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NECESSARY CONDITIONS FOR OPTIMAL BOUNDARY CONTROL PROBLEM GOVERNED BY SOME CHEMOTAXIS EQUATIONS

Sang-Uk Ryu

Abstract. This paper is concerned with the necessary conditions of the optimal boundary control for some chemotaxis equations. We obtain the existence and the necessary conditions of the optimal boundary control in the space $(H^1(0,T))^2$. Moreover, under some assumptions, we show the uniqueness of the optimal control.

1. Introduction

In this paper we consider the following optimal boundary control problem

$$(\mathbf{P}) \qquad \text{minimize} \quad J(u_1, u_2)$$

with the cost functional $J(u_1, u_2)$ of the form

$$J(u_1,u_2) = \int_0^T \|y(u_1,u_2) - y_d\|^2_{L^2(0,L)} dt \\ + \gamma\{\|u_1\|^2_{H^1(0,T)} + \|u_2\|^2_{H^1(0,T)}\}, \quad u_1,\ u_2 \in H^1(0,T),$$

where $y = y(u_1, u_2)$ is governed by the chemotaxis equations

$$\begin{aligned} &\frac{\partial y}{\partial t} = a\frac{\partial^2 y}{\partial x^2} - b\frac{\partial}{\partial x}\Big(y\frac{\partial \rho}{\partial x}\Big) \qquad \text{in } (0,L)\times(0,T], \\ &\frac{\partial \rho}{\partial t} = d\frac{\partial^2 \rho}{\partial x^2} + fy - h\rho \qquad \text{in } (0,L)\times(0,T], \\ &\frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(L,t) = 0, \quad \frac{\partial \rho}{\partial x}(0,t) = u_1(t), \quad \frac{\partial \rho}{\partial x}(L,t) = u_2(t) \qquad \text{on } (0,T], \\ &y(x,0) = y_0(x), \quad \rho(x,0) = \rho_0(x) \qquad \text{in } (0,L). \end{aligned} \tag{1.1}$$

Here, $(0,L)$ is a bounded interval in $\mathbf{R}$. a, b, d, f, $h > 0$ are given positive numbers. $y = y(x,t)$ describes the cell concentration in $(0,L)$ at time t, and

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$\rho = \rho(x,t)$ the chemoattractant concentration in $(0,L)$ at time t. $u_1(t)$ and $u_2(t)$ are the control functions. We refer to [2, 3, 5, 6] and the references for (1.1).

Many papers have already been published to study the optimal control problems for nonlinear parabolic equations(see [1, 3, 6, 7]). In [7], Ryu studied the existence of the optimal boundary control for the chemotaxis diffusion equations when the boundary control is given in the space $H^2(0,T)$. However, in this paper we obtain the existence and the necessary conditions of the optimal boundary control in the space $(H^1(0,T))^2$. Moreover, under some assumptions, we show the uniqueness of the optimal control.

2. Mathematical setting

In this section, we recall the existence and uniqueness of a local solution for (1.1)([5, 7]). To derive the existence of solutions for (1.1), we reduce (1.1) to a homogeneous problem. First we construct a lifting function for the boundary conditions,

$$\phi_U(x,t) = u_1(t)\frac{x(2L-x)}{2L} + u_2(t)\frac{x^2}{2L}.$$

Here, $u_i \in H^1_\Gamma(0,T) = \{u \in H^1(0,T) : u(0) = 0\}(i = 1,2)$ and $U = \binom{u_1}{u_2}$. Obviously

$$\Big|\frac{\partial^i \phi_U(x,t)}{\partial x^i}\Big| \le C(|u_1(t)| + |u_2(t)|), \ \forall x \in (0,L), \ \forall t \in [0,T] \ (i = 0,1,2). \tag{2.1}$$

Let us set $w(x,t) = \rho(x,t) - \phi_U(x,t)$; then the system (1.1) is equivalent to the one:

$$\begin{aligned}
&\frac{\partial y}{\partial t} = a\frac{\partial^2 y}{\partial x^2} - b\frac{\partial}{\partial x}\Big(y\frac{\partial(w+\phi_U)}{\partial x}\Big) \qquad \text{in } (0,L)\times(0,T],\\
&\frac{\partial w}{\partial t} = d\frac{\partial^2 w}{\partial x^2} + fy - hw + g_U(x,t) \qquad \text{in } (0,L)\times(0,T],\\
&\frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(L,t) = \frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(L,t) = 0 \qquad \text{on } (0,T],\\
&y(x,0) = y_0(x), \quad w(x,0) = w_0 \qquad \text{in } (0,L).
\end{aligned} \tag{2.2}$$

Here, $w_0 = \rho_0(x)$ and $g_U(x,t) = d\frac{\partial^2\phi_U}{\partial x^2} - h\phi_U - \frac{\partial \phi_U}{\partial t}$.

Now, we show the existence and uniqueness of a local solution for (2.2) as in [5, 7]. Let $A_1 = -a\frac{\partial^2}{\partial x^2} + a$ and $A_2 = -d\frac{\partial^2}{\partial x^2} + h$ with the same domain $\mathcal{D}(A_i) = H^2_n(0,L) = \{z \in H^2(0,L); \frac{\partial z}{\partial x}(0) = \frac{\partial z}{\partial x}(L) = 0\}$ $(i = 1,2)$. Then, A_i are two positive definite self-adjoint operators in $L^2(0,L)$. We set three product Hilbert spaces $\mathcal{V} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{V}'$ as $\mathcal{V} = H^1(0,L)\times H^2_n(0,L)$, $\mathcal{H} = L^2(0,L)\times H^1(0,L)$, and $\mathcal{V}' = (H^1(0,L))' \times L^2(0,L)$. Let $\mathcal{U}_{ad}$ be a closed bounded convex subset in $H^1_\Gamma \times H^1_\Gamma$.

Then, (2.2) is formulated as an abstract equation

$$\begin{aligned} &\frac{dY}{dt} + AY = F_U(Y) + G_U(t), \quad 0 < t \le T, \\ &Y(0) = Y_0 \end{aligned} \tag{2.3}$$

in the space $\mathcal{V}'$. Here, a linear isomorphism $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ is a bounded operator from $\mathcal{V}$ to $\mathcal{V}'$, and the part of A in $\mathcal{H}$ is a positive definite self-adjoint operator in $\mathcal{H}$. $F_U(\cdot) : \mathcal{V} \to \mathcal{V}'$ is the mapping

$$F_U(Y) = \begin{pmatrix} ay - b\frac{\partial}{\partial x}\left(y\frac{\partial(w+\phi_U)}{\partial x}\right) \\ fy \end{pmatrix} \quad \text{and} \quad G_U(t) = \begin{pmatrix} 0 \\ g_U(x,t) \end{pmatrix}.$$

Y_0 is defined by $Y_0 = \binom{y_0}{w_0}$.

Moreover, for $U \in \mathcal{U}_{ad}$, $F_U(\cdot)$ is a given continuous function from $\mathcal{V}$ to $\mathcal{V}'$ satisfying the following conditions([7]):

For each $\eta > 0$, there exists an increasing continuous function $\mu_\eta : [0,\infty) \to [0,\infty)$ such that for all $\widetilde{Y}, Y \in \mathcal{V}$

$$\|F_U(Y)\|_{\mathcal{V}'} \le \eta\|Y\|_{\mathcal{V}} + \mu_\eta(\|Y\|_{\mathcal{H}}), \text{ a.e. } (0,T); \tag{f.i}$$

$$\begin{aligned} &\|F_U(\widetilde{Y}) - F_U(Y)\|_{\mathcal{V}'} \le \eta\|\widetilde{Y} - Y\|_{\mathcal{V}} \\ &\quad + (\|\widetilde{Y}\|_{\mathcal{V}} + \|Y\|_{\mathcal{V}} + 1)\mu_\eta(\|\widetilde{Y}\|_{\mathcal{H}} + \|Y\|_{\mathcal{H}})\|\widetilde{Y} - Y\|_{\mathcal{H}}, \text{ a.e. } (0,T). \end{aligned} \tag{f.ii}$$

In addition, there exists a constant $C > 0$ such that for all $Y \in \mathcal{V}$, $\widetilde{U}, \overline{U} \in \mathcal{U}_{ad}$

$$\|F_{\widetilde{U}}(Y) - F_{\overline{U}}(Y)\|_{\mathcal{V}'} \le C|\widetilde{U} - \overline{U}|\|Y\|_{\mathcal{H}}, \text{ a.e. } (0,T). \tag{f.iii}$$

Since $U \in \mathcal{U}_{ad}$, we see that $G_U(\cdot) \in L^2(0,T;\mathcal{V}')$. Then, we obtain the following result([6]).

Theorem 2.1. *If $Y_0 \in \mathcal{H}$, there exists a unique weak solution*

$$Y \in H^1(0,S;\mathcal{V}') \cap \mathcal{C}([0,S];\mathcal{H}) \cap L^2(0,S;\mathcal{V})$$

to (2.3)*, the number $S \in (0,T]$ is determined by the norm $\|G_U\|_{L^2(0,T;\mathcal{V}')}$ and $\|Y_0\|_{\mathcal{H}}$.*

3. Necessary conditions for the optimal control

Let $S > 0$ be such that for each $U \in \mathcal{U}_{ad}$, (2.3) has a unique weak solution $Y(U) \in H^1(0,S;\mathcal{V}') \cap \mathcal{C}([0,S];\mathcal{H}) \cap L^2(0,S;\mathcal{V})$. Thus, the problem (**P**) is obviously formulated as follows:

$$(\overline{\mathbf{P}}) \qquad \text{minimize } J(U),$$

where

$$J(U) = \int_0^S \|DY(U) - Y_d\|^2_{\mathcal{H}} dt + \gamma\|U\|^2_{(H^1(0,S))^2}, \quad U \in \mathcal{U}_{ad}.$$

Here, $D\binom{y}{w} = \binom{y}{0}$ and $Y_d = \binom{y_d}{0}$ is a fixed element of $L^2(0,S;\mathcal{H})$ with $y_d \in L^2(0,S;L^2(0,L))$. γ is a positive constant.

Lemma 3.1. *Let $U_n = \binom{u_{1n}}{u_{2n}} \to U = \binom{u_1}{u_2}$ weakly in $(H^1(0,S))^2$. Then we have*

$$G_{U_n}(t) \to G_U(t) \text{ weakly in } L^2(0,S;\mathcal{V}'). \tag{3.1}$$

Proof. For all $\psi \in L^2(0,S;H_n^2(0,L))$,

$$\begin{aligned}
&\int_0^S \Big\langle \frac{\partial \phi_{U_n}}{\partial t} + h\phi_{U_n}, \psi \Big\rangle_{L^2(0,L),H^2(0,L)} dt \qquad (3.2)\\
&= \int_0^S \Big\langle \Big(\frac{du_{1n}(t)}{dt} + hu_{1n}(t)\Big)\frac{x(2L-x)}{2L} \\
&\quad + \Big(\frac{du_{2n}(t)}{dt} + hu_{2n}(t)\Big)\frac{x^2}{2L}, \psi\Big\rangle_{L^2(0,L),H^2(0,L)} dt \\
&= \int_0^S \Big(\frac{du_{1n}(t)}{dt} + hu_{1n}(t)\Big)\Big\langle \frac{x(2L-x)}{2L}, \psi\Big\rangle_{L^2(0,L),H^2(0,L)} dt \\
&\quad + \int_0^S \Big(\frac{du_{2n}(t)}{dt} + hu_{2n}(t)\Big)\Big\langle \frac{x^2}{2L}, \psi\Big\rangle_{L^2(0,L),H^2(0,L)} dt.
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^S \Big|\Big\langle \frac{x(2L-x)}{2L}, \psi\Big\rangle_{L^2(0,L),H^2(0,L)}\Big|^2 dt &\le \int_0^S \Big\|\frac{x(2L-x)}{2L}\Big\|^2_{L^2(0,L)} \|\psi\|^2_{H^2(0,L)} dt \\
&\le C\int_0^S \|\psi\|^2_{H^2(0,L)} dt < \infty
\end{aligned}$$

and

$$\int_0^S \Big|\Big\langle \frac{x^2}{2L}, \psi\Big\rangle_{L^2(0,L),H^2(0,L)}\Big|^2 dt < \infty,$$

we know that $\Big\langle \frac{x(2L-x)}{2L}, \psi\Big\rangle_{L^2(0,L),H^2(0,L)}, \Big\langle \frac{x^2}{2L}, \psi\Big\rangle_{L^2(0,L),H^2(0,L)} \in L^2(0,S)$. Since $U_n \to U$ weakly in $(H^1(0,S))^2$, we have

$$\int_0^S \Big(\frac{d(u_{1n}(t) - u_1(t))}{dt} + h(u_{1n}(t) - u_1(t))\Big)\Big\langle \frac{x(2L-x)}{2L}, \psi\Big\rangle_{L^2(0,L),H^2(0,L)} dt \to 0$$

and

$$\int_0^S \Big(\frac{d(u_{2n}(t) - u_2(t))}{dt} + h(u_{2n}(t) - u_2(t))\Big)\Big\langle \frac{x^2}{2L}, \psi\Big\rangle_{L^2(0,L),H^2(0,L)} dt \to 0$$

as $n \to \infty$. Therefore, we obtain from (3.2) that

$$\int_0^S \Big\langle \frac{\partial(\phi_{U_n} - \phi_U)}{\partial t} + h(\phi_{U_n} - \phi_U), \psi\Big\rangle_{L^2(0,L),H^2(0,L)} dt \to 0$$

as $n \to \infty$. Furthermore, since $U_n \to U$ strongly in $(L^2(0,S))^2$([4]), we have

$$\int_0^S \Big\langle \frac{\partial^2(\phi_{U_n} - \phi_U)}{\partial x^2}, \psi \Big\rangle_{L^2(0,L),H^2(0,L)} dt$$
$$\leq C\|U_n - U\|^2_{(L^2(0,S))^2} \to 0.$$

Hence, we obtain (3.1). □

Theorem 3.2. *There exists an optimal control $\overline{U} \in \mathcal{U}_{ad}$ for $(\overline{\mathbf{P}})$ such that $J(\overline{U}) = \min_{U \in \mathcal{U}_{ad}} J(U)$.*

Proof. By using Lemma 3.1, we obtain the existence of the optimal control $\overline{U} \in (H^1(0,S))^2$. The proof is similar to that of [7]. □

To derive the differentiability of $Y(U)$ with respect to the control U, we note that the mapping $F_U(\cdot) : \mathcal{V} \to \mathcal{V}'$ must be Fréchet differentiable with the derivative

$$F'_U(Y)Z = \begin{pmatrix} az_1 - b\frac{\partial}{\partial x}\Big(z_1 \frac{\partial(w+\phi_U)}{\partial x}\Big) - b\frac{\partial}{\partial x}\Big(y\frac{\partial z_2}{\partial x}\Big) \\ fz_1 \end{pmatrix},$$

where $Y = \binom{y}{w}$, $Z = \binom{z_1}{z_2} \in \mathcal{V}$. Then, we have the following estimates([8]);

For each $\eta > 0$, there exists a constant $C_\eta > 0$ such that for $Y, Z, P \in \mathcal{V}$

$$|\langle F'_U(Y)Z, P\rangle_{\mathcal{V}',\mathcal{V}}| \leq \begin{cases} \eta\|Z\|_{\mathcal{V}}\|P\|_{\mathcal{V}} + C_\eta(\|Y\|_{\mathcal{V}} + 1)\|Z\|_{\mathcal{H}}\|P\|_{\mathcal{V}}, \text{ a.e. } (0,S), \\ \eta\|Z\|_{\mathcal{V}}\|P\|_{\mathcal{V}} + C_\eta(\|Y\|_{\mathcal{V}} + 1)\|Z\|_{\mathcal{V}}\|P\|_{\mathcal{H}}, \text{ a.e. } (0,S). \end{cases} \tag{3.3}$$

In addition, there exists a constant $C > 0$ such that for $\widetilde{Y}, Y, Z \in \mathcal{V}$, $\widetilde{U}, \overline{U} \in \mathcal{U}_{ad}$

$$|\langle F'_U(\widetilde{Y})Z - F'_U(Y)Z, P\rangle_{\mathcal{V}',\mathcal{V}}| \leq C\|\widetilde{Y} - Y\|_{\mathcal{H}}\|Z\|_{\mathcal{V}}\|P\|_{\mathcal{V}}, \text{ a.e. } (0,S), \tag{3.4}$$

$$|\langle F'_{\widetilde{U}}(Y)Z - F'_{\overline{U}}(Y)Z, P\rangle_{\mathcal{V}',\mathcal{V}}| \leq \begin{cases} C|\widetilde{U} - \overline{U}|\|Z\|_{\mathcal{H}}\|P\|_{\mathcal{V}}, \text{ a.e. } (0,S), \\ C|\widetilde{U} - \overline{U}|\|Z\|_{\mathcal{V}}\|P\|_{\mathcal{H}}, \text{ a.e. } (0,S). \end{cases} \tag{3.5}$$

Proposition 3.3. *The mapping $U \to Y(U)$ from $\mathcal{U}_{ad}$ into $H^1(0,S;\mathcal{V}') \cap L^2(0,S;\mathcal{V})$ is differentiable in the sense*

$$\frac{Y(U+hV) - Y(U)}{h} \to Z \text{ in } H^1(0,S;\mathcal{V}') \cap L^2(0,S;\mathcal{V})$$

as $h \to 0$, where $U, V \in \mathcal{U}_{ad}$ and $U + hV \in \mathcal{U}_{ad}$. Moreover, $Z = Y'(U)V$ satisfies the linear equation

$$\frac{dZ}{dt} + AZ - F'_U(Y(U))Z = B_V(Y(U)) + G_V(t), \qquad 0 < t \leq S, \tag{3.6}$$
$$Z(0) = 0,$$

where $B_V(Y(U)) = \begin{pmatrix} -b\frac{\partial}{\partial x}\Big(y\frac{\partial \phi_V}{\partial x}\Big) \\ 0 \end{pmatrix}$ *and* $\phi_V = v_1(t)\frac{x(2L-x)}{2L} + v_2(t)\frac{x^2}{2L}$, $V = \binom{v_1}{v_2}$.

Proof. Let $U, V \in \mathcal{U}_{ad}$ and $0 \le h \le 1$. Let $Y_h = Y(U_h)$ and $Y = Y(U)$ be the solutions of (2.3) corresponding to $U_h = U + hV$ and U, respectively.

Step 1. $Y_h \to Y$ *strongly in* $\mathcal{C}([0,S];\mathcal{H})$ *as* $h \to 0$. Let $\widetilde{Y} = Y_h - Y$. Then $\widetilde{Y}$ satisfies the equation

$$\begin{aligned} &\frac{d\widetilde{Y}}{dt} + A\widetilde{Y} = (F_{U_h}(Y_h) - F_{U_h}(Y)) + (F_{U_h}(Y) - F_U(Y)) \\ &\qquad\qquad + G_{U_h}(t) - G_U(t),\ 0 < t \le S, \\ &\widetilde{Y}(0) = 0. \end{aligned} \tag{3.7}$$

From (f.iii) and (2.1), we have

$$\|F_{U_h}(Y) - F_U(Y)\|_{\mathcal{V}'} \le Ch|V(t)|\|Y(t)\|_{\mathcal{H}} \tag{3.8}$$

and

$$\|G_{U_h}(t) - G_U(t)\|_{\mathcal{V}'} = \|g_{hV}(t)\|_{L^2(\Omega)} \le Ch\Big(|V(t)| + \Big|\frac{dV(t)}{dt}\Big|\Big). \tag{3.9}$$

Taking the scalar product with $\widetilde{Y}$ to (3.7) and using (f.ii), (3.8), (3.9), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\widetilde{Y}(t)\|_{\mathcal{H}}^2 + \frac{\delta}{2}\|\widetilde{Y}(t)\|_{\mathcal{V}}^2 \\ &\quad \le (\|Y_h(t)\|_{\mathcal{V}}^2 + \|Y(t)\|_{\mathcal{V}}^2 + 1)\tilde{\mu}(\|Y_h(t)\|_{\mathcal{H}}^2 + \|Y(t)\|_{\mathcal{H}}^2)\|\widetilde{Y}(t)\|_{\mathcal{H}}^2 \\ &\qquad + Ch^2(\|Y(t)\|_{\mathcal{H}}^2 + 1)\Big(|V(t)|^2 + \Big|\frac{dV(t)}{dt}\Big|^2\Big), \end{aligned}$$

where $\tilde{\mu} : [0,\infty) \to [0,\infty)$ are some increasing continuous function.

Using Gronwall's inequality, we have

$$\|\widetilde{Y}(t)\|_{\mathcal{H}}^2 \le Ch^2\|V(t)\|_{(H^1(0,S))^2}^2(\|Y(t)\|_{L^\infty(0,S;\mathcal{H})}^2 + 1)$$

for all $t \in [0,S]$. Hence $Y_h \to Y$ strongly in $\mathcal{C}([0,S];\mathcal{H})$ as $h \to 0$.

Step 2. $\frac{Y_h - Y}{h}$ *converges strongly to the unique solution* Z *of* (3.6) *in* $H^1(0,S;\mathcal{V}') \cap L^2(0,S;\mathcal{V})$ *as* $h \to 0$. Let $W = \frac{Y_h - Y}{h} - Z$ satisfies

$$\begin{aligned} &\frac{dW}{dt} + AW - \Big(\frac{F_U(Y_h) - F_U(Y)}{h} - F_U'(Y)Z\Big) \\ &\qquad = \Big(\frac{F_{U_h}(Y_h) - F_U(Y_h)}{h} - B_V(Y)\Big), \qquad 0 < t \le S, \\ &W(0) = 0. \end{aligned} \tag{3.10}$$

By direct calculation, we have

$$\begin{aligned}
&\Big\|\frac{F_U(Y_h)-F_U(Y))}{h}-F'_U(Y)Z\Big\|_{\mathcal{V}'}\\
&\le\Big\|\int_0^1 F'_U(Y+\theta(Y_h-Y))\frac{Y_h-Y}{h}d\theta-F'_U(Y)Z\Big\|_{\mathcal{V}'}\\
&\le\Big\|\int_0^1 F'_U(Y+\theta(Y_h-Y))Wd\theta\Big\|_{\mathcal{V}'}+\Big\|\int_0^1(F'_U(Y+\theta(Y_h-Y))-F'_U(Y))Z\Big\|_{\mathcal{V}'}\\
&=I_1+I_2.
\end{aligned}$$

Then, From (3.3) and (3.4), we obtain

$$I_1\le\eta\|W\|_{\mathcal{V}}+C_\eta(\|Y_h\|_{\mathcal{V}}+\|Y\|_{\mathcal{V}}+1)\|W\|_{\mathcal{H}} \tag{3.11}$$

and

$$I_2\le C\|Y_h-Y\|_{\mathcal{H}}\|Z\|_{\mathcal{V}}. \tag{3.12}$$

Moreover, since

$$\begin{aligned}
&\Big\|\frac{\partial}{\partial x}\Big((y_h-y)\frac{\partial\phi_V}{\partial x}\Big)\Big\|_{(H^1(0,L))'}\\
=&\sup_{\|\psi\|_{H^1(0,L)}\le1}\Big\langle\frac{\partial}{\partial x}\Big((y_h-y)\frac{\partial\phi_V}{\partial x}\Big),\psi\Big\rangle_{(H^1(0,L))',H^1(0,L)}\\
\le&\sup_{\|\psi\|_{H^1(0,L)}\le1}\Big\{C\|y_h-y\|_{L^2(0,L)}\Big\|\frac{\partial\phi_V}{\partial x}\Big\|_{L^\infty(0,L)}\|\psi\|_{H^1(0,L)}\Big\}\\
\le&C\|y_h-y\|_{L^2(0,L)},
\end{aligned}$$

it is seen that

$$\Big\|\frac{F_{U_h}(Y_h)-F_U(Y_h)}{h}-B_V(Y)\Big\|_{\mathcal{V}'}\le C\|Y_h(t)-Y(t)\|_{\mathcal{H}}. \tag{3.13}$$

Taking the scalar product of the equation of (3.10) with W and using (3.11), (3.12), and (3.13) we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|W(t)\|^2_{\mathcal{H}}+\frac{\delta}{2}\|W(t)\|^2_{\mathcal{V}}\le{}& C(\|Y_h(t)\|^2_{\mathcal{V}}+\|Y(t)\|^2_{\mathcal{V}}+1)\|W(t)\|^2_{\mathcal{H}}\\
&+C(\|Z(t)\|^2_{\mathcal{V}}+1)\|Y_h(t)-Y(t)\|^2_{\mathcal{H}}.
\end{aligned}$$

From Gronwall's inequality, we obtain

$$\|W(t)\|^2_{\mathcal{H}}+\delta\int_0^t\|W(s)\|^2_{\mathcal{V}}ds\le C\|Y_h(t)-Y(t)\|^2_{L^\infty(0,S;\mathcal{H})}(\|Z\|^2_{L^2(0,S;\mathcal{V})}+1).$$

Since $Y_h\to Y$ strongly in $L^\infty(0,S;\mathcal{H})$, it follows that $\frac{Y_h-Y}{h}$ is strongly convergent to Z in $H^1(0,S;\mathcal{V}')\cap\mathcal{C}([0,S];\mathcal{H})\cap L^2(0,S;\mathcal{V})$. □

Theorem 3.4. *Let $\overline{U}$ be an optimal control of* $(\overline{\mathbf{P}})$ *and let* $\overline{Y} \in L^2(0,S;\mathcal{V}) \cap \mathcal{C}([0,S];\mathcal{H}) \cap H^1(0,S;\mathcal{V}')$ *be the optimal state, that is $\overline{Y}$ is the solution to* (2.3) *with respect to the control $\overline{U}$. Then, there exists a unique solution* $P \in L^2(0,S;\mathcal{V}) \cap \mathcal{C}([0,S];\mathcal{H}) \cap H^1(0,S;\mathcal{V}')$ *to the linear problem*

$$-\frac{dP}{dt} + AP - F'_{\overline{U}}(\overline{Y})^* P = D\overline{Y} - Y_d, \quad 0 \le t < S, \qquad (3.14)$$
$$P(S) = 0$$

in $\mathcal{V}'$. Moreover, $\overline{U}$ satisfies

$$\int_0^S \langle P, B_{V-\overline{U}}(\overline{Y}) + G_{V-\overline{U}}(t)\rangle_{\mathcal{V},\mathcal{V}'} dt + \gamma\langle \overline{U}, V - \overline{U}\rangle_{(H^1(0,S))^2} \ge 0 \quad \textit{for all } V \in \mathcal{U}_{ad}.$$

Proof. Since J is Gâteaux differentiable at $\overline{U}$ and $\mathcal{U}_{ad}$ is convex, it is seen that

$$J'(\overline{U})(V - \overline{U}) \ge 0 \qquad \text{for all } V \in \mathcal{U}_{ad}.$$

By direct calculation, we obtain

$$J'(\overline{U})(V - \overline{U}) = \int_0^S \langle D\overline{Y} - Y_d, DZ\rangle_{\mathcal{H},\mathcal{H}} dt + \gamma\langle \overline{U}, V - \overline{U}\rangle_{(H^1(0,S))^2}.$$

While,

$$\begin{aligned}
\int_0^S \langle D\overline{Y} - Y_d, DZ\rangle_{\mathcal{H},\mathcal{H}} dt &= \int_0^S \langle D\overline{Y} - Y_d, Z\rangle_{\mathcal{V}',\mathcal{V}} dt \\
&= \int_0^S \langle -\frac{dP}{dt} + AP - F'_{\overline{U}}(\overline{Y})^* P, Z\rangle_{\mathcal{V}',\mathcal{V}} dt \\
&= \int_0^S \langle P, \frac{dZ}{dt} + AZ - F'_{\overline{U}}(\overline{Y}) Z\rangle_{\mathcal{V},\mathcal{V}'} dt \\
&= \int_0^S \langle P, B_{V-\overline{U}}(\overline{Y}) + G_{V-\overline{U}}(t)\rangle_{\mathcal{V},\mathcal{V}'} dt
\end{aligned}$$

with $Z = Y'(\overline{U})(V - \overline{U})$. Hence, we obtain

$$\int_0^S \langle P, B_{V-\overline{U}}(\overline{Y}) + G_{V-\overline{U}}(t)\rangle_{\mathcal{V},\mathcal{V}'} dt + \gamma\langle \overline{U}, V - \overline{U}\rangle_{(H^1(0,S))^2} \ge 0 \quad \text{for all } V \in \mathcal{U}_{ad}.$$

□

4. Uniqueness of the optimal control

In this section we obtain the uniqueness of the optimal control for $(\overline{\mathbf{P}})$. Suppose there exist two optimal controls $U_1 = \binom{u_{11}}{u_{21}}, U_2 = \binom{u_{12}}{u_{22}} \in \mathcal{U}_{\mathrm{ad}}$ for $(\overline{\mathbf{P}})$

Lemma 4.1. *Let Y_1 and Y_2 be the solutions of* (2.3) *with respect to U_1 and U_2, respectively. Then, we have*

$$\|Y_1(t) - Y_2(t)\|^2_{L^\infty(0,S;\mathcal{H})} \le C\|U_1(t) - U_2(t)\|^2_{(H^1(0,S))^2}.$$

Proof. Let Y_1 and Y_2 be the solutions of (2.3) with respect to U_1 and U_2, respectively. Then $\widetilde{Y} = Y_1 - Y_2$ satisfies the equation

$$\begin{aligned}\frac{d\widetilde{Y}}{dt} + A\widetilde{Y} &= (F_{U_1}(Y_1) - F_{U_1}(Y_2)) + (F_{U_1}(Y_2) - F_{U_2}(Y_2)) \\ &\quad + G_{U_1}(t) - G_{U_2}(t),\ 0 < t \le S, \\ \widetilde{Y}(0) &= 0.\end{aligned} \tag{4.1}$$

From (2.1) and (f.iii), we have

$$\|F_{U_1}(Y_2) - F_{U_2}(Y_2)\|_{\mathcal{V}'} \le C|U_1(t) - U_2(t)|\|Y_2\|_{\mathcal{H}} \tag{4.2}$$

and

$$\|G_{U_1}(t) - G_{U_2}(t)\|_{\mathcal{V}'} \le C\Big(|U_1(t) - U_2(t)| + \Big|\frac{dU_1(t)}{dt} - \frac{dU_2(t)}{dt}\Big|\Big). \tag{4.3}$$

Taking the scalar product with $\widetilde{Y}$ to (4.1) and using (f.ii), (4.2), (4.3), we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\widetilde{Y}(t)\|^2_{\mathcal{H}} + \frac{\delta}{2}\|\widetilde{Y}(t)\|^2_{\mathcal{V}} \\ &\quad \le (\|Y_1(t)\|^2_{\mathcal{V}} + \|Y_2(t)\|^2_{\mathcal{V}} + 1)\tilde{\mu}(\|Y_1(t)\|^2_{\mathcal{H}} + \|Y_2(t)\|^2_{\mathcal{H}})\|\widetilde{Y}(t)\|^2_{\mathcal{H}} \\ &\qquad + C(\|Y_2(t)\|^2_{\mathcal{H}} + 1)\Big(|U_1(t) - U_2(t)|^2 + \Big|\frac{dU_1(t)}{dt} - \frac{dU_2(t)}{dt}\Big|^2\Big),\end{aligned}$$

where $\tilde{\mu} : [0, \infty) \to [0, \infty)$ are some increasing continuous function.

Using Gronwall's inequality, we obtain that

$$\begin{aligned}\|\widetilde{Y}(t)\|^2_{\mathcal{H}} &\le C(\|Y_2(t)\|^2_{L^\infty(0,S;\mathcal{H})} + 1)\|U_1(t) - U_2(t)\|^2_{(H^1(0,S))^2} \\ &\le C\|U_1(t) - U_2(t)\|^2_{(H^1(0,S))^2}\end{aligned}$$

for all $t \in [0, S]$. □

Lemma 4.2. *Let P_1 and P_2 be the corresponding adjoint equation* (3.14) *to U_1 and U_2, respectively. Then, we have*

$$\|P_1(t) - P_2(t)\|^2_{L^2(0,S;\mathcal{V})} \le C\|U_1(t) - U_2(t)\|^2_{(H^1(0,S))^2}.$$

Proof. It can be easily verified that $\widetilde{P} = P_1 - P_2$ satisfies

$$\begin{aligned}&-\frac{d\widetilde{P}}{dt} + A\widetilde{P} - (F'_{U_1}(Y_1)^* - F'_{U_2}(Y_1)^*)P_1 \\ &- F'_{U_2}(Y_1)^*\widetilde{P} - (F'_{U_2}(Y_1)^* - F'_{U_2}(Y_2)^*)P_2 = D(Y_1 - Y_2), \quad 0 \le t < S, \\ &\widetilde{P}(S) = 0.\end{aligned} \tag{4.4}$$

Taking the scalar product to (4.4) with $\widetilde{P}$ and using (3.3) and (3.4), and (3.5) we obtain

$$-\frac{1}{2}\frac{d}{dt}\|\widetilde{P}(t)\|_{\mathcal{H}}^2+\delta\|\widetilde{P}(t)\|_{\mathcal{V}}^2\le C(\|Y_1\|_{\mathcal{V}}^2+1)\|\widetilde{P}\|_{\mathcal{H}}^2 \\ +C\Big(|U_1(t)-U_2(t)|^2\|P_1\|_{\mathcal{H}}^2+(\|P_2\|_{\mathcal{V}}^2+1)\|Y_1-Y_2\|_{\mathcal{H}}^2\Big). \tag{4.5}$$

Integrating from t and S, and using Gronwall's inequality and Lemma 4.1, we have

$$\begin{aligned}\|\widetilde{P}(t)\|_{\mathcal{H}}^2 \le& C\Big(\|U_1(t)-U_2(t)\|_{(L^2(0,S))^2}^2\|P_1\|_{L^\infty(0,S;\mathcal{H})}^2 \\ &+(\|P_2\|_{L^2(0,S;\mathcal{V})}^2+1)\|Y_1-Y_2\|_{L^\infty(0,S;\mathcal{H})}^2\Big) \\ \le& C\|U_1(t)-U_2(t)\|_{(H^1(0,S))^2}^2\end{aligned}$$

for all $t\in[0,S]$. Using this result in (4.5), we obtain the desired result. □

Theorem 4.3. *If γ is large enough, then there exists a unique optimal control to $(\overline{\mathbf{P}})$.*

Proof. From Theorem 3.4, we have

$$\int_0^S\langle P_1,B_{U_2-U_1}(Y_1)+G_{U_2-U_1}(t)\rangle_{\mathcal{V},\mathcal{V}'}dt+\gamma\langle U_1,U_2-U_1\rangle_{(H^1(0,S))^2}\ge 0, \tag{4.6}$$

$$\int_0^S\langle P_2,B_{U_1-U_2}(Y_2)+G_{U_1-U_2}(t)\rangle_{\mathcal{V},\mathcal{V}'}dt+\gamma\langle U_2,U_1-U_2\rangle_{(H^1(0,S))^2}\ge 0, \tag{4.7}$$

where P_1, Y_1 are the solution with respect to U_1 and P_2, Y_2 are the solution with respect to U_2.

By adding (4.6) and (4.7), we have

$$\begin{aligned}\gamma\|U_1-U_2\|_{(H^1(0,S))^2}^2\le&\int_0^S\langle P_1-P_2,B_{U_2-U_1}(Y_1)\rangle_{\mathcal{V},\mathcal{V}'}dt \\ &+\int_0^S\langle P_2,B_{U_1-U_2}(Y_2-Y_1)\rangle_{\mathcal{V},\mathcal{V}'}dt \\ &+\int_0^S\langle P_2-P_1,G_{U_1-U_2}(t)\rangle_{\mathcal{V},\mathcal{V}'}dt.\end{aligned}$$

Since

$$\begin{aligned}\|B_{U_1-U_2}(Y_1)\|_{\mathcal{V}'}\le&\Big\|b\frac{\partial}{\partial x}\Big(y_1\frac{\partial\phi_{(U_1-U_2)}}{\partial x}\Big)\Big\|_{(H^1(0,L))'} \\ \le& C\|y_1\|_{L^2(0.L)}\Big\|\frac{\partial\phi_{(U_1-U_2)}}{\partial x}\Big\|_{L^\infty(0,L)}\le C|U_1(t)-U_2(t)|\|Y_1\|_{\mathcal{H}}\end{aligned}$$

and

$$\|G_{U_1-U_2}(t)\|_{\mathcal{V}'}\le C\Big(|U_1(t)-U_2(t)|+\Big|\frac{dU_1(t)}{dt}-\frac{dU_2(t)}{dt}\Big|\Big)$$

we have

$$\begin{aligned}\gamma\|U_1 - U_2\|^2_{(H^1(0,S))^2} \leq & C\Big(\|P_1 - P_2\|_{L^2(0,S;\mathcal{V})}\|U_1 - U_2\|_{(L^2(0,S))^2}\|Y_1\|_{L^\infty(0,S;\mathcal{H})} \\ & + \|P_2\|_{L^2(0,S;\mathcal{V})}\|U_1 - U_1\|_{(L^2(0,S))^2}\|Y_1 - Y_2\|_{L^\infty(0,S;\mathcal{H})} \\ & + \|P_1 - P_2\|_{L^2(0,S;\mathcal{V})}\|U_1 - U_2\|_{(H^1(0,S))^2}\Big).\end{aligned}$$

By using Lemma 4.1 and Lemma 4.2, we have

$$\gamma\|U_1 - U_2\|^2_{((H^1(0,S))^2} \leq C\|U_1 - U_2\|^2_{(H^1(0,S))^2}.$$

If γ is sufficiently large, we obtain the uniqueness of the optimal control. □

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Sang-Uk Ryu
Department of Mathematics, Jeju National University, Jeju 690-756, Korea
E-mail address: ryusu81@jejunu.ac.kr