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FOURIER-FEYNMAN TRANSFORM AND CONVOLUTION OF FOURIER-TYPE FUNCTIONALS ON WIENER SPACE

Byoung Soo Kim

Abstract. We develop a Fourier-Feynman theory for Fourier-type functionals $\Delta^k F$ and $\widehat{\Delta^k F}$ on Wiener space. We show that Fourier-Feynman transform and convolution of Fourier-type functionals exist. We also show that the Fourier-Feynman transform of the convolution product of Fourier-type functionals is a product of Fourier-Feynman transforms of each functionals.

1. Introduction and preliminaries

Let $C_0[0,T]$ denote the Wiener space, that is, the space of real valued continuous functions x on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote Wiener measure. $(C_0[0,T], \mathcal{M}, m)$ is a complete measure space and we denote the Wiener integral of a functional F by

$$\int_{C_0[0,T]} F(x)\, dm(x).$$

A subset E of $C_0[0,T]$ is said to be scale-invariant measurable provided ρE is Wiener measurable for every $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for every $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). Given two complex-valued functions F and G on $C_0[0,T]$, we say that $F = G$ s-a.e. if $F(\rho x) = G(\rho x)$ for m almost every $x \in C_0[0,T]$ for all $\rho > 0$.

Let $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ denote the sets of complex numbers with positive real part and nonzero complex numbers with nonnegative real part, respectively.

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Let F be a complex valued scale-invariant measurable functional on $C_0[0,T]$ such that

$$J(\lambda)=\int_{C_0[0,T]} F(\lambda^{-1/2}x)\,dm(x)$$

exists as a finite number for all real $\lambda>0$. If there exists an analytic function $J^*(\lambda)$ on $\mathbb{C}_+$ such that $J^*(\lambda)=J(\lambda)$ for all $\lambda>0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0,T]$ with parameter λ, and for $\lambda\in\mathbb{C}_+$ we write

$$\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)\,dm(x)=J^*(\lambda).$$

Let F be a functional on $C_0[0,T]$ such that $\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)\,dm(x)$ exists for all $\lambda\in\mathbb{C}_+$. If the following limit exists for nonzero real number q, then we call it the analytic Feynman integral of F over $C_0[0,T]$ with parameter q and we write

$$\int_{C_0[0,T]}^{\mathrm{anf}_q} F(x)\,dm(x)=\lim_{\lambda\to -iq}\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)\,dm(x)$$

where $\lambda\to -iq$ through $\mathbb{C}_+$.

Now we introduce the definitions of analytic Fourier-Feynman transform and convolution product for functionals defined on $C_0[0,T]$. Let $1\le p<\infty$ and let q be a nonzero real number.

Definition 1. Let F be a functional on $C_0[0,T]$. For $\lambda\in\mathbb{C}_+$ and $y\in C_0[0,T]$, let

$$T_\lambda(F)(y)=\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x+y)\,dm(x). \tag{1}$$

For $1<p<\infty$, we define the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ of F on $C_0[0,T]$ by the formula ($\lambda\in\mathbb{C}_+$)

$$T_q^{(p)}(F)(y)=\underset{\lambda\to -iq}{\mathrm{l.i.m.}}\, T_\lambda(F)(y), \tag{2}$$

whenever this limit exists; that is, for each $\rho>0$,

$$\lim_{\lambda\to -iq}\int_{C_0[0,T]} |T_\lambda(F)(\rho x)-T_q^{(p)}(F)(\rho x)|^{p'}\,dm(x)=0$$

where $1/p+1/p'=1$. We define the L_1 analytic Fourier-Feynman transform $T_q^{(1)}(F)$ of F by ($\lambda\in\mathbb{C}_+$)

$$T_q^{(1)}(F)(y)=\lim_{\lambda\to -iq} T_\lambda(F)(y), \tag{3}$$

for s-a.e. $y\in C_0[0,T]$, whenever this limit exists [2, 7, 8, 9].

By the definition of the analytic Feynman integral and the L_1 analytic Fourier-Feynman transform, it is easy to see that for a nonzero real number q,

$$T_q^{(1)}(F)(y)=\int_{C_0[0,T]}^{\mathrm{anf}_q} F(x+y)\,dm(x) \tag{4}$$

and

$$T_q^{(1)}(F)(0) = \int_{C_0[0,T]}^{\mathrm{anf}_q} F(x)\, dm(x). \tag{5}$$

Definition 2. Let F and G be functionals on $C_0[0,T]$. For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0,T]$, we define their convolution product by

$$(F * G)_\lambda(y) = \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F\Big(\frac{y+x}{\sqrt{2}}\Big) G\Big(\frac{y-x}{\sqrt{2}}\Big)\, dm(x). \tag{6}$$

Moreover if $\lambda = -iq$ for nonzero real q, the convolution product is defined by

$$(F * G)_q(y) = \int_{C_0[0,T]}^{\mathrm{anf}_q} F\Big(\frac{y+x}{\sqrt{2}}\Big) G\Big(\frac{y-x}{\sqrt{2}}\Big)\, dm(x) \tag{7}$$

if it exists [7, 8, 14, 15].

It is easy to see that commutative law holds for the convolution product [7].

Various results involving Fourier-Feynman transform on Wiener space have been established and research based on this definition is continuing at the present time [1, 3, 4, 5, 12]. Recently, Kim, Kim and Yang extended the concepts of Fourier-Feynman transform and convolution on Wiener space to the concept of Fourier-Yeh-Feynman transform and convolution on Yeh-Wiener space [10, 11]. For a detailed survey of the previous work on the Fourier-Feynman transfprm and related topics, see [13].

Now we describe the class of functionals that we work with in this paper. Recently, Chung and Tuan [6] introduced the Fourier-type functionals via the Fourier transform on Wiener space and investigate some properties of the Fourier-type functionals.

Let $\hat{f}$ be the Fourier transform of f,

$$\hat{f}(\vec{\xi}) = \Big(\frac{1}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\{i\vec{u}\cdot\vec{\xi}\}\, d\vec{u}, \quad \vec{\xi} \in \mathbb{R}^n, \tag{8}$$

where $\vec{u}\cdot\vec{\xi} = u_1\xi_1 + \cdots + u_n\xi_n$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of infinitely differentiable functions $f(\vec{u})$ decaying at infinity together with all its derivatives faster than any polynomial of $|\vec{u}|^{-1}$. Note that the Fourier transform is an isomorphism on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Also, $\Delta^k f$ and $\widehat{\Delta^k f}$ are elements of $\mathcal{S}(\mathbb{R}^n)$ for all $k = 0, 1, 2, \ldots$, where Δ denotes the Laplacian

$$\Delta = \frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \cdots + \frac{\partial^2}{\partial u_n^2}.$$

Note that $\mathcal{S}(\mathbb{R}^n)$ is a subset of $L^1(\mathbb{R}^n)$, more precisely, for $\phi \in \mathcal{S}(\mathbb{R}^n)$ we know that

$$\begin{aligned}\|\phi\|_1 =& \int_{\mathbb{R}^n} (1+|\vec{u}|^2)^{-1}(1+|\vec{u}|^2)|\phi(\vec{u})|\, d\vec{u} \\ \le& \pi^n[\|\phi\|_\infty + \||\cdot|^2\phi(\cdot)\|_\infty] < \infty.\end{aligned}$$

Now we introduce the Fourier-type functionals defined on Wiener space.

Definition 3. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an orthonormal set of functions in $L_2[0,T]$ and let k be a nonnegative integer. For $f \in \mathcal{S}(\mathbb{R}^n)$, the Fourier-type functionals $\Delta^k F$ and $\widehat{\Delta^k F}$ on Wiener space $C_0[0,T]$ are defined by

$$\Delta^k F(x) = \Delta^k f(\langle \vec{\alpha}, x\rangle) \tag{9}$$

and

$$\widehat{\Delta^k F}(x) = \widehat{\Delta^k f}(\langle \vec{\alpha}, x\rangle) \tag{10}$$

where $\langle \vec{\alpha}, x\rangle = (\langle \alpha_1, x\rangle, \ldots, \langle \alpha_n, x\rangle)$ and $\langle \alpha_j, x\rangle$ denotes the Paley-Wiener-Zigmund stochastic integral for $j = 1, 2, \ldots, n$.

In this paper, we develop a Fourier-Feynman theory for Fourier-type functionals $\Delta^k F$ and $\widehat{\Delta^k F}$ on Wiener space. In Section 2, we show that Fourier-Feynman transform of Fourier-type functionals exists. In Section 3, we show that convolution of Fourier-type functionals exists. We also show that the Fourier-Feynman transform of the convolution product of Fourier-type functionals is a product of Fourier-Feynman transforms of each functionals.

We close this section by introducing a well-known Wiener integration formula for functionals $f(\langle \vec{\alpha}, x\rangle) = f(\langle \alpha_1, x\rangle, \ldots, \langle \alpha_n, x\rangle)$;

$$\int_{C_0[0,T]} f(\langle \vec{\alpha}, x\rangle)\, dm(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\Big\{-\frac{1}{2}|\vec{u}|^2\Big\}\, d\vec{u}, \tag{11}$$

where $|\vec{u}|^2 = \sum_{j=1}^n u_j^2$.

2. Fourier-Feynman transform of Fourier-type functionals

In this section we show that the L_p analytic Fourier-Feynman transforms $T_q^{(p)}(\Delta^k F)$ and $T_q^{(p)}(\widehat{\Delta^k F})$ of the Fourier-type functionals exist. We also establish a relationship between $T_q^{(p)}(\Delta^k F)$ and $T_q^{(p)}(\widehat{\Delta^k F})$.

Theorem 2.1. *Let the Fourier-type functional $\Delta^k F$ be given by (9). Then for all p with $1 \le p < \infty$ and for all nonzero real number q, the L_p analytic Fourier-Feynman transform $T_q^{(p)}(\Delta^k F)$ exists and is given by the formula*

$$T_q^{(p)}(\Delta^k F)(y) = \Big(\frac{-iq}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{\frac{iq}{2}|\vec{u} - \langle \vec{\alpha}, y\rangle|^2\Big\}\, d\vec{u} \tag{12}$$

for s-a.e. $y \in C_0[0,T]$.

Proof. Using the Wiener integration formula (11), we obtain

$$\begin{aligned} T_\lambda(\Delta^k F)(y) &= \int_{C_0[0,T]} \Delta^k f(\lambda^{-1/2}\langle \vec{\alpha}, x\rangle + \langle \vec{\alpha}, y\rangle)\, dm(x) \\ &= \Big(\frac{\lambda}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Delta^k f(\vec{u} + \langle \vec{\alpha}, y\rangle) \exp\Big\{-\frac{\lambda}{2}|\vec{u}|^2\Big\}\, d\vec{u} \\ &= \Big(\frac{\lambda}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{-\frac{\lambda}{2}|\vec{u} - \langle \vec{\alpha}, y\rangle|^2\Big\}\, d\vec{u} \end{aligned}$$

for all $\lambda > 0$ and $y \in C_0[0,T]$. Let $\lambda \in \mathbb{C}_+^\sim$ and let $\{\lambda_n\}$ be a sequence in $\mathbb{C}_+^\sim$ which converges to λ. Since $\Delta^k f$ is an element of $\mathcal{S}(\mathbb{R}^n)$ and since $\mathcal{S}(\mathbb{R}^n)$ is a subset of $L^1(\mathbb{R}^n)$, we can apply dominated convergence theorem to show that

$$\lim_{n\to\infty} T_{\lambda_n}(\Delta^k F)(y) = T_\lambda(\Delta^k F)(y)$$

and so $T_\lambda(\Delta^k F)(y)$ is a continuous function of λ in $\mathbb{C}_+^\sim$. Let D be a closed contour in $\mathbb{C}_+$. By the Fubini theorem and the Cauchy theorem,

$$\int_D \int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{-\frac{\lambda}{2}|\vec{u} - \langle \vec{\alpha}, y\rangle|^2\Big\}\, d\vec{u}\, d\lambda = 0.$$

Hence by the Morera's theorem, $\int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\{-\frac{\lambda}{2}|\vec{u} - \langle \vec{\alpha}, y\rangle|^2\}\, d\vec{u}$ is an analytic function of λ in $\mathbb{C}_+$. Hence for $\lambda \in \mathbb{C}_+$ and $y \in C_0[0,T]$,

$$T_\lambda(\Delta^k F)(y) = \Big(\frac{\lambda}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{-\frac{\lambda}{2}|\vec{u} - \langle \vec{\alpha}, y\rangle|^2\Big\}\, d\vec{u}.$$

In case $p = 1$, by the dominated convergence theorem,

$$\begin{aligned} T_q^{(1)}(\Delta^k F)(y) &= \lim_{\lambda\to -iq} T_\lambda(\Delta^k F)(y) \\ &= \Big(\frac{-iq}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{\frac{iq}{2}|\vec{u} - \langle \vec{\alpha}, y\rangle|^2\Big\}\, d\vec{u} \end{aligned}$$

for $y \in C_0[0,T]$. If $1 < p < \infty$, again by the dominated convergence theorem, the Wiener integral

$$\int_{C_0[0,T]} \Big|\Big(\frac{-iq}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{\frac{iq}{2}|\vec{u} - \langle \vec{\alpha}, \rho y\rangle|^2\Big\}\, d\vec{u} - T_\lambda(\Delta^k F)(\rho y)\Big|^{p'}\, dm(y)$$

goes to 0 as $\lambda \to -iq$ for each $\rho > 0$. Hence $T_q^{(p)}(\Delta^k F)(y)$ exists and is given by (12) for s-a.e. $y \in C_0[0,T]$ and for all desired values of p and q. □

As we have seen in (5), the L_1 analytic Fourier-Feynman transform of F evaluated at 0 is equal to the analytic Feynman integral of F. Hence we have the following corollary.

Corollary 2.2. *Let the Fourier-type functional $\Delta^k F$ be given by (9). Then $\Delta^k F$ is analytic Wiener integrable and*

$$\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} \Delta^k F(x)\, dm(x) = \Big(\frac{\lambda}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{-\frac{\lambda}{2}|\vec{u}|^2\Big\}\, d\vec{u} \qquad (13)$$

for all $\lambda \in \mathbb{C}_+$. Moreover, $\Delta^k F$ is analytic Feynman integrable and

$$\int_{C_0[0,T]}^{\mathrm{anf}_q} \Delta^k F(x)\, dm(x) = \Big(\frac{-iq}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{\frac{iq}{2}|\vec{u}|^2\Big\}\, d\vec{u} \qquad (14)$$

for all nonzero real number q.

In the following theorem we evaluate Fourier-Feynman transform of $\widehat{\Delta^k F}$.

Theorem 2.3. *Let the Fourier-type functional* $\widehat{\Delta^k F}$ *be given by* (10). *Then for all* p *with* $1 \le p < \infty$ *and for all nonzero real number* q, *the* L_p *analytic Fourier-Feynman transform* $T_q^{(p)}(\widehat{\Delta^k F})$ *exists and is given by the formula*

$$T_q^{(p)}(\widehat{\Delta^k F})(y) = \Big(\frac{1}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{-\frac{i}{2q}|\vec{u}|^2 + i\langle\vec{\alpha}, y\rangle \cdot \vec{u}\Big\}\, d\vec{u} \qquad (15)$$

for s-a.e. $y \in C_0[0,T]$.

Proof. Using the Wiener integration formula (11) we obtain

$$\begin{aligned} T_\lambda(\widehat{\Delta^k F})(y) &= \int_{C_0[0,T]} \widehat{\Delta^k f}(\lambda^{-1/2}\langle\vec{\alpha}, x\rangle + \langle\vec{\alpha}, y\rangle)\, dm(x) \\ &= \Big(\frac{\lambda}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \widehat{\Delta^k f}(\vec{v} + \langle\vec{\alpha}, y\rangle) \exp\Big\{-\frac{\lambda}{2}|\vec{v}|^2\Big\}\, d\vec{v} \\ &= \Big(\frac{\lambda}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \widehat{\Delta^k f}(\vec{v}) \exp\Big\{-\frac{\lambda}{2}|\vec{v} - \langle\vec{\alpha}, y\rangle|^2\Big\}\, d\vec{v} \end{aligned}$$

for all $\lambda > 0$ and $y \in C_0[0,T]$. By (8) we have

$$\begin{aligned} &T_\lambda(\widehat{\Delta^k F})(y) \\ &= \Big(\frac{\lambda}{2\pi}\Big)^{n/2}\Big(\frac{1}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{-\frac{\lambda}{2}|\vec{v} - \langle\vec{\alpha}, y\rangle|^2 + i\vec{u}\cdot\vec{v}\Big\}\, d\vec{u}\, d\vec{v}. \end{aligned}$$

But since

$$\begin{aligned} &\int_{\mathbb{R}^n} \exp\Big\{-\frac{\lambda}{2}|\vec{v} - \langle\vec{\alpha}, y\rangle|^2 + i\vec{u}\cdot\vec{v}\Big\}\, d\vec{v} \\ &= \int_{\mathbb{R}^n} \exp\Big\{-\frac{\lambda}{2}\Big|\vec{v} - \Big[\langle\vec{\alpha}, y\rangle + \frac{i}{\lambda}\vec{u}\Big]\Big|^2 + i\langle\vec{\alpha}, y\rangle\cdot\vec{u} - \frac{1}{2\lambda}|\vec{u}|^2\Big\}\, d\vec{v} \\ &= \Big(\frac{2\pi}{\lambda}\Big)^{n/2} \exp\Big\{-\frac{1}{2\lambda}|\vec{u}|^2 + i\langle\vec{\alpha}, y\rangle\cdot\vec{u}\Big\}, \end{aligned}$$

we have

$$T_\lambda(\widehat{\Delta^k F})(y) = \Big(\frac{1}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{-\frac{1}{2\lambda}|\vec{u}|^2 + i\langle\vec{\alpha}, y\rangle\cdot\vec{u}\Big\}\, d\vec{u}$$

for all $\lambda > 0$ and $y \in C_0[0,T]$. Let $\lambda \in \mathbb{C}_+^\sim$ and let $\{\lambda_n\}$ be a sequence in $\mathbb{C}_+^\sim$ which converges to λ. Since $\Delta^k f$ is integrable on $\mathbb{R}^n$, we can apply dominated convergence theorem to show that

$$\lim_{n\to\infty} T_{\lambda_n}(\widehat{\Delta^k F})(y) = T_\lambda(\widehat{\Delta^k F})(y)$$

and so $T_\lambda(\widehat{\Delta^k F})(y)$ is a continuous function of λ in $\mathbb{C}_+^\sim$. Let D be a closed contour in $\mathbb{C}_+$. By the Fubini theorem and the Cauchy theorem,

$$\int_D\int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{-\frac{1}{2\lambda}|\vec{u}|^2 + i\langle\vec{\alpha}, y\rangle\cdot\vec{u}\Big\}\, d\vec{u}\, d\lambda = 0.$$

Hence by the Morera's theorem, $\int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\{-\frac{1}{2\lambda}|\vec{u}|^2 + i\langle\vec{\alpha}, y\rangle \cdot \vec{u}\}\, d\vec{u}$ is an analytic function of λ in $\mathbb{C}_+$. Hence for $\lambda \in \mathbb{C}_+$ and $y \in C_0[0,T]$,

$$T_\lambda(\widehat{\Delta^k F})(y) = \Big(\frac{1}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{-\frac{1}{2\lambda}|\vec{u}|^2 + i\langle\vec{\alpha}, y\rangle \cdot \vec{u}\Big\}\, d\vec{u}.$$

In case $p = 1$, by the dominated convergence theorem,

$$\begin{aligned} T_q^{(1)}(\widehat{\Delta^k F})(y) &= \lim_{\lambda \to -iq} T_\lambda(\widehat{\Delta^k F})(y) \\ &= \Big(\frac{1}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{-\frac{i}{2q}|\vec{u}|^2 + i\langle\vec{\alpha}, y\rangle \cdot \vec{u}\Big\}\, d\vec{u} \end{aligned}$$

for $y \in C_0[0,T]$. If $1 < p < \infty$, again by the dominated convergence theorem, the Wiener integral

$$\begin{aligned} \int_{C_0[0,T]} \Big| \Big(\frac{1}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{-\frac{i}{2q}|\vec{u}|^2 + i\langle\vec{\alpha}, \rho y\rangle \cdot \vec{u}|^2\Big\}\, d\vec{u} \\ - T_\lambda(\widehat{\Delta^k F})(\rho y)\Big|^{p'}\, dm(y) \end{aligned}$$

goes to 0 as $\lambda \to -iq$ for each $\rho > 0$. Hence $T_q^{(p)}(\widehat{\Delta^k F})(y)$ exists and is given by (15) for s-a.e. $y \in C_0[0,T]$ and for all desired values of p and q. □

The following corollary is a parallel result of Corollary 2.2 for the Fourier-type functional $\widehat{\Delta^k F}$. It can be obtained by (15) and (5).

Corollary 2.4. *Let the Fourier-type functional $\widehat{\Delta^k F}$ be given by* (10). *Then $\widehat{\Delta^k F}$ is analytic Wiener integrable and*

$$\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} \widehat{\Delta^k F}(x)\, dm(x) = \Big(\frac{1}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Delta^k f(\vec{v}) \exp\Big\{-\frac{1}{2\lambda}|\vec{v}|^2\Big\}\, d\vec{v} \tag{16}$$

for all $\lambda \in \mathbb{C}_+$. Moreover, $\widehat{\Delta^k F}$ is analytic Feynman integrable and

$$\int_{C_0[0,T]}^{\mathrm{anf}_q} \widehat{\Delta^k F}(x)\, dm(x) = \Big(\frac{1}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Delta^k f(\vec{u}) \exp\Big\{-\frac{i}{2q}|\vec{u}|^2\Big\}\, d\vec{u} \tag{17}$$

for any nonzero real number q.

Comparing equations (12) and (15), we obtain a relationship between the Fourier-Feynman transforms of $\Delta^k F$ and $\widehat{\Delta^k F}$ as in the following theorem.

Theorem 2.5. *Let the Fourier-type functionals $\Delta^k F$ and $\widehat{\Delta^k F}$ be given by* (9) *and* (10), *respectively. Let $1 \le p < \infty$ and let q be any nonzero real number. Then we have*

$$T_q^{(p)}(\widehat{\Delta^k F})(y) = (-iq)^{n/2} T_{-1/q}^{(p)}(\Delta^k F)(qy) \exp\Big\{\frac{iq}{2}|\langle\vec{\alpha}, y\rangle|^2\Big\} \tag{18}$$

for s-a.e. $y \in C_0[0,T]$.

Proof. Replacing q with $-1/q$ and y with qy in (12), we have

$$\begin{aligned}&T^{(p)}_{-1/q}(\Delta^k F)(qy)\\&=\Big(\frac{i}{2\pi q}\Big)^{n/2}\int_{\mathbb{R}^n}\Delta^k f(\vec{u})\exp\Big\{-\frac{i}{2q}|\vec{u}-q\langle\vec{\alpha},y\rangle|^2\Big\}\,d\vec{u}\\&=\Big(\frac{i}{2\pi q}\Big)^{n/2}\int_{\mathbb{R}^n}\Delta^k f(\vec{u})\exp\Big\{-\frac{i}{2q}|\vec{u}|^2+i\langle\vec{\alpha},y\rangle\cdot\vec{u}-\frac{iq}{2}|\langle\vec{\alpha},y\rangle|^2\Big\}\,d\vec{u}.\end{aligned}$$

Finally by (15) we have the desired result. □

If we take $y=0$ in (18) above, by (5) we have a relationship between the analytic Feynman integrals of $\Delta^k F$ and $\widehat{\Delta^k F}$ as follows.

Corollary 2.6. *Let the Fourier-type functionals* $\Delta^k F$ *and* $\widehat{\Delta^k F}$ *be given by* (9) *and* (10)*, respectively. Then we have*

$$\int^{\mathrm{anf}_q}_{C_0[0,T]}\widehat{\Delta^k F}(x)\,dm(x)=(-iq)^{n/2}\int^{\mathrm{anf}_{-1/q}}_{C_0[0,T]}\Delta^k F(x)\,dm(x) \tag{19}$$

for any nonzero real number q.

3. Convolution of Fourier-type functionals

In this section we will show the existence of the convolution product for the Fourier-type functionals on Wiener space. We also show that the Fourier-Feynman transform of the convolution product of Fourier-type functionals is a product of Fourier-Feynman transforms of each functionals.

Lemma 3.1. *Let* $f,g\in\mathcal{S}(\mathbb{R}^n)$ *and let* k *be a nonnegative integer. For* $\lambda\in\mathbb{C}_+^{\sim}$ *and* $\vec{w}\in\mathbb{R}^n$*, let*

$$h(\lambda,\vec{w})=\Big(\frac{\lambda}{2\pi}\Big)^{n/2}\int_{\mathbb{R}^n}\Delta^k f\Big(\frac{\vec{w}+\vec{u}}{\sqrt{2}}\Big)\,\Delta^k g\Big(\frac{\vec{w}-\vec{u}}{\sqrt{2}}\Big)\exp\Big\{-\frac{\lambda}{2}|\vec{u}|^2\Big\}\,d\vec{u}. \tag{20}$$

Then $h(\lambda,\langle\vec{\alpha},y\rangle)$ *exists for a.e.* $y\in C_0[0,T]$.

Proof. First note that

$$\int_{\mathbb{R}^n}|h(\lambda,\vec{w})|\,d\vec{w}\le\Big(\frac{|\lambda|}{2\pi}\Big)^{n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big|\Delta^k f\Big(\frac{\vec{w}+\vec{u}}{\sqrt{2}}\Big)\,\Delta^k g\Big(\frac{\vec{w}-\vec{u}}{\sqrt{2}}\Big)\Big|\,d\vec{u}\,d\vec{w}.$$

Letting $\vec{v}=(\vec{w}+\vec{u})/\sqrt{2}$ and $\vec{r}=(\vec{w}-\vec{u})/\sqrt{2}$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}|h(\lambda,\vec{w})|\,d\vec{w}&\le\Big(\frac{|\lambda|}{2\pi}\Big)^{n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\Delta^k f(\vec{v})||\Delta^k g(\vec{r})|\,d\vec{v}\,d\vec{r}\\&=\Big(\frac{|\lambda|}{2\pi}\Big)^{n/2}\|\Delta^k f\|_1\|\Delta^k g\|_1.\end{aligned}$$

But as we have seen in Section 1, $\Delta^k f$ and $\Delta^k g$ are elements of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is a subset of $L^1(\mathbb{R}^n)$. Hence the right hand side of the last inequality is a finite number and so $h(\lambda,\vec{w})$ exists for a.e. $\vec{w}\in\mathbb{R}^n$. Since the Paley-Wiener-Zigmund integral $\langle\alpha_j,y\rangle$, for $j=1,\dots,n$, exists for a.e. $y\in C_0[0,T]$, now it is easy to see that $h(\lambda,\langle\vec{\alpha},y\rangle)$ exists for a.e. $y\in C_0[0,T]$. □

Next we show that the convolution product of the Fourier-type functionals exists. Since we have two kinds of Fourier-type functionals, the existence theorem of the convolution product is divided into three cases. In Theorem 3.2 below, we consider the convolution product of Δ^kF and Δ^kG. While in Theorem 3.3, we consider the convolution product of Δ^kF and $\widehat{\Delta^kG}$. Finally in Theorem 3.4, we consider the convolution product of $\widehat{\Delta^kF}$ and $\widehat{\Delta^kG}$.

Theorem 3.2. *Let the Fourier-type functionals Δ^kF and Δ^kG are given by* (9) *with corresponding f and g, respectively. Then for all nonzero real number q, the convolution product $(\Delta^kF*\Delta^kG)_q$ exists and is given by the formula*

$$(\Delta^kF*\Delta^kG)_q(y)=h(-iq,\langle\vec{\alpha},y\rangle), \tag{21}$$

for a.e. $y\in C_0[0,T]$, where h is given by (20).

Proof. For all $\lambda>0$, by the Wiener integration formula (11), we have

$$\begin{aligned}&(\Delta^kF*\Delta^kG)_\lambda(y)\\&=\int_{C_0[0,T]}\Delta^kf\Big(\frac{\langle\vec{\alpha},y\rangle}{\sqrt{2}}+\frac{\langle\vec{\alpha},x\rangle}{\sqrt{2\lambda}}\Big)\,\Delta^kg\Big(\frac{\langle\vec{\alpha},y\rangle}{\sqrt{2}}-\frac{\langle\vec{\alpha},x\rangle}{\sqrt{2\lambda}}\Big)\,dm(x)\\&=\Big(\frac{\lambda}{2\pi}\Big)^{n/2}\int_{\mathbb{R}^n}\Delta^kf\Big(\frac{\langle\vec{\alpha},y\rangle+\vec{u}}{\sqrt{2}}\Big)\,\Delta^kg\Big(\frac{\langle\vec{\alpha},y\rangle-\vec{u}}{\sqrt{2}}\Big)\,\exp\Big\{-\frac{\lambda}{2}|\vec{u}|^2\Big\}\,d\vec{u}\\&=h(\lambda,\langle\vec{\alpha},y\rangle)\end{aligned}$$

where h is given by (20). Let $\lambda\in\mathbb{C}_+^\sim$ and let $\{\lambda_n\}$ be a sequence in $\mathbb{C}_+^\sim$ which converges to λ. Then

$$\begin{aligned}&\Big|\Delta^kf\Big(\frac{\langle\vec{\alpha},y\rangle+\vec{u}}{\sqrt{2}}\Big)\,\Delta^kg\Big(\frac{\langle\vec{\alpha},y\rangle-\vec{u}}{\sqrt{2}}\Big)\,\exp\Big\{-\frac{\lambda_n}{2}|\vec{u}|^2\Big\}\Big|\\&\le\Big|\Delta^kf\Big(\frac{\langle\vec{\alpha},y\rangle+\vec{u}}{\sqrt{2}}\Big)\,\Delta^kg\Big(\frac{\langle\vec{\alpha},y\rangle-\vec{u}}{\sqrt{2}}\Big)\,\exp\Big\{-\frac{c}{2}|\vec{u}|^2\Big\}\Big|,\end{aligned}$$

for all $n=1,2,\dots$, where $c=\min\{\mathrm{Re}(\lambda_n):n=1,2,\dots\}\ge0$. By the same argument as in the proof of Lemma 3.1, we know that the right hand side of the above inequality is an integrable function of $\vec{u}$ on $\mathbb{R}^n$ for a.e. $y\in C_0[0,T]$. Hence we can apply dominated convergence theorem to show that

$$\lim_{n\to\infty}h(\lambda_n,\langle\vec{\alpha},y\rangle)=h(\lambda,\langle\vec{\alpha},y\rangle),$$

and so $h(\lambda, \langle\vec{\alpha}, y\rangle)$ is a continuous function of λ in $\mathbb{C}_+^\sim$. Let D be a closed contour in $\mathbb{C}_+$. Then

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_D\Big|\Big(\frac{\lambda}{2\pi}\Big)^{n/2}\Delta^k f\Big(\frac{\langle\vec{\alpha}, y\rangle+\vec{u}}{\sqrt{2}}\Big)\,\Delta^k g\Big(\frac{\langle\vec{\alpha}, y\rangle-\vec{u}}{\sqrt{2}}\Big)\exp\Big\{-\frac{\lambda}{2}|\vec{u}|^2\Big\}\Big|\,d\lambda\,d\vec{u}\\ &\le\Big(\frac{a}{2\pi}\Big)^{n/2}\int_{\mathbb{R}^n}\int_D\Big|\Delta^k f\Big(\frac{\langle\vec{\alpha}, y\rangle+\vec{u}}{\sqrt{2}}\Big)\,\Delta^k g\Big(\frac{\langle\vec{\alpha}, y\rangle-\vec{u}}{\sqrt{2}}\Big)\Big|\,d\lambda\,d\vec{u}\\ &=\Big(\frac{a}{2\pi}\Big)^{n/2}l(D)\int_{\mathbb{R}^n}\Big|\Delta^k f\Big(\frac{\langle\vec{\alpha}, y\rangle+\vec{u}}{\sqrt{2}}\Big)\,\Delta^k g\Big(\frac{\langle\vec{\alpha}, y\rangle-\vec{u}}{\sqrt{2}}\Big)\Big|\,d\vec{u},\end{aligned}$$

where $a=\max\{|\lambda|:\lambda\in D\}$ and $l(D)$ denotes the length of the contour D. By the same argument as in the proof of Lemma 3.1, we know that the integral on the right hand side of the last inequality is finite for a.e. $y\in C_0[0,T]$. Hence we can apply Fubini theorem to show that

$$\begin{aligned}&\int_D h(\lambda, \langle\vec{\alpha}, y\rangle)\,d\lambda\\ &=\int_{\mathbb{R}^n}\int_D\Big(\frac{\lambda}{2\pi}\Big)^{n/2}\Delta^k f\Big(\frac{\langle\vec{\alpha}, y\rangle+\vec{u}}{\sqrt{2}}\Big)\,\Delta^k g\Big(\frac{\langle\vec{\alpha}, y\rangle-\vec{u}}{\sqrt{2}}\Big)\exp\Big\{-\frac{\lambda}{2}|\vec{u}|^2\Big\}\,d\lambda\,d\vec{u}.\end{aligned}$$

But since the integrand on the right hand side of the above equation is an analytic function of λ in $\mathbb{C}_+$, the inner integral is equal to 0. Hence by the Morera's theorem, $h(\lambda, \langle\vec{\alpha}, y\rangle)$ is an analytic function of λ in $\mathbb{C}_+$. Finally, we can apply dominated convergence theorem once more to show that

$$(\Delta^k F*\Delta^k G)_q(y)=\lim_{\lambda\to -iq}h(\lambda, \langle\vec{\alpha}, y\rangle)=h(-iq, \langle\vec{\alpha}, y\rangle)$$

for a.e. $y\in C_0[0,T]$ as we wished to prove. □

A careful look at the proofs of Lemma 3.1 and Theorem 3.2, we see that the essential conditions to ensure the results are the integrability of $\Delta^k f$ and $\Delta^k g$. Since $\widehat{\Delta^k f}$ and $\widehat{\Delta^k g}$ are also elements of $L^1(\mathbb{R}^n)$, we have the following theorems. We just state them without proofs.

Theorem 3.3. *Let the Fourier-type functionals* $\Delta^k F$ *and* $\widehat{\Delta^k G}$ *are given by* (9) *and* (10) *with corresponding* f *and* g *in* $\mathcal{S}(\mathbb{R}^n)$*, respectively. Then for all nonzero real number* q*, the convolution product* $(\Delta^k F*\widehat{\Delta^k G})_q$ *exists and is given by the formula*

$$\begin{aligned}&(\Delta^k F*\widehat{\Delta^k G})_q(y)\\ &=\Big(\frac{\lambda}{2\pi}\Big)^{n/2}\int_{\mathbb{R}^n}\Delta^k f\Big(\frac{\langle\vec{\alpha}, y\rangle+\vec{u}}{\sqrt{2}}\Big)\,\widehat{\Delta^k g}\Big(\frac{\langle\vec{\alpha}, y\rangle-\vec{u}}{\sqrt{2}}\Big)\exp\Big\{-\frac{\lambda}{2}|\vec{u}|^2\Big\}\,d\vec{u},\end{aligned}\tag{22}$$

for a.e. $y\in C_0[0,T]$.

As we noted in Section 1, the convolution product for the Fourier-Feynman transform is commutative. Hence we know that $(\widehat{\Delta^k F}*\Delta^k G)_q$ also exists.

Theorem 3.4. *Let the Fourier-type functionals $\widehat{\Delta^k F}$ and $\widehat{\Delta^k G}$ are given by (10) with corresponding f and g in $\mathcal{S}(\mathbb{R}^n)$, respectively. Then for all nonzero real number q, the convolution product $(\widehat{\Delta^k F} * \widehat{\Delta^k G})_q$ exists and is given by the formula*

$$(\widehat{\Delta^k F} * \widehat{\Delta^k G})_q(y) = \Big(\frac{\lambda}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \widehat{\Delta^k f}\Big(\frac{\langle \vec{\alpha}, y\rangle + \vec{u}}{\sqrt{2}}\Big)\, \widehat{\Delta^k g}\Big(\frac{\langle \vec{\alpha}, y\rangle - \vec{u}}{\sqrt{2}}\Big) \exp\Big\{-\frac{\lambda}{2}|\vec{u}|^2\Big\}\, d\vec{u}, \tag{23}$$

for a.e. $y \in C_0[0,T]$.

Although the right hand sides of (22) and (23) can be expressed further using the equations (8) for the Fourier transform $\widehat{\Delta^k F}$ and $\widehat{\Delta^k G}$, we will not give them here.

In our next theorem we show that the Fourier Feynman transform of convolution product of Fourier-type functionals $\Delta^k F$ and $\Delta^k G$ is the product of transforms of each functionals.

Theorem 3.5. *Let the Fourier-type functionals $\Delta^k F$ and $\Delta^k G$ are given by (9) with corresponding f and g in $\mathcal{S}(\mathbb{R}^n)$, respectively. Let $1 \le p < \infty$ and let q be a nonzero real number. Then*

$$T_q^{(p)}((\Delta^k F * \Delta^k G)_q)(y) = T_q^{(p)}(\Delta^k F)\Big(\frac{y}{\sqrt{2}}\Big)\, T_q^{(p)}(\Delta^k G)\Big(\frac{y}{\sqrt{2}}\Big), \tag{24}$$

for a.e. $y \in C_0[0,T]$.

Proof. For all $\lambda > 0$, by the expression (21) for convolution product of $\Delta^k F$ and $\Delta^k G$, we have

$$\begin{aligned} T_\lambda((\Delta^k F * \Delta^k G)_q)(y) &= \int_{C_0[0,T]} (\Delta^k F * \Delta^k G)_q(\lambda^{-1/2}x + y)\, dm(x) \\ &= \int_{C_0[0,T]} h(-iq, \lambda^{-1/2}\langle \vec{\alpha}, x\rangle + \langle \vec{\alpha}, y\rangle)\, dm(x). \end{aligned}$$

By the Wiener integration formula (11) and the expression (20) for h, we have

$$\begin{aligned} &T_\lambda((\Delta^k F * \Delta^k G)_q)(y) \\ &= \Big(\frac{\lambda}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} h(-iq, \vec{v} + \langle \vec{\alpha}, y\rangle) \exp\Big\{-\frac{\lambda}{2}|\vec{v}|^2\Big\}\, d\vec{v} \\ &= \Big(\frac{\lambda}{2\pi}\Big)^{n/2}\Big(\frac{-iq}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Delta^k f\Big(\frac{\vec{v} + \vec{u} + \langle \vec{\alpha}, y\rangle}{\sqrt{2}}\Big) \Delta^k g\Big(\frac{\vec{v} - \vec{u} + \langle \vec{\alpha}, y\rangle}{\sqrt{2}}\Big) \\ &\quad \exp\Big\{-\frac{\lambda}{2}(|\vec{v}|^2 + |\vec{u}|^2)\Big\}\, d\vec{u}\, d\vec{v}. \end{aligned}$$

Letting $\vec{w} = (\vec{v} + \vec{u})/\sqrt{2}$ and $\vec{r} = (\vec{v} - \vec{u})/\sqrt{2}$, we have

$$
\begin{aligned}
&T_\lambda((\Delta^k F * \Delta^k G)_q)(y) \\
&= \Big(\frac{\lambda}{2\pi}\Big)^{n/2}\Big(\frac{-iq}{2\pi}\Big)^{n/2}\Big(\int_{\mathbb{R}^n} \Delta^k f(\vec{w}) \exp\Big\{-\frac{\lambda}{2}\Big|\vec{w} - \frac{\langle\vec{\alpha}, y\rangle}{\sqrt{2}}\Big|^2\Big\}\, d\vec{w}\Big) \\
&\Big(\int_{\mathbb{R}^n} \Delta^k g(\vec{r}) \exp\Big\{-\frac{\lambda}{2}\Big|\vec{r} - \frac{\langle\vec{\alpha}, y\rangle}{\sqrt{2}}\Big|^2\Big\}\, d\vec{r}\Big).
\end{aligned}
$$

Now by the same argument as in the proof of Theorem 2.1, we can show that the last expression is analytic in $\lambda \in \mathbb{C}_+^\sim$ and we have

$$
\begin{aligned}
T_q^{(p)}((\Delta^k F * \Delta^k G)_q)(y) = \Big(\frac{-iq}{2\pi}\Big)^n &\Big(\int_{\mathbb{R}^n} \Delta^k f(\vec{w}) \exp\Big\{\frac{iq}{2}\Big|\vec{w} - \frac{\langle\vec{\alpha}, y\rangle}{\sqrt{2}}\Big|^2\Big\}\, d\vec{w}\Big) \\
&\Big(\int_{\mathbb{R}^n} \Delta^k g(\vec{r}) \exp\Big\{\frac{iq}{2}\Big|\vec{r} - \frac{\langle\vec{\alpha}, y\rangle}{\sqrt{2}}\Big|^2\Big\}\, d\vec{r}\Big).
\end{aligned}
$$

Finally by Theorem 2.1 we complete the proof. □

Considering Theorems 3.3, 3.4 and the proof of Theorem 3.5, we see that the relationship (24) holds for the Fourier-type functionals $\widehat{\Delta^k F}$ and $\widehat{\Delta^k G}$. We state the results in Theorems 3.6 and 3.7 below without proofs.

Theorem 3.6. *Let the Fourier-type functionals $\Delta^k F$ and $\widehat{\Delta^k G}$ are given by (9) and (10) with corresponding f and g in $\mathcal{S}(\mathbb{R}^n)$, respectively. Let $1 \le p < \infty$ and let q be a nonzero real number. Then*

$$
T_q^{(p)}((\Delta^k F * \widehat{\Delta^k G})_q)(y) = T_q^{(p)}(\Delta^k F)\Big(\frac{y}{\sqrt{2}}\Big)\, T_q^{(p)}(\widehat{\Delta^k G})\Big(\frac{y}{\sqrt{2}}\Big), \tag{25}
$$

for a.e. $y \in C_0[0, T]$.

Theorem 3.7. *Let the Fourier-type functionals $\widehat{\Delta^k F}$ and $\widehat{\Delta^k G}$ are given by (10) with corresponding f and g in $\mathcal{S}(\mathbb{R}^n)$, respectively. Let $1 \le p < \infty$ and let q be a nonzero real number. Then*

$$
T_q^{(p)}((\widehat{\Delta^k F} * \widehat{\Delta^k G})_q)(y) = T_q^{(p)}(\widehat{\Delta^k F})\Big(\frac{y}{\sqrt{2}}\Big)\, T_q^{(p)}(\widehat{\Delta^k G})\Big(\frac{y}{\sqrt{2}}\Big), \tag{26}
$$

for a.e. $y \in C_0[0, T]$.

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School of Liberal Arts, Seoul National University of Science and Technology, Seoul 139-743, Korea
E-mail address: mathkbs@seoultech.ac.kr