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SEDENION FUNCTIONS OF HYPERCOMPLEX VARIABLES IN THE SENSE OF CLIFFORD ANALYSIS

Su Hyun Park and Kwang Ho Shon*

Abstract. The aim of this paper is to define hyperholomorphic functions with sedenion variables in $\mathbb{C}^8$ and research the properties of hyperholomorphic functions of sedenion variables. We generalize the properties of hyperholomorphic functions in sedenionic analysis.

1. Introduction

Deavours [1] has developed a theory of quaternion analysis. Naser [4] and Nôno [5] gave some properties of quaternionic hyperholomorphic functions. In 2004, Kajiwara, Li and Shon [2] obtained some results for the regeneration in complex , quaternion and Clifford analysis, and for the inhomogeneous Cauchy-Riemann system of quaternion and Clifford analysis in ellipsoid. Nôno [6, 7] gave some properties of characterizations of domains of holomorphy by the existence of hyper-conjugate harmonic functions and domains of hyperholomorphic in $\mathbb{C}^4$. In 2013, Lim and Shon [3] obtained some properties of hyperholomorphic functions on $\mathcal{O}$ and hyper-conjugate harmonic functions of octonion variables. We investigate some properties of quaternion, octonion, sedenion functions of complex variables in the sense of Clifford analysis.

2. Notations on Sedenion analysis

The field $\mathcal{S} \cong \mathbb{C}^8$ of sedenions

$$z = \sum_{l=0}^{15} e_l x_l, \ x_l(l = 0, 1, ..., 15) \in \mathbb{R} \tag{1}$$

is a sixteen dimensional non-commutative $\mathbb{R}$-field generated by sixteen base elements $e_l(l = 0, 1, ..., 15)$ with the following non-commutative multiplication rules:

$$e_i^2 = -1, \ e_ie_j = -e_je_i, \ (e_ie_j)e_k = e_i(e_je_k) \ (i \neq j \neq k, i \neq 0, j \neq 0, k \neq 0).$$

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*Corresponding author.

The multiplication of these unit sedenion follows:

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$	e_9	$-e_8$	e_{11}	$-e_{10}$	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$
e_2	e_2	$-e_3$	-1	e_1	e_6	$-e_7$	$-e_4$	e_5	e_{10}	$-e_{11}$	$-e_8$	e_9	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}
e_3	e_3	e_2	$-e_1$	-1	e_7	e_6	$-e_5$	$-e_4$	e_{11}	e_{10}	$-e_9$	$-e_8$	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_5	e_4	e_7	$-e_6$	$-e_1$	-1	e_3	$-e_2$	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_6	$-e_7$	e_4	e_5	$-e_2$	$-e_3$	-1	e_1	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	e_7	e_6	$-e_5$	e_4	$-e_3$	e_2	$-e_1$	-1	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	-1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_9	e_8	e_{11}	$-e_{10}$	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	-1	e_3	$-e_2$	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{10}	$-e_{11}$	e_8	e_9	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	$-e_3$	-1	e_1	$-e_6$	$-e_7$	e_4	e_5
e_{11}	e_{11}	e_{10}	$-e_9$	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	e_2	$-e_1$	-1	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{12}	$-e_{13}$	$-e_{14}$	$-e_{15}$	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	-1	e_1	e_2	e_3
e_{13}	e_{13}	e_{12}	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	$-e_1$	-1	e_3	$-e_2$
e_{14}	e_{14}	$-e_{15}$	e_{12}	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	$-e_2$	$-e_3$	-1	e_1
e_{15}	e_{15}	e_{14}	$-e_{13}$	e_{12}	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	$-e_3$	$-e_2$	$-e_1$	-1

The element e_0 is the identity of $\mathcal{S}$ and e_1 identify the imaginary unit $\sqrt{-1}$ in the $\mathbb{C}$-field of complex numbers. A sedenion z given by (1) is regarded as

$$z = z_1 + z_2e_2 + z_3e_4 + z_4e_6 + z_5e_8 + z_6e_{10} + z_7e_{12} + z_8e_{14} \in \mathcal{S},$$

where $z_1 = x_0 + e_1x_1,\ z_2 = x_2 + e_1x_3,\ z_3 = x_4 + e_1x_5,\ z_4 = x_6 + e_1x_7,$ $z_5 = x_8 + e_1x_9,\ z_6 = x_{10} + e_1x_{11},\ z_7 = x_{12} + e_1x_{13}$ and $z_8 = x_{14} + e_1x_{15}$ are complex numbers in $\mathbb{C}$. Thus, we identify $\mathcal{S}$ with $\mathbb{C}^8$.

We write the sedenion $z = \sum_{l=0}^{15} e_lx_l$ and the sedenion conjugate $z^* = x_0 - \sum_{l=1}^{15} e_lx_l$. Also, the absolute value $|z|$ of z and an inverse z^{-1} of z in $\mathcal{S}$ are defined by

$$|z| = \sqrt{\sum_{l=1}^{8} |z_l|^2},\ z^{-1} = \frac{z^*}{|z|^2}\ (z \neq 0).$$

Thus, the sedenion $z \in \mathcal{S}$ have the following forms:

$$\begin{aligned} z &= \sum_{l=0}^{15} e_lx_l \\ &= z_1 + z_2e_2 + z_3e_4 + z_4e_6 + z_5e_8 + z_6e_{10} + z_7e_{12} + z_8e_{14} \\ &= Z_1 + Z_2e_4 + Z_3e_8 + Z_4e_{12} \\ &= P_1 + P_2e_8 \in \mathcal{S} \end{aligned}$$

and

$$\begin{aligned} z^* &= x_0 - \sum_{l=1}^{15} e_lx_l \\ &= \overline{z_1} - z_2e_2 - z_3e_4 - z_4e_6 - z_5e_8 - z_6e_{10} - z_7e_{12} - z_8e_{14} \\ &= Z_1^* - Z_2e_4 - Z_3e_8 - Z_4e_{12} \\ &= P_1^* - P_2e_8 \in \mathcal{S}, \end{aligned}$$

where $Z_1 = z_1 + z_2e_2, Z_2 = z_3 + z_4e_2, Z_3 = z_5 + z_6e_2$ and $Z_4 = z_7 + z_8e_2$ are quaternion numbers in $\mathbb{C}^2$, and ,$P_1 = Z_1 + Z_2e_4$ and $P_2 = Z_3 + Z_4e_4$ are octonion numbers in $\mathbb{C}^4$.

We use the following differential operators:

$$\begin{aligned} \frac{\partial}{\partial Z_1} &:= \frac{\partial}{\partial z_1} - e_2\frac{\partial}{\partial z_2},\ \frac{\partial}{\partial Z_1^*} = \frac{\partial}{\partial \overline{z_1}} + e_2\frac{\partial}{\partial z_2}, \\ \frac{\partial}{\partial Z_2} &:= \frac{\partial}{\partial z_3} - e_2\frac{\partial}{\partial z_4},\ \frac{\partial}{\partial Z_2^*} = \frac{\partial}{\partial \overline{z_3}} + e_2\frac{\partial}{\partial z_4}, \\ \frac{\partial}{\partial Z_3} &:= \frac{\partial}{\partial z_5} - e_2\frac{\partial}{\partial z_6},\ \frac{\partial}{\partial Z_3^*} = \frac{\partial}{\partial \overline{z_5}} + e_2\frac{\partial}{\partial z_6}, \\ \frac{\partial}{\partial Z_4} &:= \frac{\partial}{\partial z_7} - e_2\frac{\partial}{\partial z_8},\ \frac{\partial}{\partial Z_4^*} = \frac{\partial}{\partial \overline{z_7}} + e_2\frac{\partial}{\partial z_8}, \end{aligned}$$

where $\partial/\partial z_l$, $\partial/\partial \overline{z_l}$ $(l = 1, 2, ..., 8)$ are usual differential operators used in complex analysis.

And we use the following differential operators:

$$\frac{\partial}{\partial P_1} := \frac{\partial}{\partial Z_1} - e_4 \frac{\partial}{\partial Z_2}, \ \frac{\partial}{\partial P_1^*} = \frac{\partial}{\partial Z_1^*} + e_4 \frac{\partial}{\partial Z_2},$$
$$\frac{\partial}{\partial P_2} := \frac{\partial}{\partial Z_3} - e_4 \frac{\partial}{\partial Z_4}, \ \frac{\partial}{\partial P_2^*} = \frac{\partial}{\partial Z_3^*} + e_4 \frac{\partial}{\partial Z_4}.$$

Also, we use the following sedenion differential operators:

$$D := \frac{\partial}{\partial P_1} - e_8 \frac{\partial}{\partial P_2}, \ D^* = \frac{\partial}{\partial P_1^*} + e_8 \frac{\partial}{\partial P_2}.$$

The operator

$$DD^* = \sum_{l=1}^{8} \frac{\partial^2}{\partial z_l \partial \overline{z_l}} = \frac{1}{4} \sum_{l=0}^{15} \frac{\partial^2}{\partial x_l^2}$$

is the usual complex Laplacian Δ.

3. Some properties of hyperholomorphic functions on $\mathcal{S}$

Let Ω be an open set in $\mathbb{C}^8$. The function $f(z)$ is defined on Ω with values in $\mathcal{S}$ as follows:

$$\begin{aligned} f(z) &= \sum_{l=0}^{15} u_l e_l \\ &= f_1(z) + f_2(z)e_2 + f_3(z)e_4 + f_4(z)e_6 \\ &\quad + f_5(z)e_8 + f_6(z)e_{10} + f_7(z)e_{12} + f_8(z)e_{14}. \end{aligned}$$

Definition 1. Let Ω be an open set in $\mathbb{C}^8$. A function $f(z)$ is said to be L(R)-hyperholomorphic on Ω, if
(a) $f_k(z)$ $(k = 1, 2, ..., 8)$ are continuously differential functions on Ω,
(b)

$$D^* f = 0 \ (f D^* = 0) \text{ on } \Omega. \tag{2}$$

The function $f(z)$ is a L-hyperholomorphic function on $\Omega \subset \mathbb{C}^8$, simply we say that $f(z)$ is a hyperholomorphic function on $\Omega \subset \mathbb{C}^8$.

The equation (2) operate to $f(z)$ as follows:

$$\begin{aligned} D^*f &= (\frac{\partial}{\partial P_1^*} + e_8\frac{\partial}{\partial P_2})(f_1(z) + f_2(z)e_2 + f_3(z)e_4 + f_4(z)e_6 \\ &\quad + f_5(z)e_8 + f_6(z)e_{10} + f_7(z)e_{12} + f_8(z)e_{14}) \\ &= (\frac{\partial f_1}{\partial P_1^*} - \frac{\partial \overline{f_5}}{\partial P_2^*}) + (\frac{\partial f_2}{\partial P_1^*} + \frac{\partial \overline{f_6}}{\partial P_2^*})e_2 + (\frac{\partial f_3}{\partial P_1^*} + \frac{\partial \overline{f_7}}{\partial P_2^*})e_4 \\ &\quad + (\frac{\partial f_4}{\partial P_1^*} + \frac{\partial \overline{f_8}}{\partial P_2^*})e_6 + (\frac{\partial f_5}{\partial P_1^*} + \frac{\partial \overline{f_1}}{\partial P_2^*})e_8 + (\frac{\partial f_6}{\partial P_1^*} - \frac{\partial \overline{f_2}}{\partial P_2^*})e_{10} \\ &\quad + (\frac{\partial f_7}{\partial P_1^*} - \frac{\partial \overline{f_3}}{\partial P_2^*})e_{12} + (\frac{\partial f_8}{\partial P_1^*} - \frac{\partial \overline{f_4}}{\partial P_2^*})e_{14}. \end{aligned}$$

If the following equations

$$\begin{gathered} \frac{\partial f_1}{\partial P_1^*} = \frac{\partial \overline{f_5}}{\partial P_2^*},\ \frac{\partial f_2}{\partial P_1^*} = -\frac{\partial \overline{f_6}}{\partial P_2^*},\ \frac{\partial f_3}{\partial P_1^*} = -\frac{\partial \overline{f_7}}{\partial P_2^*},\ \frac{\partial f_4}{\partial P_1^*} = -\frac{\partial \overline{f_8}}{\partial P_2^*}, \\ \frac{\partial f_5}{\partial P_1^*} = -\frac{\partial \overline{f_1}}{\partial P_2^*},\ \frac{\partial f_6}{\partial P_1^*} = \frac{\partial \overline{f_2}}{\partial P_2^*},\ \frac{\partial f_7}{\partial P_1^*} = \frac{\partial \overline{f_3}}{\partial P_2^*},\ \frac{\partial f_8}{\partial P_1^*} = \frac{\partial \overline{f_4}}{\partial P_2^*} \end{gathered} \tag{3}$$

are satisfied, the function $f(z)$ is a hyperholomorphic function on Ω. These are the corresponding s-Cauchy-Riemann equations on $\mathbb{C}^8$.

Remark 1. We redefine the equations (3) in $\mathbb{C}^4$ as follows:

$$\begin{gathered} \frac{\partial f_1}{\partial Z_1^*} = \frac{\partial \overline{f_5}}{\partial Z_3^*},\ \frac{\partial f_1}{\partial Z_2} = \frac{\partial \overline{f_5}}{\partial Z_4},\ \frac{\partial f_2}{\partial Z_1^*} = -\frac{\partial \overline{f_6}}{\partial Z_3^*},\ \frac{\partial f_2}{\partial Z_2} = -\frac{\partial \overline{f_6}}{\partial Z_4}, \\ \frac{\partial f_3}{\partial Z_1^*} = -\frac{\partial \overline{f_7}}{\partial Z_3^*},\ \frac{\partial f_3}{\partial Z_2} = -\frac{\partial \overline{f_7}}{\partial Z_4},\ \frac{\partial f_4}{\partial Z_1^*} = -\frac{\partial \overline{f_8}}{\partial Z_3^*},\ \frac{\partial f_4}{\partial Z_2} = -\frac{\partial \overline{f_8}}{\partial Z_4}, \\ \frac{\partial f_5}{\partial Z_1^*} = -\frac{\partial \overline{f_1}}{\partial Z_3^*},\ \frac{\partial f_5}{\partial Z_2} = -\frac{\partial \overline{f_1}}{\partial Z_4},\ \frac{\partial f_6}{\partial Z_1^*} = \frac{\partial \overline{f_2}}{\partial Z_3^*},\ \frac{\partial f_6}{\partial Z_2} = \frac{\partial \overline{f_2}}{\partial Z_4}, \\ \frac{\partial f_7}{\partial Z_1^*} = \frac{\partial \overline{f_3}}{\partial Z_3^*},\ \frac{\partial f_7}{\partial Z_2} = \frac{\partial \overline{f_3}}{\partial Z_4},\ \frac{\partial f_8}{\partial Z_1^*} = \frac{\partial \overline{f_4}}{\partial Z_3^*},\ \frac{\partial f_8}{\partial Z_2} = \frac{\partial \overline{f_4}}{\partial Z_4}. \end{gathered} \tag{4}$$

Remark 2. We redefine the equations (4) in $\mathbb{C}^8$ as follows:

$$\begin{aligned}
&\frac{\partial f_1}{\partial \overline{z}_1}=\frac{\partial \overline{f_5}}{\partial \overline{z_5}},\ \frac{\partial f_1}{\partial z_2}=\frac{\partial \overline{f_5}}{\partial z_6},\ \frac{\partial f_1}{\partial z_3}=\frac{\partial \overline{f_5}}{\partial z_7},\ \frac{\partial f_1}{\partial z_4}=\frac{\partial \overline{f_5}}{\partial z_8},\\
&\frac{\partial f_2}{\partial \overline{z_1}}=-\frac{\partial \overline{f_6}}{\partial \overline{z_5}},\ \frac{\partial f_2}{\partial z_2}=-\frac{\partial \overline{f_6}}{\partial z_6},\ \frac{\partial f_2}{\partial z_3}=-\frac{\partial \overline{f_6}}{\partial z_7},\ \frac{\partial f_2}{\partial z_4}=-\frac{\partial \overline{f_6}}{\partial z_8},\\
&\frac{\partial f_3}{\partial \overline{z}_1}=-\frac{\partial \overline{f_7}}{\partial \overline{z}_5},\ \frac{\partial f_3}{\partial z_2}=-\frac{\partial \overline{f_7}}{\partial z_6},\ \frac{\partial f_3}{\partial z_3}=-\frac{\partial \overline{f_7}}{\partial z_7},\ \frac{\partial f_3}{\partial z_4}=-\frac{\partial \overline{f_7}}{\partial z_8},\\
&\frac{\partial f_4}{\partial \overline{z}_1}=-\frac{\partial \overline{f_8}}{\partial \overline{z}_5},\ \frac{\partial f_4}{\partial z_2}=-\frac{\partial \overline{f_8}}{\partial z_6},\ \frac{\partial f_4}{\partial z_3}=-\frac{\partial \overline{f_8}}{\partial z_7},\ \frac{\partial f_4}{\partial z_4}=-\frac{\partial \overline{f_8}}{\partial z_8},\\
&\frac{\partial f_5}{\partial \overline{z}_1}=-\frac{\partial \overline{f_1}}{\partial \overline{z_5}},\ \frac{\partial f_5}{\partial z_2}=-\frac{\partial \overline{f_1}}{\partial z_6},\ \frac{\partial f_5}{\partial z_3}=-\frac{\partial \overline{f_1}}{\partial z_7},\ \frac{\partial f_5}{\partial z_4}=-\frac{\partial \overline{f_1}}{\partial z_8},\\
&\frac{\partial f_6}{\partial \overline{z_1}}=\frac{\partial \overline{f_2}}{\partial \overline{z_5}},\ \frac{\partial f_6}{\partial z_2}=\frac{\partial \overline{f_2}}{\partial z_6},\ \frac{\partial f_6}{\partial z_3}=\frac{\partial \overline{f_2}}{\partial z_7},\ \frac{\partial f_6}{\partial z_4}=\frac{\partial \overline{f_2}}{\partial z_8},\\
&\frac{\partial f_7}{\partial \overline{z}_1}=\frac{\partial \overline{f_3}}{\partial \overline{z}_5},\ \frac{\partial f_7}{\partial z_2}=\frac{\partial \overline{f_3}}{\partial z_6},\ \frac{\partial f_7}{\partial z_3}=\frac{\partial \overline{f_3}}{\partial z_7},\ \frac{\partial f_7}{\partial z_4}=\frac{\partial \overline{f_3}}{\partial z_8},\\
&\frac{\partial f_8}{\partial \overline{z}_1}=\frac{\partial \overline{f_4}}{\partial \overline{z}_5},\ \frac{\partial f_8}{\partial z_2}=\frac{\partial \overline{f_4}}{\partial z_6},\ \frac{\partial f_8}{\partial z_3}=\frac{\partial \overline{f_4}}{\partial z_7},\ \frac{\partial f_8}{\partial z_4}=\frac{\partial \overline{f_4}}{\partial z_8}.
\end{aligned} \tag{5}$$

We call that the equations (5) are the condition of harmonicity of the hyperholomorphic function $f(z)$ on Ω in $\mathbb{C}^8$.

Lemma 3.1. *Let Ω be an open set in $\mathbb{C}^8$. If the function $f(z)$ is a hyperholomorphic in Ω , then the functions $f_k(z)$ $(k = 1, 2, ..., 8)$ are of class $\mathcal{C}^\infty$ in Ω.*

Proof. Fix $z = \sum_{l=0}^{15} e_l x_l$ in Ω and let δ be chosen such that a ball with center z and with radius δ, $\mathrm{B}(z;\delta) \subset \Omega$. Then f_k has a harmonic conjugate $f_{k+4}(k = 1, 2, 3, 4)$ on $\mathrm{B}(z;\delta)$. That is, $f = \sum_{l=1}^{8} f_l(z)e_{2(l-1)}$ is hyperholomorphic and hence infinitely differentiable on $\mathrm{B}(z;\delta)$. It follows that $f_k(k = 1, 2, ..., 8)$ is of class $\mathcal{C}^\infty$. □

Theorem 3.2. *If the function $f(z)$ satisfies the condition of harmonicity* (5) *in an open set Ω in $\mathbb{C}^8$, then the functions $f_k(z)(k = 1, 2, ..., 8)$ are harmonic on Ω.*

Proof. From the condition of harmony (5), we have

$$\begin{aligned} DD^* f_1 &= \sum_{l=1}^{8} \frac{\partial^2 f_1}{\partial z_l \partial \bar{z}_l} \\ &= \frac{\partial}{\partial z_1}(\frac{\partial \overline{f_5}}{\partial \bar{z}_5}) + \frac{\partial}{\partial \bar{z}_2}(\frac{\partial \overline{f_5}}{\partial z_6}) + \frac{\partial}{\partial \bar{z}_3}(\frac{\partial \overline{f_5}}{\partial z_7}) + \frac{\partial}{\partial \overline{z_4}}(\frac{\partial \overline{f_5}}{\partial z_8}) \\ &\quad + \frac{\partial}{\partial \bar{z}_5}(-\frac{\partial \overline{f_5}}{\partial z_1}) + \frac{\partial}{\partial z_6}(-\frac{\partial \overline{f_5}}{\partial \bar{z}_2}) + \frac{\partial}{\partial z_7}(-\frac{\partial \overline{f_5}}{\partial \bar{z}_3}) + \frac{\partial}{\partial z_8}(-\frac{\partial \overline{f_5}}{\partial \overline{z_4}}) \\ &= 0. \end{aligned}$$

And the functions $f_k(z)$ $(k = 2, 3, ..., 8)$ are proved by the similar method as in the proof of the case of f_1. □

References

[1] C. A . Deavours, *The quaternion calculus,* Amer. Math. Monthly **80** (1973), 995–1008.

[2] J. Kajiwara, X. D. Li, and K. H. Shon, *Regeneration in Complex, Quaternion and Clifford analysis*, Proc. the 9th(2001) Internatioal Conf. on Finite or Infinite Dimensional Complex Analysis and Applications, Advances in Complex Analysis and Its Applications Vol. 2, Kluwer Academic Publishers (2004), 287-298.

[3] S. J. Lim and K. H. Shon, *Hyperholomorphic functions and hyper-conjugate harmonic functions of octonion variables,* J. Inequa. Appl. **77** (2013), 1-8.

[4] M. Naser, *Hyperholomorphic functions,* Siberian Math. J. **12** (1971), 959-968.

[5] K. Nôno, *Hyperholomorphic functions of a quaternion variable*, Bull. Fukuoka Univ. Ed. **32** (1983), 21-37.

[6] ———, *Characterization of domains of holomorphy by the existence of hyper-conjugate harmonic functions*, Rev. Roumaine Math. Pures Appl. **31** (1986), no. 2, 159-161.

[7] ———, *Domains of Hyperholomorphic in* $\mathbb{C}^2 \times \mathbb{C}^2$, Bull. Fukuoka Univ. Ed. **36** (1987), 1-9.

Su Hyun Park
Department of Mathematics, Pusan National University, Busan 609-735, Korea
E-mail address: d2hyun@hanmail.net

Kwang Ho Shon
Department of Mathematics, Pusan National University, Busan 609-735, Korea
E-mail address: khshon@pusan.ac.kr