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ON HÖLDER'S INEQUALITY FOR THREE SEQUENCES

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Abstract. Recent results of Wu and Tian on refined Hölder's inequality for two sequences are extended to the case of three sequences.

1. Statements of results

Classical Hölder's inequality for two positive sequences $a = \{a_i\}_{i=1}^n$ and $b = \{b_i\}_{i=1}^n$ states that

$$\sum_{i=1}^n a_i b_i \le \left(\sum_{i=1}^n (a_i)^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n (b_i)^q\right)^{\frac{1}{q}},$$

where positive indexes p, q are related to be $\frac{1}{p} + \frac{1}{q} = 1$.

There are a lot of extensions of this inequality. We focus on a result of Wu([3]) which extends this inequality by finding a quantity $g(a, b)$, $0 < g(a, b) \le 1$, satisfying

$$\sum_{i=1}^n a_i b_i \le \left(\sum_{i=1}^n (a_i)^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n (b_i)^q\right)^{\frac{1}{q}} \cdot g(a, b)$$

when p, q are positive with $\frac{1}{p} + \frac{1}{q} = 1$, and a result of Tian([2]) stating reversely :

$$\sum_{i=1}^n a_i b_i \ge \left(\sum_{i=1}^n (a_i)^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n (b_i)^q\right)^{\frac{1}{q}} \cdot g(a, b)$$

when one of p, q are negative with $\frac{1}{p} + \frac{1}{q} = 1$.

We, in this note, extend results of Wu and Tian to the case of three sequences as follows.

Theorem 1.1. *Let $a = \{a_i\}_{i=1}^n, b = \{b_i\}_{i=1}^n, c = \{c_i\}_{i=1}^n$ be positive sequences. Let $\{e_i\}_{i=1}^n$ satisfy $1 - e_j + e_k \ge 0$ for all j, k, $1 \le j, k \le n$. Let $p \ge q > 0$ and*

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$r > 0$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Then

$$\sum_{i=1}^{n} a_i b_i c_i \leq \left(\sum_{i=1}^{n} (a_i)^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} (b_i)^q \right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} (c_i)^r \right)^{\frac{1}{r}} \cdot g(a,b),$$

where

$$g(a,b) = \left\{ 1 - \left(\frac{\sum_{i=1}^{n} (b_i)^q e_i}{\sum_{i=1}^{n} (b_i)^q} - \frac{\sum_{i=1}^{n} (a_i)^p e_i}{\sum_{i=1}^{n} (a_i)^p} \right)^2 \right\}^{\frac{1}{2p}}.$$

Theorem 1.2. *Let $a = \{a_i\}_{i=1}^n, b = \{b_i\}_{i=1}^n, c = \{c_i\}_{i=1}^n$ be positive sequences. Let $\{e_i\}_{i=1}^n$ satisfy $1 - e_j + e_k \geq 0$ for all j, k, $1 \leq j, k \leq n$. Let $p \leq q < 0$ and $r > 0$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Then*

$$\sum_{i=1}^{n} a_i b_i c_i \geq \left(\sum_{i=1}^{n} (a_i)^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} (b_i)^q \right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} (c_i)^r \right)^{\frac{1}{r}} \cdot g(a,b),$$

where

$$g(a,b) = \left\{ 1 - \left(\frac{\sum_{i=1}^{n} (b_i)^q e_i}{\sum_{i=1}^{n} (b_i)^q} - \frac{\sum_{i=1}^{n} (a_i)^p e_i}{\sum_{i=1}^{n} (a_i)^p} \right)^2 \right\}^{\frac{1}{2p}}.$$

We give the proofs by modifying methods used in [3] and [2]. Theorem 1.1 will be proven in Section 2. Theorem 1.2 will be proven in Section 4 after preparing a lemma in Section 3. We refer to [1] for general theory on inequalities.

2. Proof of Theorem 1.1

Note simply that

$$\begin{aligned}
&\left(\sum_{i=1}^{n} a_i b_i c_i \right)^2 = \sum_{i=1}^{n} a_i b_i c_i \sum_{i=1}^{n} a_i b_i c_i = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_i c_i a_j b_j c_j \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_i c_i a_j b_j c_j - \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_i c_i a_j b_j c_j e_i + \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_i c_i a_j b_j c_j e_j \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_i c_i a_j b_j c_j \left(1 - e_i + e_j\right) \\
&= \sum_{i=1}^{n} a_i b_i c_i \sum_{j=1}^{n} a_j b_j c_j \left(1 - e_i + e_j\right).
\end{aligned}$$

Applying Hölder's inequality, the last quantity is at most

$$\sum_{i=1}^{n} a_i b_i c_i \left\{ \sum_{j=1}^{n} (a_j)^p (1-e_i+e_j) \right\}^{\frac{1}{p}} \left\{ \sum_{j=1}^{n} (b_j)^q (1-e_i+e_j) \right\}^{\frac{1}{q}} \left\{ \sum_{j=1}^{n} (c_j)^r (1-e_i+e_j) \right\}^{\frac{1}{r}}$$

$$= \sum_{i=1}^{n} a_i b_i c_i \left\{ \sum_{j=1}^{n} (a_j)^p (1-e_i+e_j) \right\}^{\frac{1}{p}} \left\{ \sum_{j=1}^{n} (b_j)^q (1-e_i+e_j) \right\}^{\frac{1}{q}-\frac{1}{p}}$$

$$\cdot \left\{ \sum_{j=1}^{n} (b_j)^q (1-e_i+e_j) \right\}^{\frac{1}{p}} \left\{ \sum_{j=1}^{n} (c_j)^r (1-e_i+e_j) \right\}^{\frac{1}{r}}$$

$$= \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} (b_i)^q (a_j)^p (1-e_i+e_j) \right\}^{\frac{1}{p}} \left\{ \sum_{j=1}^{n} (b_i)^q (b_j)^q (1-e_i+e_j) \right\}^{\frac{1}{q}-\frac{1}{p}}$$

$$\cdot \left\{ \sum_{j=1}^{n} (a_i)^p (b_j)^q (1-e_i+e_j) \right\}^{\frac{1}{p}} \left\{ \sum_{j=1}^{n} (c_i)^r (c_j)^r (1-e_i+e_j) \right\}^{\frac{1}{r}}.$$

Applying Hölder's inequality one more time, the last quantity is at most

$$\left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (b_i)^q (a_j)^p (1-e_i+e_j) \right\}^{\frac{1}{p}} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (b_i)^q (b_j)^q (1-e_i+e_j) \right\}^{\frac{1}{q}-\frac{1}{p}}$$

$$\cdot \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i)^p (b_j)^q (1-e_i+e_j) \right\}^{\frac{1}{p}} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (c_i)^r (c_j)^r (1-e_i+e_j) \right\}^{\frac{1}{r}}$$

$$= \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (b_i)^q (a_j)^p - \sum_{i=1}^{n} \sum_{j=1}^{n} (b_i)^q (a_j)^p e_i + \sum_{i=1}^{n} \sum_{j=1}^{n} (b_i)^q (a_j)^p e_j \right\}^{\frac{1}{p}}$$

$$\cdot \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (b_i)^q (b_j)^q - \sum_{i=1}^{n} \sum_{j=1}^{n} (b_i)^q (b_j)^q e_i + \sum_{i=1}^{n} \sum_{j=1}^{n} (b_i)^q (b_j)^q e_j \right\}^{\frac{1}{q}-\frac{1}{p}}$$

$$\cdot \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i)^p (b_j)^q - \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i)^p (b_j)^q e_i + \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i)^p (b_j)^q e_j \right\}^{\frac{1}{p}}$$

$$\cdot \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (c_i)^r (c_j)^r - \sum_{i=1}^{n} \sum_{j=1}^{n} (c_i)^r (c_j)^r e_i + \sum_{i=1}^{n} \sum_{j=1}^{n} (c_i)^r (c_j)^r e_j \right\}^{\frac{1}{r}}$$

$$=\left\{\sum_{i=1}^{n}(b_i)^q\right\}^{2\left(\frac{1}{q}-\frac{1}{p}\right)}\left\{\sum_{i=1}^{n}(c_i)^r\right\}^{\frac{2}{r}}$$

$$\cdot\left[\left\{\sum_{i=1}^{n}(a_i)^p\sum_{j=1}^{n}(b_j)^q\right\}^2-\left\{\sum_{i=1}^{n}(b_i)^q\,e_i\sum_{j=1}^{n}(a_j)^p-\sum_{i=1}^{n}(b_i)^q\sum_{j=1}^{n}(a_j)^p\,e_j\right\}^2\right]^{\frac{1}{p}}.$$

Thus we obtain

$$\sum_{i=1}^{n}a_ib_ic_i\le\left\{\sum_{i=1}^{n}(b_i)^q\right\}^{\left(\frac{1}{q}-\frac{1}{p}\right)}\left\{\sum_{i=1}^{n}(c_i)^r\right\}^{\frac{1}{r}}$$

$$\cdot\left[\left\{\sum_{i=1}^{n}(a_i)^p\sum_{j=1}^{n}(b_j)^q\right\}^2-\left\{\sum_{i=1}^{n}(b_i)^q\,e_i\sum_{j=1}^{n}(a_j)^p-\sum_{i=1}^{n}(b_i)^q\sum_{j=1}^{n}(a_j)^p\,e_j\right\}^2\right]^{\frac{1}{2p}}.$$

Therefore

$$\sum_{i=1}^{n}a_ib_ic_i$$

$$\le\left(\sum_{i=1}^{n}(a_i)^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}(b_i)^q\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n}(c_i)^r\right)^{\frac{1}{r}}\left\{1-\left(\frac{\sum_{i=1}^{n}(b_i)^q\,e_i}{\sum_{i=1}^{n}(b_i)^q}-\frac{\sum_{i=1}^{n}(a_i)^p\,e_i}{\sum_{i=1}^{n}(a_i)^p}\right)^2\right\}^{\frac{1}{2p}}.$$

The proof is complete.

3. Lemmas

We need the following easy lemma in proving Theorem 1.2. We include its proof for the convenience of the reader.

Lemma 3.1. *Let* $a_i,b_i,c_i>0$, $(i=1,2,\cdots,n)$, $\lambda_j<0$ $(j=1,2)$ *and* $\lambda_1+\lambda_2+\lambda_3=1$. *Then*

$$\sum_{i=1}^{n}a_i^{\lambda_1}b_i^{\lambda_2}c_i^{\lambda_3}\ge\left(\sum_{i=1}^{n}a_i\right)^{\lambda_1}\left(\sum_{i=1}^{n}b_i\right)^{\lambda_2}\left(\sum_{i=1}^{n}c_i\right)^{\lambda_3}. \tag{1}$$

Proof of Lemma 3.1. Since $\lambda_j<0$ $(j=1,2)$ and $\lambda_1+\lambda_2+\lambda_3=1$, we have

$$\frac{1}{\lambda_3}>0,-\frac{\lambda_1}{\lambda_3}>0,-\frac{\lambda_2}{\lambda_3}>0\quad\text{with}\quad\frac{1}{\lambda_3}+\left(-\frac{\lambda_1}{\lambda_3}\right)+\left(-\frac{\lambda_2}{\lambda_3}\right)=1.$$

Factorizing c_i as

$$c_i=\left(a_i^{\frac{\lambda_1}{\lambda_3}}\right)\left(b_i^{\frac{\lambda_2}{\lambda_3}}\right)\left(c_i^{\frac{\lambda_3}{\lambda_3}}\right)\left(a_i^{-\frac{\lambda_1}{\lambda_3}}\right)\left(b_i^{-\frac{\lambda_2}{\lambda_3}}\right)$$

and then using Hölder's inequality, we have

$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} \left(a_i^{\frac{\lambda_1}{\lambda_3}}\right) \left(b_i^{\frac{\lambda_2}{\lambda_3}}\right) \left(c_i^{\frac{\lambda_3}{\lambda_3}}\right) \left(a_i^{-\frac{\lambda_1}{\lambda_3}}\right) \left(b_i^{-\frac{\lambda_2}{\lambda_3}}\right)$$

$$\leq \left(\sum_{i=1}^{n} a_i^{\lambda_1} b_i^{\lambda_2} c_i^{\lambda_3}\right)^{\frac{1}{\lambda_3}} \left(\sum_{i=1}^{n} a_i\right)^{-\frac{\lambda_1}{\lambda_3}} \left(\sum_{i=1}^{n} b_i\right)^{-\frac{\lambda_2}{\lambda_3}}.$$

Thus,

$$\sum_{i=1}^{n} a_i^{\lambda_1} b_i^{\lambda_2} c_i^{\lambda_3} \geq \left(\sum_{i=1}^{n} a_i\right)^{\lambda_1} \left(\sum_{i=1}^{n} b_i\right)^{\lambda_2} \left(\sum_{i=1}^{n} c_i\right)^{\lambda_3},$$

which completes the proof. □

4. Proof of Theorem 1.2

Applying Lemma 3.1 twice, we have

$$\left(\sum_{i=1}^{n} a_i b_i c_i\right)^2 = \sum_{i=1}^{n} a_i b_i c_i \sum_{j=1}^{n} a_j b_j c_j \left(1 - e_i + e_j\right)$$

$$\geq \sum_{i=1}^{n} a_i b_i c_i \left\{\sum_{j=1}^{n} (a_j)^p \left(1 - e_i + e_j\right)\right\}^{\frac{1}{p}} \left\{\sum_{j=1}^{n} (b_j)^q \left(1 - e_i + e_j\right)\right\}^{\frac{1}{q}} \left\{\sum_{j=1}^{n} (c_j)^r \left(1 - e_i + e_j\right)\right\}^{\frac{1}{r}}$$

$$= \sum_{i=1}^{n} a_i b_i c_i \left\{\sum_{j=1}^{n} (a_j)^p \left(1 - e_i + e_j\right)\right\}^{\frac{1}{p}} \left\{\sum_{j=1}^{n} (b_j)^q \left(1 - e_i + e_j\right)\right\}^{\frac{1}{q}-\frac{1}{p}}$$

$$\cdot \left\{\sum_{j=1}^{n} (b_j)^q \left(1 - e_i + e_j\right)\right\}^{\frac{1}{p}} \left\{\sum_{j=1}^{n} (c_j)^r \left(1 - e_i + e_j\right)\right\}^{\frac{1}{r}}$$

$$= \sum_{i=1}^{n} \left\{\sum_{j=1}^{n} (b_i)^q (a_j)^p \left(1 - e_i + e_j\right)\right\}^{\frac{1}{p}} \left\{\sum_{j=1}^{n} (b_i)^q (b_j)^q \left(1 - e_i + e_j\right)\right\}^{\frac{1}{q}-\frac{1}{p}}$$

$$\cdot \left\{\sum_{j=1}^{n} (a_i)^p (b_j)^q \left(1 - e_i + e_j\right)\right\}^{\frac{1}{p}} \left\{\sum_{j=1}^{n} (c_i)^r (c_j)^r \left(1 - e_i + e_j\right)\right\}^{\frac{1}{r}}$$

$$\geq \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (b_i)^q (a_j)^p (1 - e_i + e_j) \right\}^{\frac{1}{p}} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (b_i)^q (b_j)^q (1 - e_i + e_j) \right\}^{\frac{1}{q} - \frac{1}{p}}$$

$$\cdot \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i)^p (b_j)^q (1 - e_i + e_j) \right\}^{\frac{1}{p}} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (c_i)^r (c_j)^r (1 - e_i + e_j) \right\}^{\frac{1}{r}}$$

$$= \left\{ \sum_{i=1}^{n} (b_i)^q \right\}^{2\left(\frac{1}{q} - \frac{1}{p}\right)} \left\{ \sum_{i=1}^{n} (c_i)^r \right\}^{\frac{2}{r}}$$

$$\cdot \left[\left\{ \sum_{i=1}^{n} (a_i)^p \sum_{j=1}^{n} (b_j)^q \right\}^2 - \left\{ \sum_{i=1}^{n} (b_i)^q e_i \sum_{j=1}^{n} (a_j)^p - \sum_{i=1}^{n} (b_i)^q \sum_{j=1}^{n} (a_j)^p e_j \right\}^2 \right]^{\frac{1}{p}}.$$

Thus we obtain

$$\sum_{i=1}^{n} a_i b_i c_i \geq \left\{ \sum_{i=1}^{n} (b_i)^q \right\}^{\left(\frac{1}{q} - \frac{1}{p}\right)} \left\{ \sum_{i=1}^{n} (c_i)^r \right\}^{\frac{1}{r}}$$

$$\cdot \left[\left\{ \sum_{i=1}^{n} (a_i)^p \sum_{j=1}^{n} (b_j)^q \right\}^2 - \left\{ \sum_{i=1}^{n} (b_i)^q e_i \sum_{j=1}^{n} (a_j)^p - \sum_{i=1}^{n} (b_i)^q \sum_{j=1}^{n} (a_j)^p e_j \right\}^2 \right]^{\frac{1}{2p}}.$$

Therefore

$$\sum_{i=1}^{n} a_i b_i c_i$$

$$\geq \left(\sum_{i=1}^{n} (a_i)^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} (b_i)^q \right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} (c_i)^r \right)^{\frac{1}{r}} \left\{ 1 - \left(\frac{\sum_{i=1}^{n} (b_i)^q e_i}{\sum_{i=1}^{n} (b_i)^q} - \frac{\sum_{i=1}^{n} (a_i)^p e_i}{\sum_{i=1}^{n} (a_i)^p} \right)^2 \right\}^{\frac{1}{2p}}.$$

The proof is complete.

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