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A Decomposition of the Work of Leading Mathematical Discussions with Single Case Questions

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Mathematical discussion has been highlighted so that what students do actually guides their learning of mathematics and mathematical practice. However, the work of leading mathematical discussions has not yet been specified in such a way that it can be adequately studied and taught to teachers. This study analyzes a teacher's lessons that show full engagement in leading discussions, and examines the work of leading mathematical discussions in elementary classrooms. It identifies and illustrates the central tasks of leading mathematical discussions with single case questions with five steps. This article argues several key issues in leading mathematical discussions: helping students engage in struggling with important mathematical ideas, treating mathematical connections in an explicit and public way to have coherent and structured discussions, and parsing the work of teaching at a grain size that is usable in educating teachers

I. Introduction

Mathematical discussions have been studied and highlighted as constituting the environment in which communicating mathematics and doing mathematical work occurs (e.g., Cobb & Bauersfeld, 1995; Cobb, Yackel, & McClain, 2000; Lampert, 1990). Through discussions, students learn both linguistic forms and the values and beliefs of the community of mathematics (Gee, 1990; Lampert, 1990; McNair, 1998). Furthermore, discussions serve as a powerful tool to develop students' own thinking and ability to contribute their own views (Kozulin, 1990; Sfard, 2000). Therefore, teachers need to have the ability to lead mathematical discussions. For example, teachers are expected to create a classroom environment where students can develop their mathematical understanding (Ball & Bass, 2000b; Lampert, 2001). All in all, how to lead mathematical discussion is an important object that should be taught and learned in teacher education. An important issue involved in leading such discussions is that they have an intricate nature of practice and include many tasks and moves which are invisible to a casual observer (Ball & Forzani, 2009; Lewis, 2007). However, few studies have

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examined in detail what is involved in leading mathematical discussions. It is a difficult job to manage mathematical discussions so that they have the pedagogical purposes that are typically expected in teaching practice.

Leading mathematical discussions is a question of what to do when. For example, knowing what a rectangle is does not require knowing how to act and how to communicate that knowledge. What steps make up leading mathematical discussions and what tasks of teaching are expected to be performed in each step are not specified. Furthermore, in terms of teaching, it seems apparent that leading a mathematical discussion is important and relevant to improving student learning. However, it is difficult to explain how it matters (Hiebert & Grouws, 2007). Teachers create mathematical learning opportunities for students through considering students' entry level knowledge, the nature and purpose of the activities, and the likelihood of engagement (Hiebert & Grouws, 2007). Therefore, articulating what is involved in leading mathematical discussions could enable the study of the connection between teaching and student achievement.

Furthermore, understanding the work of leading mathematical discussions has implications for the preparation of teachers. An awareness of the need to focus teacher education on practice has come into the spotlight (Ball & Forzani, 2007; Franke, Kazemi, & Battey, 2007; Grossman et al., 2009). One of demands on this approach to teacher education requires unpacking and breaking down the complicated and incorporated work of teaching into its constituent parts and, thus, studying, analyzing, and rehearsing it. Grossman and her colleagues (2009) call it the "decomposition of practice." The purpose of this study is to decompose the work of leading mathematical discussions with single case mathematics questions which contain typical tasks that teachers often teach through the use of textbooks in many elementary schools; conceptualize steps for leading mathematical discussions; and specify the core aspect of teaching in each step. Two research questions frames the research:

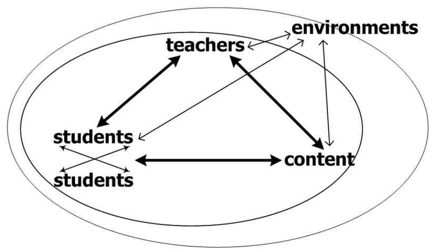
1. What are the steps for leading classroom discussions with single case mathematical questions?
2. What tasks of teaching would be carried out to lead mathematical discussions?

This study is grounded in empirical data, but it is a study of teaching not teachers (Hiebert & Grouws, 2007; Sleep, 2012). According to the clarification by Sleep (2012) about a way to approach a study of teaching, I analyzed data from lessons taught by a particular teacher. This research sometimes describes an aspect of a lesson. The purpose of such an example is not to claim that a particular teacher was not able to teach mathematics well or to highlight the quality of the people who do that work. Rather, this research aims to describe the work of teaching, in particular, the work of leading classroom discussions.

II. Theoretical Framework

This research is based on Cohen's (2011) identification of discussion in terms of instructional discourse, Cohen, Raudenbush, and Ball's (2003) specification about interactions in instruction where mathematics discussion is occurring, and Sleep's (2012) clarification of types of the work of teaching in the instructional context. Cohen (2011) specifies that instructional discourse is a socially organized means to extend and exchange knowledge, and identifies four types of direct discourse in instruction: individual seatwork, lecture, recitation, and discussion. Among them, discussion is less restrictive, and it offers teachers plenty of opportunities to make connections with learning because what students make of instruction is vividly presented. Students also can shape the discourse by arguing, explaining, and questioning. Although discussion could be disrupted by students, teachers have to control this disruption by discipline. In other words, if participation remains high and relevant, few discipline problems arise that teachers must manage apart from discussion (p.145). To lead a discussion, teachers require more specialized resources to manage complex interaction, and, thus, invite students to work as novice inquirers. Cohen points out that in discussion uncertainty becomes central to instruction because the explanation and justification of ideas open up different ways to think about issues and make those differences central to the class's work and because students' participation increases (p.159). Teachers need to have a good deal of student commentary to attend to, some of which is puzzling, and they must make many decisions about the conduct of instruction.

Apparently, discussion depends on interactions among teachers, students, and content in a certain environment. Cohen et al. (2003) introduce instructional triangle as shown in Figure II-1. In this model, they emphasize interactions between teachers, students, content, and environments in teaching and learning. Interaction refers to no particular form of discourse but instead to teachers' and students' connected work, extending through, days, weeks, and months (Cohen et al., 2003, p. 122). In brief, teachers interact with students and with content, and manage interactions between students and content. Therefore, the work of teaching is what is involved in managing these dynamic relationships as students do the complementary work of making a relationship with the content to learn it (Sleep, 2012, p. 937). The work of teaching includes the activities in which teachers engage and the responsibilities that they have to teach mathematics both inside and outside of the classroom (Ball & Forzani, 2009).



[Figure II-1] Instructional Triangle (Cohen et al., 2003)

Sleep (2012) distinguishes three interdependent types of the work of teaching mathematics: articulating the mathematics; orienting the instructional activity; and steering the instruction. According to her definition, the first two are about the specification and coordination of goals and plans, and they can occur both before and during instruction. Articulating the content and orienting the instructional activity aim at a clarification of the learning goals for students, an understanding of how the activity is intended to move students toward those goals, and a detailing of the task and possible teacher moves that position the activity so it is more likely to engage students with the intended mathematics (Sleep, 2012, p. 938). On the other hand, steering instruction is deploying teaching moves during a lesson in order to help students remain engaged with the intended mathematics. Sleep emphasizes intimate relationships among the three types of the work. To steer the instruction, the learning goals and the details of the activity must be known. The work of steering also updates that the activity may vary. The current study is related to steering the instruction as the work of teaching, in particular, leading mathematical discussions during a lesson.

In this study, clarification and conceptualization of the work of teaching in the instruction does not depend on a distinct educational philosophy or policy, teaching style, or use of materials (Sleep, 2012). However, this study assumes that the purpose of mathematics education is mathematical proficiency that is necessary for everyone to learn mathematics successfully, as defined by the National Research Council (2001). They discuss its five intertwined and interdependent strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (p. 116). In the analysis of classroom discussions, I consider ways in which the work supports the development of students' mathematical proficiency.

III. Data and Method

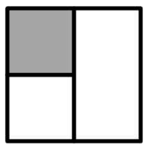
I took a grounded approach to the data (Strauss & Corbin, 2008). Its purpose is to develop a conceptual framework about leading mathematical discussions with a simple case question. In particular, I identified in the videotaped lessons tasks of leading classroom discussions of underserved students from a high-needs school district, which discussions were carefully led by an experienced classroom teacher who is skilled at making her teaching public. These were grounded in particular moments and episodes in the video clips. While these classrooms might not involve the same type of instructional demands as school instruction, they offer valuable resources to investigate how to lead mathematical discussions in order to have a desirable and stable framework that can function well for a pedagogical goal of teacher education. This inductive process generated five steps to lead mathematical discussions and different tasks of teaching in each step. I revisited each lesson again to elaborate each descriptive summary into an analytical explanation of each step and the work of teaching. Ideas generated in the data analysis were further examined by repeating the analysis of the data through a constant and comparative process. Important episodes were identified and coded with relevant justification about the tasks of teaching. The characterization of the tasks of teaching in which the teacher lead mathematical discussions was crystallized. Thus, although grounded in empirical data, this study is primarily conceptual.

1. Data

The data for this study is derived from Deborah Ball's fifth grade classes in the Elementary Mathematics Laboratory (EML) at the University of Michigan in 2006 and 2007. The classes in both years consisted of approximately twenty-five students, who were ethnically and racially diverse. Students' entering levels of mathematics achievement varied widely in terms of both mathematical skills and concepts. The class was two and half hours long and met daily for two weeks. The class worked on approximately two topics in each class, which were carefully selected by the teacher and researchers to be generative and rich with mathematical possibility and opportunity. Ball has many years of teaching experience in the elementary school, and her research focuses on mathematics instruction and on interventions designed to improve its quality and effectiveness. Moreover, many video recordings from the school where she taught third graders have been researched in diverse studies (e.g., Schoenfeld, 2008; Stylianides & Ball, 2008) and used as examples (e.g., Ball, 1993; Ball & Bass, 2000a, 2000b, 2003). Given her own extensive background in teaching and research and the analysis of her work in the existing literature, Ball is considered a highly experienced and expert teacher of mathematics to elementary students.

The present study used these video recordings to decompose the work of teaching shown in Ball's classroom discussions with single case questions, which generally have one answer with one standardardized mathematical reason. This research concentrates on two classes, one from 2006 and one from 2007, that targeted the understanding of the concept of fractions.

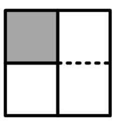
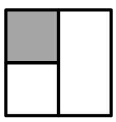
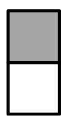
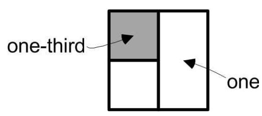
The question from the 2007 EML class used Figure III-1, as seen below, and asked the question: "What fraction of the big rectangle is shaded gray?" In this question, a fraction is a number that shows how many parts are shaded in of all the equal parts of the unit as a whole. The whole is a big square, and the shaded part is one of four small parts in the big square. Students in the 2007 EML suggested and discussed four claims as shown in <Table III-1>.



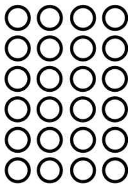
[Figure III-1] Unequally Divided Square for "What fraction of the big rectangle is shaded gray?"

The question from the 2006 EML class was based on Figure III-2 and asked the question, "Show what one-eighth means." In this question, a fraction means a part of a collection of the same objects. One-eighth represents the quantity where the whole is divided into eight equal parts. As the whole consists of twenty-four circles, dividing twenty-four circles into eight equal groups results in three circles for each group. Thus, shading three

<Table III-1> Students' Four Claims for Interpreting Representation of Figure III-1

one-fourth	one-third	one-half	one-and-one-third
			

circles is the mathematically appropriate response. As shown in Table III-2, students in the 2006 EML suggested and discussed four claims.



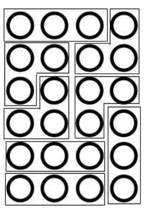
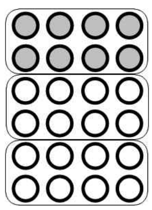
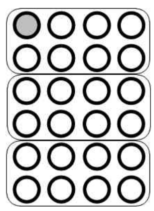
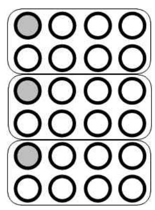
[Figure III-2] Twenty-Four Circles for “Show what one-eighth means”

While the students’ claims and reasons differed at the beginning step, they eventually approached four basic concepts for fractions by the end of the discussions: identifying the whole; clarifying the equal parts; counting all equal parts for the denominator; and counting the shaded parts for the numerator. The important questions are how the teacher led discussions so that students presented their ideas confidently even though some of them were mathematically inappropriate, and what tasks of teaching the teacher carried out in order to reach mathematically valid claims and reasons with students to develop their mathematical understanding, with respect to concepts related to fractions, specifically.

2. Data Analysis

Data analysis was based on multiple observations of all the taped lessons and on repeated reading of the transcripts. Data were analyzed using a method of open coding at the descriptive level to identify tasks of teaching associated with leading mathematical discussions (Strauss & Corbin, 2008). I did a microanalysis to generate ideas and delve more deeply into the data that seemed relevant, but whose meaning remained elusive. I examined each statement in depth, and then extracted certain themes. My approach of looking at this data in depth is based on my belief that it is detailed analysis like this that leads to rich descriptions. Themes were developed into concepts, and I gave labels to each of these concepts. I then categorized the concepts according to shared properties and established a hierarchy within the data. As categories of meaning emerged, I searched for those that had internal convergence and external divergence because the categories should be internally consistent but distinct from one another (Marshall & Rossman, 2006). Furthermore, I explored the context of the instruction of the definition of fraction. Finally, I integrated the categories to attain a full picture of how the teacher led discussions to help students’

<Table III-2> Students’ Four Claims for Representing One-Eighth in Figure III-2

three circles	eight circles	one circle	three circles
			

mathematical understanding and to specify what tasks of teaching are undertaken to lead mathematical discussions. I also selected episodes to depict the nature of each step of discussions. My aim in the selection of episodes was to illustrate specific features of tasks of teaching to lead mathematical discussion in detail, while preserving the complexity and richness of the context from which they was drawn (Miles & Huberman, 1994). The episodes were brief descriptions intended to rebuild and represent the context (Erickson, 1986). Moreover, providing actual excerpts from video clips helps readers reach their own conclusions about the precision of the analysis of the data and the viability and utility of the conceptual framework and implications drawn from it (Lincoln & Guba, 1985).

IV. Findings

A conceptual framework of the work of leading mathematical discussions with a single case question has five steps, as shown in Figure IV-1. This figure shows all the steps for leading a discussion as well as the tasks of teaching involved in each step. To show the elaborated work of teaching in each step, this section mainly uses episodes from the discussion with Figure III-1. I hope it maximizes the possibility of bringing readers to the context of the analysis.

1. Disclosing Students' Claims and Reasons

A discussion starts with facilitating students to present their claims and reasons. In the first two steps, the main work of teaching is creating a mathematically comfortable classroom environment, keeping the students' attention, and encouraging them to listen carefully and understand what the presenters claimed and how they reasoned. Moreover, this work would help a teacher gain extensive knowledge about what mathematical claims and reasons students have. The two steps, presentation and restatement, are repeatedly used until all claims are collected, in the data, claims as shown in Table III-1 and Table III-2.

A. Step 1: Presentation

The beginning of classroom discussion is a field for encouraging students to present their claims and explain why they are valid. The following excerpt from the video focuses on the discussion of one-and-one-third, as illustrated in the fourth representation in Table III-1, in the question of "What fraction of the big rectangle is shaded gray?"[1)]

Teacher: Okay, *everyone should be listening to Jane right now.*

Jane: Well, it's one-and-one-third, I think, because (she points to the unshaded rectangle) this is one, and (she points to the unshaded small square) this is a half, these two (the two small squares) so this is one right here (the unshaded rectangle), so all these are three sides (one unshaded rectangle and two small squares), so that'll be one-and-one-third all together.

Teacher: *Okay, go through the pieces again so everyone can hear you.* Pay very careful attention

1) All students' names in this paper are pseudonyms.

	Steps	Tasks of Teaching
Disclosing students' claims and reasons	Step1: Presentation	• Creating mathematically comfortable classroom environment • Gathering students' attention to encourage them to understand what they listen to • Respecting what a student presents • Not adding a teacher's decisions even if a student's claim and reason are mathematically inappropriate • Supporting a student to clearly present his/her claim and to explain how to reach them • Asking specific questions to illustrate a student's mathematical reason • Figuring out the claim and reason that a student presents • Generally, a teacher and a presenter talk each other.
	Step2: Restatement	• Encouraging the audience to restate the claim and reason that they heard and understood • Asking specific questions to examine mathematical reasoning that the students understood • Generally, a teacher and the students who were the audience in Step 1 talk each other.
	• *Repeating these two steps until all students' claims and reasons are collected and the audience understood them* • *Decomposing students' mathematical claims and reasons in these two steps*	
Investigating the claims and reasons	Step3: Comparison	• Choosing appreciate claims for emphasizing a certain concept • Asking about the differences among selected claims and reasons • Using specific questions if students seem to feel difficulty to find these difference • Generally, the teacher repeats this step until concepts that the teacher intends are illustrated
	Step4: Clarification	• Reviewing and confirming concepts and mathematical reasoning that students found in Step 3 • Asking specific questions related to the unpacked concepts
	Step5: Conclusion	• Briefly summarizing mathematical reasoning that the students found

[Figure IV-1] Framework for Leading Mathematical Discussions with a Single Case Question

to what Jane's saying. S*ay it one- go through the whole thing*.

Jane: It's three-

Teacher: Through the whole thing.

Jane: It's three pieces (one unshaded rectangle and two small squares) so that's a third (the shaded one small square). And (she points to the unshaded rectangle) this is one, so that would be one. And that's another half of that so it'll be one-and-one-third.

Teacher: *One-and-one-third?*

Jane: (She nods) Yeah. (Emphasis added)

Before this scene, many students raised their hands, and the teacher called on one of them. After the teacher got the students' attention, Jane came in front of the classroom and stated and justified her claim. The teacher then asked her to present her claim and reason again. The second statement of the student was typically shorter and more concise than the first. In fact, one-and-one-third is a mathematically inappropriate claim to the question, but Jane appeared very confident while showing and validating her claim to her classmates. It is noticeable that there is no interruption about the student's presentation, none of the teacher's own judgment in this step, nor comments along the line of "that's not correct." It seems to be based on the consideration on the student's claim with respect and a trial to understand why the student has the claim. Praising students' presentations is noteworthy, such as "You had a good reason," "It makes sense," and "You did a good job of thinking about that." Unlike this scene, when a student presented a mathematically correct claim, there was no response in a way that only accepted the answer. In any case, the main task of leading classroom discussion is asking specific questions about mathematical factors and reasons that led to the claim. According to Lampert (1990), the humility and courage, which Lakatos and Polya refer to, are essential to do mathematics through making and testing mathematical hypotheses. To accomplish this, it is necessary to create a classroom environment that is a comfortable one in which students can express and share their thoughts, and set up classroom norms that support mathematical reasons as the primary source of legitimacy for ideas and claims. Moreover, in this moment, it is interesting to talk only with the student who proposed the claim. The other students seem to be expected to carefully listen to and understand the student's mathematical explanation. This would be a cognitive opportunity for students to explore and experience another student's claim and reason and to acknowledge significant mathematical factors in the question provided.

It is also critical to uncover the mathematically incorrect claims and how students attain them. Mathematically erroneous claims are not merely erroneous, but have their own reasons (Russell, 1999). When students disclose their inappropriate claims and reasons, teachers are able to think about how to assist them. Since other students may also have gone through similar kinds of mathematical reasons, drawing out any of the incorrect claims and their reasons is a very significant component of developing students' mathematical understanding. Moreover, mathematically inappropriate claims and the reasons for them provide issues to be discussed that can lead to discovering valid claims. Through these discussions, students are able to probe how the claims were reached, explore how to refute them, and suggest valid claims and their reasons. This process helps all students broaden and deepen their mathematical understanding individually and collectively.

B. Step 2: Restatement

The second step is for the other students, the audience of the presenting student, to restate the claim and reason given by the student who made the claim. The following excerpt shows one student's restatement after having listened to the student's presentation about one-and‐one-third.

Teacher: *We're not arguing with Jane. We're just trying to understand what she thinks.* Have any questions for what she just said- *do you understand why she says it's one and one- third? Can someone who's not Jane explain what she* *just said? Why does Jane think this is one and one third?* Iris?

Iris: Because she's counting all the pieces together and then she's counting the whole, which is one. And then all the pieces is three and then there's one shaded, so she has one and one-third.

Teacher: Okay. Did that get it? *Very nice listening. Iris. Very nice.*(Emphasis added)

In this scene it is shown the teacher's attempt to have students grasp how the presenter arrived at the claim rather than mathematically correcting students' claims, as particularly shown in the above comment that "We're not arguing with Jane. We're just trying to understand what she thinks." Another task of teaching is asking the students to paraphrase what they understood about this claim and its reason. Restatement plays the role of stimulating students to understand and share one another's mathematical claim and reason and articulate their understanding. Furthermore, it helps a teacher recognize how well students understand the claim and its reason. After this moment, it is seen to check whether the other students understand the student's restatement and to add affirming words of praise to the students who restate mathematical reason.

When a student presented a mathematically correct claim, it is still shown to encourage students to restate the mathematical reason. However, it is noticeable to scrutinize the claim and its reason with the students who were the audience, in particular, asking specific questions to identify the reasons which are asserted and to see what students understood about the reasons for the claim. This step, rather than being stagnant, deepens the appreciation of claims and reasons.

These two steps provides students with the opportunities to freely present and justify their mathematical claims and reasons without fear of criticism as well as understand and share other classmates' ideas. While it would be important to have an open attitude when listening to students' ideas, it is not always acceptable for students to present at all times.

Teacher: Let's get some comments now. Abby?

Abby: Well, I'm wondering maybe if there's a different way to do it.

Teacher: *Okay, but first let's comment on this and then we can see a different way....* (After finishing the discussion of the first claim) *Abby, go ahead. ... Oh, let me give you a new drawing. I'm sorry. There you go. Can people see Abby?* (Abby then presented her claim and reason.) (Emphasis added)

This scene shows that during the discussion about three circles in the question, "Show what one-eighth means," one student suddenly attempted to present her claim. However, it is shown not to let the student present her claim at that moment. After finishing the prior issue, the student had a chance to present her claim and reason.

2. Investigating the Claims and Reasons

After collecting all the different claims and reasons, the class investigates them carefully. The work of teaching in the last three steps is guiding students to investigate all the claims and reasons, to recognize identifying the whole and illuminating the equal parts as significant concepts of fractions, and to decide what more mathematically appropriate reasons are.

A. Step 3: Comparison

Identifying differences is a traditional method for conceiving and defining ideas (Apostle, 1952). Moreover, attempting to compare various ideas and judging their similarities and differences provides students with additional learning opportunities (Yackel & Cobb, 1996). A main purpose in this step is identifying concepts of fractions by comparing diverse claims and reasons. Among the four claims which are one-fourth, one-third, one-half, and one-and-one-third, in the question, “What fraction of the big rectangle is shaded gray?” the teacher chose one-fourth and one-third with two representations, as shown in the first and second representations of Table III-1.

> Teacher: [T]he people who are thinking that this is one-third are reasoning differently than the people who think it’s one-fourth. *What’s the difference between the people who say one-third and one-fourth?* These people are counting three parts, and these people are also counting three parts. *Why do these people get a different answer than the people who say one-third? What’s the difference in what they’re paying attention to? John?*
>
> John: Well what they’re doing is- one-fourth, they’re saying that the recta- the big rectangle, which is half of the shape, it can be cut into to resemble two- two squares. But with a third- they’re say- they’re treating the rectangle like one little square, so then they say that it’s three. That it’s one-third.
>
> Teacher: *Can someone pick up from where John is? What’s the difference between counting three parts and counting four parts?* Everybody’s counting parts, everybody’s worried about how much is shaded in. *What’s the difference between the one-third and the one-fourth?* Ann?
>
> Ann: It matters how the length is equal.
>
> Teacher: Say it again?
>
> Ann: How many- how all the parts are equal. (Emphasis added)

In this scene, the teacher asked questions, such as what the difference between one-third and one-fourth is and what the difference between counting three parts and four parts is. Rather than criticizing the inappropriate claim or closing the discussion with the teacher’s own summary or conclusion, the class compared the two claims and found equal partitioning as one of the concepts of fractions. Her selection, one-third and one-fourth, was important to guide students to acknowledge the concept of dividing into equal parts in fractions. After this scene, she motivated students to compare one-half and one-and-one-third for identifying the whole as one of concepts of fractions.

This step shows one of the ways to guide students to explore claims and reasons: using students’ claims and reasons selectively and leading students to understand a concept. It is not shown to consider all concepts at once or to let students work alone on investigations. The main work of teaching is encouraging students by asking specific questions. The intent of these questions would not acquire students’ simple answers but help students obtain an understanding of the concepts the teacher intended in this class. Thus, students would have a chance to analyze mathematically both correct and incorrect claims and reasons, and acknowledge how it works or does not work and where it go well or does not go well. This step is repeated until a teacher concludes illustrating the concepts a teacher intended to explore, in this case, such as identifying the whole and deciding denominators and numerators.

B. Step 4: Clarification

After investigating all claims and reasons through the process of comparisons, the discussion moves to clarify the main concepts related to a problem. One of tasks of leading discussions in this step is using the students' claims as examples to reach mathematically appropriate concepts. The following excerpt relates to the question, "What fraction of the big rectangle is shaded gray?"

Teacher: Okay, it matters if the parts are equal. (She points to the second representation of Table III-1.) *In this one, are all the parts equal?*

Students: No.

Teacher: So, people who think it's one-third are counting how many parts, and noticing one is shaded. The people who call it one-fourth are counting how many parts, but *what are they doing to figure out the parts first?*

Kelly: They're making it equal?

Joy: Cutting it up

Teacher: *They're cutting it up further to do what?*

Kelly: To make it equal.

……

Teacher: This is the other main idea we need to have today is- You have to think, what is the whole? … What is the whole thing? So let's practice with the- what the whole question is. (She points to the fourth representation of Table III-1.) *In Jane's diagram, the way she was looking at it, what was the whole?* October?

October: The rectangle.

Teacher: (pointing to the rectangle of the fourth representation of Table III-1) This rectangle, right? (She points to the first representation of Table III-1.) And, *this one right here that Eddie talked about, what was the whole?*

Kane: The whole square. (Emphasis added)

This step is based on the concepts that students gained an understanding of in the previous step of comparison. In this scene, the students recognized the concepts of equal partitioning and identifying the whole for fractions. The teacher used very specific questions to point out these two concepts, and she reviewed and confirmed what the students found. However, the teachers did not use such terms as "correct" nor did she use negative terms such as "wrong" to indicate her evaluation of the particular mathematical reasoning. Since the students analyzed, understood and evaluated all the claims and reasons for them, and, therefore, appreciated for themselves important concepts for fractions, students did not need to depend on the teacher's interpretation or judgments about claims and their reasons, and students could find and establish appropriate mathematical understanding for fractions through the discussions.

An important feature of this step is the continuous emphasis on mathematical reasons rather than finding the correct answer to the question. Furthermore, the students were expected to recognize the importance of the discussion about their claims and reasons and to realize that they could attain an understanding of significant mathematical concepts through the discussion (Lampert, 1990).

C. Step 5: Conclusion

In the last step, the main task of teaching is briefly providing a recapitulation of the concepts that students found throughout the discussion. The following excerpt is parts of the teacher's comment in this step.

> Teacher: To make it equal. Then they count the parts, and see that one is shaded. So equal parts is a very important idea in fractions, and that's why you're quite right that this picture is trying to trick you. Because the parts aren't equal. So it tricks you into thinking you just have to count parts. (She draws additional line through square as shown in Figure III-1.) But you actually do have to make them equal. And once you make them equal, then you have four equal parts and one of them is shaded. (Emphasis added)

The main work of teaching in this scene is briefly summarizing what the students ultimately had discovered on their own. In particular, it is shown to explain why the provided question had certain complications, noting the concepts that students had formulated themselves, and clarifying the numerator and the denominator in the representation. There are no additional mathematical reasons or new interpretations. At this step, the teacher's explanation is reasonable to students because of summing up the results and the process of all discussions by students. This approach shows respect on students' roles in the discussion, the cooperative thinking process the class engaged in to find appropriate mathematical claims and reasons, and students' ability to develop their mathematical understanding.

V. Discussion and Conclusions

Discussion gives opportunities to students for argument, explanations, and questions, but it requires management by teachers for complex interactions between students and mathematics (Cohen, 2011). Lack of ability to skillfully lead a mathematical discussion hinders students' learning because of the unstructured instruction, waste of time, and shallow mathematical work. Just saying that teachers should be able to lead mathematical discussions does not articulate what is involved in doing this work. Decomposition of practice parses and describes the work of teaching so it can be studied and practiced in education for teachers (Grossman et al., 2009). This study decomposed the work of leading a mathematical discussion based on a single case question by specifying its major tasks and steps to manage classroom discussions. Throughout the examination of practice, I found a couple of key issues in the work of leading mathematical discussions.

The first key issue is helping students engage in struggling or wrestling with important mathematical ideas. It is important to make students expend effort to make sense of mathematics that is not immediately apparent (Hiebert & Grouws, 2007). It is shown in the data that mathematically incorrect claims and reasons are critical to the discussions. Without immediately distinguishing students' claims and reasons as correct and incorrect, strategically using and managing them guides students toward key concepts underlying the problem. Russell (1999) refers to mathematically incorrect reasoning as flawed reasoning, which, in its very flaws, provides opportunities to develop mathematical reasoning skills; he asserts that instances of students' flawed reasoning present the opportunity for students to critique and test diverse mathematical claims and reasons including their own, and respond to refutations of their claims. Flawed reasoning can also highlight mathematical issues that are relevant to the whole class and need to be addressed. The present study indicates that, teachers' responses to and questions about flawed reasoning, geared specifically to different steps in the reasoning process, as described above, can help students address main concepts, in this case, identifying the whole and partitioning in equal parts in fractions.

Second, treating mathematical connections in an explicit and public way is critical to having coherent and structured discussions of the key ideas of mathematics. According to Hiebert and Grouws (2007), it includes asking questions about how different claims and reasons are similar to and different from each other, attending to the relationships among mathematical ideas, and reminding students about the main point of the lesson and how this point fits within the current sequence of lessons and ideas. The framework found in this study for leading mathematical discussions includes these features systemically in the five steps. Tasks of teaching in each step are elaborated and specified with regard to what work of teaching formulates these features in mathematical discussions.

Third, the present study indicates that the tasks of teaching for leading discussions suggest specific methods that teachers can employ. The students in this research did not simply gain certain knowledge, but explained their claims and reasons, investigated and shared one another's mathematical reasoning, and created relationships between their own mathematical ideas, which were uncovered in diverse claims in the presenting step, and the valid mathematical reasoning which they ultimately chose. This research apparently shows that there is the specific work of teaching which initiates, revitalizes and strengthens students' use of mathematical reasoning. In other words, students' development of mathematical reasoning depends on how a teacher facilitates students' engagement in discussions, what kinds of questions teachers use, how they respond to students' questions, and what kinds of comments they give. A major contribution of this study is an identification of an important aspect of the work of teaching at a grain size that would be usable in the education of teachers.

This study has a number of limitations. First, although the data was from classes based on the topics carefully selected by researchers to be generative and rich with mathematical possibility and opportunity and taught by a highly experienced and expert teacher, it is still a small data set from just one teacher's two classes. Thus, the findings are likely missing aspects of the work that were invisible in the settings I observed. The tasks of teaching may be different in the context of mathematics education. Other limitations are caused by the topics researched. While I clarified the rationale of focusing on a single case question with fractions, it is just one possible type of question that teachers use for teaching mathematics. This fact might influence the conceptualization for the teaching of mathematics. However, I attempted to use the data to add the detailed examination of the phenomena for teaching mathematics. In regard to methods of analysis, this research is based on my observation, description and analysis of the data. Even though I clarified my reasons at each stage, this study is limited to the threads I recognized in the data. While limited in scope, this study is an important step. It could be used for providing recommendations for developing a curriculum for the education of teachers based on the results of the research proposed here.

The tasks articulated in this study can inform research on teaching and teacher education; teacher education should prepare teachers with the knowledge and skills necessary for students to have mathematical proficiency. I expect to conduct further research on the work of teaching with various kinds of questions, such as open-ended questions, proofs, or contextualized questions, in various contexts, such as schools in higher or lower social economical statuses. This further research will look into how different types of questions would initiate different work of teaching to lead mathematical discussions, and how different social situations would influence mathematical discussion in classrooms.

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단답형 문제를 이용한 수학 토론에 수반된 교수 업무 분석

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수학 토론은 학생들의 수학 학습 및 수학적 관행을 신장시키는 방안으로서 강조되어 왔다. 그러나, 수학 토론의 진행에 수반되는 교수 업무가 무엇인지 명시되지 않아서, 토론하는 방법을 교사들에게 가르치는데 어려움이 있다. 본 연구는 수학 토론 수업을 분석하여, 토론의 다섯 단계에 따라 수학 토론에 수반된 교수 업무를 구체적으로 구분하였다. 수학적 토론을 진행하는 주요 사안으로, 본 연구는 학생들이 중요한 수학적 아이디어를 탐험하도록 하며, 일관성있는 토론을 위해 수학적 관련성을 명백하게 밝혀야 하며, 교사 교육의 관점에서 교수 업무 분석의 중요성을 주장한다. 향후 연구로서 다양한 문제 유형에 대한 수학 토론과 관련된 교수 업무 분석을 제안한다.

* 주제어 : 수학 토론 (mathematical discussion), 수학 지도 (mathematics instruction), 교수 업무(work of teaching), 교사 교육 (teacher education)

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