\|\varphi^{j+\frac{1}{2}}\|_{e_k}+\|\xi^j\|_{e_k}+\|\xi^{j-1}\|_{e_k})\|[\eta^{j+\frac{1}{2}}]\|_{e_k}\\\leq&C\sum_{i=1}^{N_h}(\|\nabla\xi^{j+\frac{1}{2}}\|_{L^\infty(K_i)})h^{-\frac{1}{2}}(\|\eta(t^{j+\frac{1}{2}})\|_{K_i}+h\|\nabla\eta(t^{j+\frac{1}{2}})\|_{K_i}+\|\varphi^{j+\frac{1}{2}}\|_{K_i}\\&+\|\xi^j\|_{K_i}+\|\xi^{j-1}\|_{K_i})h^{-\frac{1}{2}}(\|\eta^{j+\frac{1}{2}}\|_{K_i}+h\|\nabla\eta^{j+\frac{1}{2}}\|_{K_i})\\\leq&C\sum_{i=1}^{N_h}(\|\nabla\xi^{j+\frac{1}{2}}\|_{K_i}h^{-\frac{d}{2}})h^{-1}(h^\mu+(\Delta t)^2+\|\xi^j\|_{K_i}+\|\xi^{j-1}\|_{K_i})h^\mu.\end{aligned}$$

Since $\mu \geq \frac{d}{2} + 1$, we obtain

$$|I_{43}| \leq C(h^{2\mu} + (\Delta t)^4 + \|\xi^j\|_0^2 + \|\xi^{j-1}\|_0^2) + \epsilon \|\xi^{j+\frac{1}{2}}\|_1^2.$$

From the bounds of $|I_{4i}|,\ 1 \leq i \leq 3$, we get

$$|I_4| \leq C(h^{2\mu} + (\Delta t)^4 + \|\xi^j\|_0^2 + \|\xi^{j-1}\|_0^2) + 3\epsilon \|\xi^{j+\frac{1}{2}}\|_1^2.$$

Now we compute the bound of I_5

$$\begin{aligned}
|I_5| &= |(f(x, t^{j+\frac{1}{2}}, u(t^{j+\frac{1}{2}})) - f(x, t^{j+\frac{1}{2}}, EU^j), \xi^{j+\frac{1}{2}})| \\
&\leq C\|u(t^{j+\frac{1}{2}}) - EU^j\|_0^2 + \|\xi^{j+\frac{1}{2}}\|_0^2 \\
&\leq C(\|u(t^{j+\frac{1}{2}}) - \tilde{u}(t^{j+\frac{1}{2}})\|_0^2 + \|\tilde{u}(t^{j+\frac{1}{2}}) - E(\tilde{u}^j)\|_0^2 + \|E\xi^j\|_0^2) \\
&\quad + C(\|\xi^{j+1}\|_0^2 + \|\xi^j\|_0^2) \\
&\leq C((\Delta t)^4 + h^{2\mu} + \|\xi^{j+1}\|_0^2 + \|\xi^j\|_0^2 + \|\xi^{j-1}\|_0^2).
\end{aligned}$$

Substituting the bounds of $I_i, 1 \leq i \leq 5$, into (4.10), we get

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\xi^{j+1}\|_0^2 - \|\xi^j\|_0^2) + \frac{\tilde{c}}{2}\|\xi^{j+\frac{1}{2}}\|_1^2 \\
&\leq C\Big(h^{2\mu} + (\Delta t)^4 + \|\xi^{j+1}\|_0^2 + \|\xi^j\|_0^2 + \|\xi^{j-1}\|_0^2\Big),
\end{aligned} \tag{4.12}$$

for sufficiently small ϵ. If we sum both sides of (4.12) from $j = 1$ to $N - 1$, then we obtain

$$\|\xi^N\|_0^2 - \|\xi^1\|_0^2 \leq C\Big\{(\Delta t)\sum_{j=1}^{N-1}\Big(h^{2\mu} + (\Delta t)^4\Big) + (\Delta t)\sum_{j=0}^{N}\|\xi^j\|_0^2\Big\},$$

which implies

$$\|\xi^N\|_0^2 \leq \|\xi^1\|_0^2 + C(\Delta t)\sum_{j=1}^{N-1}\Big(h^{2\mu} + (\Delta t)^4\Big) + C(\Delta t)\sum_{j=0}^{N}\|\xi^j\|_0^2,$$

where Δt is sufficiently small. By applying the discrete version of Gronwall's inequality, we have

$$\|\xi^N\|_0^2 \leq C\Big(\|\xi^1\|_0^2 + \Delta t\sum_{j=1}^{N-1}\big(h^{2\mu} + (\Delta t)^4\big)\Big).$$

Therefore we prove by mathematical induction that

$$\|\xi^n\|_0 \leq C(h^\mu + (\Delta t)^2), \qquad n = 0, 1, \cdots, N,$$

which implies that

$$\|e\|_{\ell^\infty(L^2)} := \max_{0 \leq n \leq N}\|e^n\|_0 \leq C(h^\mu + (\Delta t)^2),$$

that is, we obtain the optimal $\ell^\infty(L^2)$ error estimation of the fully discrete solutions. This completes the proof. □

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