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Division Algorithm in SuanXue QiMeng

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The Division Algorithm is known to be the fundamental foundation for Number Theory and it leads to the Euclidean Algorithm and hence the whole theory of divisibility properties. In JiuZhang SuanShu(九章算術), greatest common divisiors are obtained by the exactly same method as the Euclidean Algorithm in Elements but the other theory on divisibility was not pursued any more in Chinese mathematics. Unlike the other authors of the traditional Chinese mathematics, Zhu ShiJie(朱世傑) noticed in his SuanXue QiMeng(算學啓蒙, 1299) that the Division Algorithm is a really important concept. In [4], we claimed that Zhu wrote the book with a far more deeper insight on mathematical structures. Investigating the Division Algorithm in SuanXue QiMeng in more detail, we show that his theory of Division Algorithm substantiates his structural apporaches to mathematics.

Keywords: mathematical structures, Division Algorithm, Zhu ShiJie, SuanXue QiMeng; 朱世傑, 算學啓蒙.

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0 Introduction

A primitive form of the Division Algorithm appeared in the first two theorems of the 7th book of Elements [2] compiled by Euclid(c. 300 B.C.). We recall that the Euclidean Algorithm for determining the greatest common divisor of two natural numbers is processed by means of successive subtractions and that the exactly same process was introduced in the first chapter FangTian(方田) of JiuZhang SuanShu(九章算術). The number of successive subtractions in the Euclidean Algorithm is the quotient in the division of a bigger number by a smaller number.

We state the exact statement of the Division Algorithm as follows.

Given integers a and b with $b > 0$, there are unique integers q and r satisfying

$$a = bq + r, \quad 0 \le r < b.$$

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In this paper, we deal with the Division Algorithm for natural numbers a, b and hence q, r are also natural numbers, called the quotient and the remainder in the division of a by b, respectively as usual.

It is well known that the Division Algorithm implies the Euclidean Algorithm and acts as a basic foundation for Number Theory.

As mentioned in [3], Chinese mathematicians mostly concerned with problems dealing with practical applications and numbers in Chinese mathematical sources are concrete numbers, mostly denominate numbers. Thus they could hardly develop a pure Number Theory.

Since the Division Algorithm also relates to the theory of fractions, it was frequently used in fractions but implicitly. It is quite notable that the Division Algorithm was used in the second chapter SuMi(粟米) of JiuZhang SuanShu to deal with GuiJian(貴賤) problems, i.e., the superior and inferior quality items problems. We note that fractions and their operations were already introduced in the first chapter of JiuZhang SuanShu. Readers may have a difficulty to appreciate the Division Algorithm in the book, for they may gather that the final results were obtained by manipulating in the field $\mathbb{Q}$ of rational numbers rather than in the set $\mathbb{N}$ of natural numbers [1, 2].

Unlike JiuZhang SuanShu, Zhu ShiJie first introduced the Division Algorithm way before the GuiJian problems and applied it to various problems. In SuanXue QiMeng(算學啓蒙, 1299), Zhu put GuiJian problems in the last chapter, GuiJian FanLuMen(貴賤反率門) of the second book and then introduced the theory of fractions in the first chapter, ZhiFen QiTongMen(之分齊同門) of the third book based on the Division Algorithm(also see [4]). Zhu adopted GuiJian problems from those problems in JiuZhang SuanShu but he precisely indicated that problems of Dan(石), Jun(鈞), Cheng(秤), Jin(斤), Liang(兩) (see problem 40 – 43 in JiuZhang) should convert those into problems in the setting of $\mathbb{N}$.

This paper is a sequel to our previous paper [4] on the mathematical structural approaches in SuanXue QiMeng. Investigating the Division Algorithm in SuanXue QiMeng in more detail, we show that on top of its own importance, it also substantiates Zhu's view on mathematical structures in SuanXue QiMeng as before.

The reader may find all the Chinese sources of this paper in ZhongGuo KeXue JiShu DianJi TongHui ShuXueJuan(中國科學技術典籍通彙 數學卷) [2] and hence they will not be numbered as an indivisual reference.

1 Division Algorithm in SuanXue QiMeng

We first note that the traditional Chinese system of weights and measures contains non-decimal ones. Among them, the most complicated one is the system of weights as follows:

1 Liang(兩) = 24 Zhu, 1 Jin(斤) = 16 Liang, 1 Cheng(秤) = 15 Jin,
1 Jun(鈞) = 2 Cheng = 30 Jin, 1 Dan(石, or 碩 in QiMeng) = 4 Jun = 120 Jin.

For a further development in this paper, we include the following table of conversions of the above system in the unit Zhu:

1 Liang = 24 Zhu, 1 Jin = 384 Zhu, 1 Cheng = 5,760 Zhu, 1 Jun = 11,520 Zhu,
1 Dan = 46,080 Zhu.

Thus the conversions of compound numbers into a single unit and their reverse processes involve rather complicated calculations and they occur frequently in practical problems in Chinese mathematical works.

We quote two problems omitting the answer part(答曰) in SuanXue QiMeng which indicate the Division Algorithm.

今有片腦五斤七兩一十八銖 直錢七釐二毫 問直銀幾何
術曰 列五斤 身外加六 通兩內子七 得八十七兩 以二十四乘之
得數加入一十八銖 共得二千一百六銖 於上以七釐二毫乘之合問
(留頭乘法門, 第六問)

今有銀一十五兩一錢六分三釐二毫 每銀七釐二毫換片腦一銖 問得幾何
術曰 列銀數爲實 以七釐二毫爲法除之 得二千一百六銖
以斤銖法三百八十四約之 得五斤 不滿法者 以兩數法二十四約之
得七兩 不滿法者 命之合問
(九歸除法門, 第十四問)

In the first problem, the conversion of 5 Jin 7 Liang 18 Zhu into 2,106 Zhu is explained as follows:

5 Jin 7 Liang = $(5 \times 16) + 7 = 87$ (通兩內子) and then
87 Liang 18 Zhu = $(87 \times 24) + 18$ = 2,106 Zhu.

As Zhu's typical arrangement of problems, he put the second problem in the above quote using the same numbers in the first problem to explain the reverse processes. Indeed, he obtained 2,106 Zhu which is converted to the compound number 5 Jin 7 Liang 18 Zhu as follows:

2,106 Zhu = $(384 \times 5) + 186$ (384 Zhu = 1 Jin, 斤銖法),

where 186 is called BuManFaZhe(不滿法者), the remainder in the Division Algorithm. The whole process is called "以b約a" by which one obtains the quotient and the remainder.

Zhu ShiJie didn't introduce the terminology "quotient". And then he applied the Division Algorithm to 186 and 24.

186 = $(24 \times 7) + 18$ (1 Liang = 24 Zhu, 兩銖法).

In all, he has the answer 5 Jin 7 Liang 18 Zhu (BuManFaZhe MingZhi(命之)).

The above example shows Zhu's excellent heuristic presentation for beginners. In the very next problem of the conversion, Zhu put the case where a is divisible by b. Furthermore, Zhu put many more problems as above which use the Division Algorithm.

Zhu dealt with problems in the setting of $\mathbb{N}$ in the first two books of QiMeng and introduced the notion of decimal fractions up to 10^{-128}. In those two books, there are many problems which have soultions of repeating decimal fractions and the method of representing them was not introduced in the traditional Chinese mathematics(see also [3]). Thus Zhu has to use fractions for repeating decimal fractions. There are 13 problems with solutions of fractions. Among them, there are only two cases in which the solutions are finite decimal fractions. One is related to the MiLu(密率), $\pi = \dfrac{22}{7}$ (see Problem 16 in 田畝形段門) and the other to an arithmetic sequence with common difference $\dfrac{5}{27}$ (see Problem 9 in 求差分和門). Thus we can easily guess the reason why he used the fractions instead of decimal fractions. The first case of fraction solutions appears in the last problem of 庫務解稅門 in the first book, where the solution is $\dfrac{4,500,000}{1,363} = 3,301\dfrac{737}{1,363}$. Using the Division Algorithm, $4,500,000 = (1,363 \times 3,301) + 737$, he has the solution as follows:

四百五十萬爲實 . . . 一千三百六十三爲法 實如法而一 不滿法者命分

Combining the above discussions regarding conversions of compound numbers and fractions, one can easily gather that Zhu ShiJie fully understands the structure of the Division Algorithm involved in both problems.

To illustrate the role of the Division Algorithm in GuiJian problems, we include the first problem in GuiJianFanLuMen(貴賤反率門) of the second book in SuanXue QiMeng although it follows exactly the same method in JiuZhang SuanShu.

今有錢三百四十五文 共買檀乳香一百四十兩
只云乳香兩價貴如檀香價一文 問二色各幾何
荅曰 檀香 七十五兩 兩價 二文 乳香 六十五兩 兩價 三文 術曰 列錢數爲實
以一百四十兩爲法 實如法而一 得二文乃檀香兩價 加一文卽乳香兩價 餘實
六十五爲乳香數也 反減下法餘七十五卽檀香數也 合問

Let $a(= 140), b(= 345)$ denote the total number of two items and total amount of price, respectively. Further let z be the unit price of the inferior quality (= 檀香) and x, y the numbers of the superior items (= 乳香) and inferior items, respectively. Then one has the following equations

$$x + y = a, \quad x(z + 1) + yz = b$$

which implies $b = az + x,\ \ 0 < x, y < a$.

Thus using the Division Algorithm, $345 = (140 \times 2) + 65$, and hence we have the solution $z = 2, x = 65, y = 75$.

This method is called QiLu(其率) in JiuZhang SuanShu.

On the other hand, the number of the superior items(inferior items, resp.) which can be bought by one unit coin, is $p(p + 1$, resp.). Then using the same notions a, b, x, y as above, one has

$$x + y = a,\ \ \frac{x}{p} + \frac{y}{p+1} = b$$

which implies $a = bp + y_1,\ \ 0 < y_1 < b\ (y_1 = \dfrac{y}{p+1} \in \mathbb{N})$.

Using the Division Algorithm for a, b again, one can solve the given problem and this method is called FanQiLu(反其率) in JiuZhang , QiFanLu(其反率) in QiMeng.

As we mentioned in the previous section, there are four problems (Problem 40–43) for which comments by Liu Hui(劉徽) are not complete. We take the third problem in GuiJianFanLuMen(貴賤反率門) where Zhu ShiJie converted the problem in the set $\mathbb{N}$ and then applied the Division Algorithm.

今有錢一十六貫五百文 買漆一石三鈞一秤四斤五兩六銖
欲其貴賤石率之 問各幾何
荅曰 其一石三鈞四兩一十八銖 石價八千六百三十四文
其一秤四斤一十二銖 石價八千六百三十三文
術曰 列漆通銖 得八萬八千六十二爲法 列錢以四萬六千八十乘之
得七億六千三十二萬爲實 實如法而一 得八千六百三十三文爲賤石價
加一文卽貴石價 不盡八萬七百五十四
反減下法餘七千三百八 以秤斤銖法除之 得一秤四斤一十二銖爲賤數
其不盡八萬七百五十四 以石鈞兩銖法除之 得貴數 合問

Using the same notions as above, a = 1 Dan 3 Jun 1 Cheng 4 Jin 5 Liang 9 Zhu $= 88,062$ Zhu and $b = 16,500$ but the price is paid by the unit Dan. Thus a should be $\dfrac{88,062}{46,080}$ which is not a natural number (1 Dan $= 46,080$ Zhu). Zhu converts the given problem into the one with $a = 88,062,\ b = 16,500 \times 46,080 = 760,320,000$.

Now Zhu applied the Division Algorithm to the new problem and had

$$760,320,000 = 88,062 \times 8,633 + 80,754,$$

which implies that 8,633(文) is the unit price for the inferior item.

We now return the above identity into the original problem as follows

$$\frac{760,320,000}{46,080} = \frac{88,062}{46,080} \times 8,633 + \frac{80,754}{46,080}.$$

Using $\frac{x}{46,080}$ Dan $= x$ Zhu together with the above identity, one has

$$16,500 = 88,062 \text{ (Zhu)} \quad \times 8,633 + 80,754 \text{ (Zhu)}.$$

Zhu ShiJie converted the amounts in the unit Zhu(銖) into the corresponding compound numbers by the Division Algorithm which are included in his answer. Using the exactly same argument as above in the next 4 problems, he solved them.

The above discussion shows that Zhu ShiJie did understand the mathematical structure of the Division Algorithm precisely in ℕ.

2 Conclusion

The Chinese mathematicians had used to reveal the mathematical structures by a series of practical problems. We note that the Division Algorithm was already used in GuiJian(貴賤) problems of JiuZhang SuanShu, where Liu Hui commented that it relates to the set ℕ of natural numbers as "其率知 欲令无分" [1, 2], but he missed this fact in some GuiJian problems.

As claimed in [4], Zhu ShiJie always approached his mathematics along mathematical structures. He introduced the Division Algorithm in the setting of multiplications and divisions in ℕ and then used it in GuiJian problems. Furthermore, Zhu showed that the Division Algorithm is precisely a mathematical structure in ℕ. Moreover, he introduced the theory of fractions just after the Division Algorithm in GuiJian problems. This shows that his theory of Division Algorithm also strongly substantiates his structural apporaches to mathematics.

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