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REEB FLOW INVARIANT UNIT TANGENT SPHERE BUNDLES

Jong Taek Cho and Sun Hyang Chun*

Abstract. For unit tangent sphere bundles T_1M with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$, we have two fundamental operators that is, $h = \frac{1}{2}\pounds_\xi\phi$ and $\ell = \bar{R}(\cdot, \xi)\xi$, where $\pounds_\xi$ denotes Lie differentiation for the Reeb vector field ξ and $\bar{R}$ denotes the Riemmannian curvature tensor of T_1M. In this paper, we study the Reeb flow invariancy of the corresponding $(0, 2)$-tensor fields H and L of h and ℓ, respectively.

1. Introduction

For a given contact metric structure $(\eta, \bar{g}, \phi, \xi)$, a symmetry type occurs when the geodesic flow generated by ξ, which is called the *Reeb flow*, leaves some structure tensors invariant. This is always the case for ξ and η since $\pounds_\xi\xi = 0$ and $\pounds_\xi\eta = 0$. The metric $\bar{g}$ is left invariant by the Reeb flow (or equivalently, the flow consists of local isometries or ξ is a Killing vector field) if and only if ϕ is preserved under the Reeb flow. Apart from the defining structure tensors η, $\bar{g}$, ϕ and ξ, two other operators play a fundamental role in contact metric geometry, namely, the structural operator $h = \frac{1}{2}\pounds_\xi\phi$ and the characteristic Jacobi operator $\ell = \bar{R}(\cdot, \xi)\xi$, where $\pounds_\xi$ denotes Lie differentiation in the characteristic direction ξ.

An important topic in the study of the contact metric structure $(\eta, \bar{g}, \phi, \xi)$ on unit tangent sphere bundles T_1M is to determine those Riemannian manifolds (M, g) for which the corresponding contact metric structure enjoys such a symmetry along the Reeb flow. In fact, Y. Tashiro ([11]) proved that ξ is a Killing vector on the unit tangent

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*Corresponding author

sphere bundle T_1M if and only if (M, g) has constant curvature $c = 1$. E. Boeckx and the present authors ([6]) proved that T_1M satisfies $\pounds_\xi h = 0$ if and only if (M, g) is of constant curvature $c = 1$ and T_1M satisfies $\pounds_\xi \ell = 0$ if and only if (M, g) is of constant curvature $c = 0$ or $c = 1$.

In the present paper, we define the (0,2)-tensor fields L and H by $L(\bar{X}, \bar{Y}) = g(\ell\bar{X}, \bar{Y})$ and $H(\bar{X}, \bar{Y}) = g(h\bar{X}, \bar{Y})$ for any vector fields $\bar{X}$ and $\bar{Y}$ on $\bar{M}$ and we investigate when the (0,2)-tensor fields L and H on T_1M are preserved by the geodesic flow. Namely, we prove:

Theorem 1. *Let T_1M be the unit tangent sphere bundle with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. Then T_1M satisfies $\pounds_\xi L = 0$ if and only if (M, g) is of constant curvature $c = -4$, $c = 0$ or $c = 1$.*

Theorem 2. *Let T_1M be the unit tangent sphere bundle with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. Then T_1M satisfies $\pounds_\xi H = 0$ if and only if (M, g) is of constant curvature $c = -1$ or $c = 1$.*

From the results in [6] and Theorems 1 and 2, we find an evident distinction of the Reeb flow invariancy between h, ℓ and the corresponding $(0, 2)$-tensor fields H, L, respectively. That is, $T_1\mathbb{H}(-1)$ satisfies $\pounds_\xi H = 0$, but $\pounds_\xi h \neq 0$. And $T_1\mathbb{H}(-4)$ satisfies $\pounds_\xi L = 0$, but $\pounds_\xi \ell \neq 0$.

2. The standard contact metric structure on a unit tangent sphere bundle

We start by reviewing some fundamental facts on contact metric manifolds. We refer to [1] for more details. All manifolds are assumed to be connected and of class C^∞. A $(2n-1)$-dimensional manifold $\bar{M}$ is said to be a *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^{n-1} \neq 0$ everywhere on $\bar{M}$, where the exponent denotes the $(n-1)$-th exterior power of the exterior derivative $d\eta$ of η. We call such η a *contact form* of $\bar{M}$. It is well known that given a contact form η, there exists a unique vector field ξ, which is called the *characteristic vector field* or the *Reeb vector field*, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, \bar{X}) = 0$ for any vector field $\bar{X}$ on $\bar{M}$. A Riemannian metric $\bar{g}$ on $\bar{M}$ is an associated metric to a contact form η if there exists a $(1, 1)$-tensor field ϕ satisfying

$$(2.1)\quad \eta(\bar{X}) = \bar{g}(\bar{X}, \xi), \quad d\eta(\bar{X}, \bar{Y}) = \bar{g}(\bar{X}, \phi\bar{Y}), \quad \phi^2\bar{X} = -\bar{X} + \eta(\bar{X})\xi,$$

where $\bar{X}$ and $\bar{Y}$ are vector fields on $\bar{M}$. From (2.1) it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \bar{g}(\phi\bar{X}, \phi\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \eta(\bar{X})\eta(\bar{Y}).$$

A Riemannian manifold $\bar{M}$ equipped with structure tensors $(\eta, \bar{g}, \phi, \xi)$ satisfying (2.1) is said to be a *contact metric manifold.* Given a contact metric manifold $\bar{M}$, we define the *structural operator* h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes Lie differentiation. Then we may observe that h is self-adjoint and satisfies

$$h\xi = 0 \quad \text{and} \quad h\phi = -\phi h, \tag{2.2}$$

$$\bar{\nabla}_{\bar{X}}\xi = -\phi\bar{X} - \phi h\bar{X}, \tag{2.3}$$

where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$. From (2.2) and (2.3) we see that each trajectory of ξ is a geodesic. We denote by $\bar{R}$ the Riemannian curvature tensor defined by

$$\bar{R}(\bar{X}, \bar{Y})\bar{Z} = \bar{\nabla}_{\bar{X}}(\bar{\nabla}_{\bar{Y}}\bar{Z}) - \bar{\nabla}_{\bar{Y}}(\bar{\nabla}_{\bar{X}}\bar{Z}) - \bar{\nabla}_{[\bar{X},\bar{Y}]}\bar{Z}$$

for all vector fields $\bar{X}, \bar{Y}, \bar{Z}$ on $\bar{M}$. Along a trajectory of ξ, the Jacobi operator $\ell = \bar{R}(\cdot, \xi)\xi$ is a symmetric $(1,1)$-tensor field. We call it the *characteristic Jacobi operator.* We have

$$\ell = \phi\ell\phi - 2(h^2 + \phi^2), \tag{2.4}$$

$$\bar{\nabla}_\xi h = \phi - \phi\ell - \phi h^2. \tag{2.5}$$

A contact metric manifold for which ξ is Killing is called a *K-contact manifold.* It is easy to see that a contact metric manifold is K-contact if and only if $h = 0$ or, equivalently, $\ell = I - \eta \otimes \xi$.

The basic facts and fundamental formulae about tangent bundles are well-known (cf. [7], [9], [13]). We briefly review some notations and definitions. Let (M, g) be an n-dimensional Riemannian manifold and ∇ the associated Levi-Civita connection. R denotes its Riemannian curvature tensor. The tangent bundle over (M, g) is denoted by TM and consists of pairs (p, u), where p is a point in M and u a tangent vector to M at p. The mapping $\pi : TM \to M$, $\pi(p, u) = p$, is the natural projection from TM onto M. For a vector field X on M, its *vertical lift* X^v on TM is the vector field defined by $X^v\omega = \omega(X) \circ \pi$, where ω is a 1-form on M. For the Levi-Civita connection ∇ on M, the *horizontal lift* X^h of X is defined by $X^h\omega = \nabla_X\omega$. The tangent bundle TM can be endowed in a natural way with a Riemannian metric $\tilde{g}$, the so-called *Sasaki metric*, depending only on the Riemannian metric g on M. It is determined by

$$\tilde{g}(X^h, Y^h) = \tilde{g}(X^v, Y^v) = g(X, Y) \circ \pi, \quad \tilde{g}(X^h, Y^v) = 0$$

for all vector fields X and Y on M. Also, TM admits an almost complex structure tensor J defined by $JX^h = X^v$ and $JX^v = -X^h$. Then $\tilde{g}$ is a Hermitian metric for the almost complex structure J.

The unit tangent sphere bundle $\bar{\pi} : T_1M \to M$ is a hypersurface of TM given by $g_p(u, u) = 1$. Note that $\bar{\pi} = \pi \circ i$, where i is the immersion of T_1M into TM. A unit normal vector field $N = u^v$ to T_1M is given by the vertical lift of u for (p, u). The horizontal lift of a vector is tangent to T_1M, but the vertical lift of a vector is not tangent to T_1M in general. So, we define the *tangential lift* of X to $(p, u) \in T_1M$ by

$$X^t_{(p,u)} = (X - g(X, u)u)^v.$$

Clearly, the tangent space $T_{(p,u)}T_1M$ is spanned by vectors of the form X^h and X^t, where $X \in T_pM$.

We now define the standard contact metric structure of the unit tangent sphere bundle T_1M over a Riemannian manifold (M, g). The metric g' on T_1M is induced from the Sasaki metric $\tilde{g}$ on TM. Using the almost complex structure J on TM, we define a unit vector field ξ', a 1-form η' and a (1,1)-tensor field ϕ' on T_1M by

$$\xi' = -JN, \quad \phi' = J - \eta' \otimes N.$$

Since $g'(\bar{X}, \phi'\bar{Y}) = 2d\eta'(\bar{X}, \bar{Y})$, (η', g', ϕ', ξ') is not a contact metric structure. Rectifying this by

$$\xi = 2\xi', \quad \eta = \frac{1}{2}\,\eta', \quad \phi = \phi', \quad \bar{g} = \frac{1}{4}\,g',$$

we get the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. Here the tensor ϕ is explicitly given by

$$\phi X^t = -X^h + \frac{1}{2}\,g(X, u)\xi, \quad \phi X^h = X^t, \tag{2.6}$$

where X and Y are vector fields on M.

From now on, we consider $T_1M = (T_1M; \eta, \bar{g}, \phi, \xi)$ with the standard contact metric structure. Then the Levi-Civita connection $\bar{\nabla}$ of T_1M is described by

$$\begin{aligned} \bar{\nabla}_{X^t}Y^t &= -g(Y, u)X^t, \\ \bar{\nabla}_{X^t}Y^h &= \frac{1}{2}\,(R(u, X)Y)^h, \\ \bar{\nabla}_{X^h}Y^t &= (\nabla_X Y)^t + \frac{1}{2}\,(R(u, Y)X)^h, \\ \bar{\nabla}_{X^h}Y^h &= (\nabla_X Y)^h - \frac{1}{2}\,(R(X, Y)u)^t \end{aligned} \tag{2.7}$$

for all vector fields X and Y on M (cf. [2], [3]).

For the Riemannian curvature tensor $\bar{R}$, we give only the two expressions we need for the characteristic Jacobi operator ℓ:

$$
\begin{aligned}
\bar{R}(X^t,Y^h)Z^h =& -\frac{1}{2}\{R(Y,Z)(X-g(X,u)u)\}^t \\
&+\frac{1}{4}\{R(Y,R(u,X)Z)u\}^t \\
&-\frac{1}{2}\{(\nabla_Y R)(u,X)Z\}^h, \\
\bar{R}(X^h,Y^h)Z^h =&(R(X,Y)Z)^h+\frac{1}{2}\{R(u,R(X,Y)u)Z\}^h \\
&-\frac{1}{4}\{R(u,R(Y,Z)u)X-R(u,R(X,Z)u)Y\}^h \\
&+\frac{1}{2}\{(\nabla_Z R)(X,Y)u\}^t
\end{aligned}
\tag{2.8}
$$

for all vector fields X, Y and Z on M. From $\xi=2u^h$ and (2.7), it follows

$$
\bar{\nabla}_{X^t}\xi=-2\phi X^t-(R_uX)^h,\quad \bar{\nabla}_{X^h}\xi=-(R_uX)^t \tag{2.9}
$$

where $R_u=R(\cdot,u)u$ is the Jacobi operator associated with the unit vector u. From (2.3) and (2.9), it follows that

$$
\begin{aligned}
hX^t &= X^t-(R_uX)^t, \\
hX^h &= -X^h+\frac{1}{2}g(X,u)\xi+(R_uX)^h.
\end{aligned}
\tag{2.10}
$$

Using the formulae (2.8), we get

$$
\begin{aligned}
\ell X^t &= (R_u^2X)^t+2(R_u'X)^h, \\
\ell X^h &= 4(R_uX)^h-3(R_u^2X)^h+2(R_u'X)^t
\end{aligned}
\tag{2.11}
$$

where $R_u'=(\nabla_uR)(\cdot,u)u$ and $R_u^2=R(R(\cdot,u)u,u)u$. By using (2.5), (2.6) and (2.8) we obtain

$$
\begin{aligned}
(\bar{\nabla}_\xi h)X^t &= -2(R_uX)^h+2(R_u^2X)^h-2(R_u'X)^t, \\
(\bar{\nabla}_\xi h)X^h &= -2(R_uX)^t+2(R_u^2X)^t+2(R_u'X)^h.
\end{aligned}
\tag{2.12}
$$

Finally, from (2.7) and (2.11) we compute

$$
\begin{aligned}
(\bar{\nabla}_\xi \ell)X^t &= 4(R_u'R_uX+R_uR_u'X)^t+4(R_u''X+R_u^2X-R_u^3X)^h, \\
(\bar{\nabla}_\xi \ell)X^h &= 8(R_u'X-R_u'R_uX-R_uR_u'X)^h+4(R_u''X+R_u^2X-R_u^3X)^t.
\end{aligned}
\tag{2.13}
$$

The above formulae (2.9) $\sim$ (2.13) are also found in [4], [5], [6].

We define the (0,2)-tensor fields H and L by $H(\bar{X},\bar{Y}) = g(h\bar{X},\bar{Y})$ and $L(\bar{X},\bar{Y}) = g(\ell\bar{X},\bar{Y})$ for any vector fields $\bar{X}$ and $\bar{Y}$ on $\bar{M}$, respectively.

3. Proofs of Theorems

Proof of Theorem 1.

At first, from the definition of Lie differentiation and (2.3), we have

$$\begin{aligned}
&(\pounds_\xi L)(\bar{X},\bar{Y}) \\
&= \xi L(\bar{X},\bar{Y}) - L(\pounds_\xi \bar{X},\bar{Y}) - L(\bar{X},\pounds_\xi \bar{Y}) \\
&= g((\bar{\nabla}_\xi \ell)\bar{X},\bar{Y}) + g(\ell(\bar{\nabla}_{\bar{X}}\xi),\bar{Y}) + g(\ell\bar{X},\nabla_{\bar{Y}}\xi) \\
&= g((\bar{\nabla}_\xi \ell)\bar{X},\bar{Y}) + g(-\ell\phi\bar{X} - \ell\phi h\bar{X},\bar{Y}) + g(\ell\bar{X},-\phi\bar{Y} - \phi h\bar{Y}).
\end{aligned} \tag{3.1}$$

From (3.1), we see that the condition $\pounds_\xi L = 0$ is equivalent to

$$\bar{\nabla}_\xi \ell = \ell\phi - \phi\ell + \ell\phi h - h\phi\ell. \tag{3.2}$$

Now we suppose that T_1M satisfies $\pounds_\xi L = 0$. Then from (3.2), by a straightforward calculation, we have two equations:

$$\begin{aligned}
0 =& (4R'_uX + R'_uR_uX + R_uR'_uX)^t \\
&+ (2R''_uX - 3R^2_uX - R^3_uX + 4R_uX)^h,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
0 =& (4R'_uX - 5R'_uR_uX - 5R_uR'_uX)^h \\
&+ (2R''_uX - 3R^2_uX - R^3_uX + 4R_uX)^t.
\end{aligned} \tag{3.4}$$

These equations are equivalent to the conditions:

$$4R'_uX + R'_uR_uX + R_uR'_uX = 0, \tag{3.5}$$

$$4R'_uX - 5R'_uR_uX - 5R_uR'_uX = 0, \tag{3.6}$$

$$2R''_uX - 3R^2_uX - R^3_uX + 4R_uX = 0 \tag{3.7}$$

for all vector fields X on M. From (3.5) and (3.6), we obtain $R'_uX = 0$. This implies that (M,g) is a locally symmetric space ([8], [12]). Further, we see from (3.7) that the eigenvalues of R_u are constant and equal to 0 or 1 or -4, i.e., (M,g) is a globally Osserman space (i.e., the eigenvalues of R_u do not depend on the point p and not on the choice of unit vector u at p). However, a locally symmetric globally Osserman space is locally flat or locally isometric to a rank one symmetric space ([10]). Therefore, we conclude that M is a space of constant curvature $c = 0$ or $c = 1$ or -4. Conversely, if (M,g) is of constant curvature c, then we can calculate the following explicit expressions for h, ℓ, $\bar\nabla_\xi h$ and $\bar\nabla_\xi \ell$ from (2.10) $\sim$ (2.13):

$$
\begin{aligned}
&hX^t = (1-c)X^t, \quad hX^h = (c-1)(X^h - \frac{1}{2}g(X,u)\xi),\\
&\ell X^t = c^2X^t, \qquad\quad \ell X^h = (4c-3c^2)(X^h - \frac{1}{2}g(X,u)\xi),\\
&(\bar\nabla_\xi h)X^t = 2(c^2-c)(X^h - \frac{1}{2}g(X,u)\xi), \quad (\bar\nabla_\xi h)X^h = 2(c^2-c)X^t,\\
&(\bar\nabla_\xi \ell)X^t = 4(c^2-c^3)(X^h - \frac{1}{2}g(X,u)\xi), \quad (\bar\nabla_\xi \ell)X^h = 4(c^2-c^3)X^t
\end{aligned} \tag{3.8}
$$

for vector fields X on M. From (3.8), we easily check that T_1M satisfies (3.2) when $c=-4$, $c=0$ or $c=1$. □

Proof of Theorem 2.

From the definition of Lie differentiation and (2.3), we have

$$
\begin{aligned}
&(\pounds_\xi H)(\bar X,\bar Y)\\
&= \xi H(\bar X,\bar Y) - H(\pounds_\xi \bar X,\bar Y) - H(\bar X,\pounds_\xi \bar Y)\\
&= g((\bar\nabla_\xi h)\bar X,\bar Y) + g(h(\bar\nabla_{\bar X}\xi),\bar Y) + g(h\bar X,\bar\nabla_{\bar Y}\xi)\\
&= g((\bar\nabla_\xi h)\bar X,\bar Y) + g(-h\phi\bar X - h\phi h\bar X,\bar Y) + g(h\bar X,-\phi\bar Y - \phi h\bar Y)\\
&= g((\bar\nabla_\xi h)\bar X,\bar Y) + 2g(\phi h\bar X,\bar Y).
\end{aligned} \tag{3.9}
$$

From (3.9), we see that the condition $\pounds_\xi H = 0$ is equivalent to

$$\bar\nabla_\xi h = 2h\phi. \tag{3.10}$$

We suppose that T_1M satisfies $\pounds_\xi H = 0$. Then from (3.10), by a straightforward calculation, we have two equations:

$$(R_u^2X - X)^h - (R'_uX)^t = 0, \tag{3.11}$$

$$(R_u^2X - X)^t + (R'_uX)^h = 0 \tag{3.12}$$

for any vector field X perpendicular to u on M. From (3.11) and (3.12), we obtain $R_u^2X - X = 0$ and $R'_uX = 0$. Using the similar arguments as in the proof of Theorem 1, we can conclude that the base manifold (M,g) must be locally symmetric and of constant curvature 1 or -1. Conversely, when (M,g) has constant curvature $c=-1$ or $c=1$, we show that (3.10) holds. □

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Jong Taek Cho
Department of Mathematics, Chonnam National University,
Gwangju 500-757, Korea.
E-mail: jtcho@chonnam.ac.kr

Sun Hyang Chun
Department of Mathematics, Chosun University,
Gwangju 501-759, Korea.
E-mail: shchun@chosun.ac.kr