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ON A DIFFUSIVE PREDATOR-PREY MODEL WITH STAGE STRUCTURE ON PREY

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Abstract. In this paper, we consider a diffusive delayed predator-prey model with Beddington-DeAngelis type functional response under homogeneous Neumann boundary conditions, where the discrete time delay covers the period from the birth of immature preys to their maturity. We investigate the global existence of nonnegative solutions and the long-term behavior of the time-dependent solution of the model.

1. Introduction

Recently, the results have been obtained for a stage structured predator prey models with a time-delay term in the functional responses in a homogeneous environment. For example, Liu and Zhang [2] studied the following model, which is a stage structured predator-prey model of the Beddington-DeAngelis type functional response based on a consideration of time the prey took from birth to maturity:

$$(1.1)\quad \begin{cases} x_i'(t) = bx_m(t) - d_i x_i(t) - be^{-d_i\tau}x_m(t-\tau), \\ x_m'(t) = be^{-d_i\tau}x_m(t-\tau) - ax_m^2(t) - \dfrac{mx_m(t)y(t)}{1+k_1x_m(t)+k_2y(t)}, \\ y'(t) = \dfrac{nmx_m(t)y(t)}{1+k_1x_m(t)+k_2y(t)} - dy(t), \end{cases}$$

where $x_m(\theta) > 0$ is continuous on $-\tau \leq \theta \leq 0$ and $x_i(0)$, $x_m(0)$, $y(0) > 0$. Here x_m and y represent mature prey and predator densities, respectively, while x_i is the immature prey density. All constants are positive, and b means the birth rate of the mature prey; d_i is a mortality rate of immature prey; n is the birth rate of predator; m is the

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capture rate(1/time); k_1 is the handling time(1/prey); k_2 is a constant describing the magnitude of interference among predators(1/predator); $e^{-d_i\tau}$ is the surviving rate of each immature prey to become mature prey; a is the death rate of mature prey; d is the death rate of predator. In there, the permanence, extinction of species and the global stability of the equilibria were demonstrated.

In this paper, we extend the above model (1.1) as the following a diffusive predator-prey model of Beddington-DeAngelis type functional response with stage structure on prey:

$$(1.2)\quad \begin{cases} \dfrac{\partial u_i(x,t)}{\partial t} = bu(x,t) - d_i u_i(x,t) - be^{-d_i\tau}u(x,t-\tau), \\ \dfrac{\partial u(x,t)}{\partial t} = D_u\Delta u(x,t) + be^{-d_i\tau}u(x,t-\tau) - au^2(x,t) \\ \qquad\qquad -\dfrac{mu(x,t)v(x,t)}{1+k_1u(x,t)+k_2v(x,t)}, \\ \dfrac{\partial v(x,t)}{\partial t} = D_v\Delta v(x,t) + \dfrac{nmu(x,t)v(x,t)}{1+k_1u(x,t)+k_2v(x,t)} \\ \qquad\qquad -dv(x,t) \quad \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial u_i}{\partial \nu} = 0 \quad \text{on } \partial\Omega\times(0,\infty), \\ u(x,\theta) = u_\theta(x) \geq 0,\ v(x,\theta) = v_\theta(x) \geq 0 \quad \text{in } \Omega\times[-\tau,0], \\ u(x,0) \not\equiv 0,\ v(x,0) \not\equiv 0 \quad \text{in } \Omega, \\ u_i(x,0) = b\displaystyle\int_{-\tau}^{0} e^{d_i s}u(x,s)ds, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$; u_i and u represent the densities of immature and mature prey, respectively; and v denotes the density of mature predator. D_u and D_v are the diffusion coefficients of mature prey and mature predator, respectively; $\partial/\partial\nu$ denotes the outward normal derivative on the boundary $\partial\Omega$ of Ω.

We assume that the predator consumes the mature prey with the functional response of Beddington-DeAngelis type. Furthermore, the following is assumed due to the continuity of solutions of the model,

$$(1.3)\qquad u_i(x,0) = b\int_{-\tau}^{0} e^{d_i s}u(x,s)ds.$$

From the first equation in model (1.2) with the initial condition (1.3), we have

$$u_i(x,t) = b\int_{-\tau}^{0} e^{d_i s} u(x,t+s)ds.$$

Thus, $u_i(x,t)$ is completely determined by $u(x,t)$. For the convenience, we denote

$$r = be^{-d_i\tau},\ \ K = \frac{be^{-d_i\tau}}{a},\ \ u_\tau = u(x,t-\tau).$$

So, we need to focus on the following subsystem:

$$(1.4)\quad \begin{cases} \dfrac{\partial u(x,t)}{\partial t} = D_u\Delta u(x,t) + ru_\tau - \dfrac{r}{K}u^2(x,t) \\ \qquad\qquad -\dfrac{mu(x,t)v(x,t)}{1+k_1u(x,t)+k_2v(x,t)}, \\ \dfrac{\partial v(x,t)}{\partial t} = D_v\Delta v(x,t) - \dfrac{nmu(x,t)v(x,t)}{1+k_1u(x,t)+k_2v(x,t)} \\ \qquad\qquad -dv(x,t) \quad \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega\times(0,\infty), \\ u(x,\theta) = u_\theta(x) \ge 0,\ v(x,\theta) = v_\theta(x) \ge 0 \quad \text{in } \Omega\times[-\tau,0], \\ u(x,0) \not\equiv 0,\ v(x,0)\not\equiv 0 \quad \text{in } \Omega. \end{cases}$$

In this paper, we examine the global existence and the long-term behavior of solutions to a diffusive delayed predator-prey model (1.2) with Beddington-DeAngelis type functional response under homogeneous Neumann boundary conditions, where the discrete time delay covers the period from the birth of immature preys to their maturity.

This paper is organized as follows. In the next section, we study the global existence of nonnegative solutions and the long-term behavior of the time-dependent solution of (1.4), particularly its uniformly persistent property.

2. Global existence and persistence

In this section, we show the existence, uniqueness result, and persistence of solutions for the system (1.4) by using the Positivity Lemma [3] and comparison argument.

First, we define the coupled upper and lower solution to system (1.4) as follows by [3].

Definition 2.1. A pair of nonnegative functions $\widetilde{\mathbf{u}} = (\widetilde{u}, \widetilde{v})$, $\widehat{\mathbf{u}} = (\widehat{u}, \widehat{v})$ in $C(\overline{\Omega} \times [-\tau, T]) \cap C^{2,1}(\Omega \times (0, T])$ are called *coupled upper and lower solutions* of (1.4) if $\widehat{\mathbf{u}} \leq \widetilde{\mathbf{u}}$ in $\overline{\Omega} \times [-\tau, T]$ and it satisfies:

$$(2.1)\quad \begin{cases} \widetilde{u}_t - D_u \Delta \widetilde{u} \geq r\widetilde{u}_\tau - \dfrac{r}{K}\widetilde{u}^2 - \dfrac{m\widetilde{u}\widehat{v}}{1 + k_1\widetilde{u} + k_2\widehat{v}}, & \\ \widehat{u}_t - D_u \Delta \widehat{u} \leq r\widehat{u}_\tau - \dfrac{r}{K}\widehat{u}^2 - \dfrac{m\widehat{u}\widetilde{v}}{1 + k_1\widehat{u} + k_2\widetilde{v}}, & \\ \widetilde{v}_t - D_v \Delta \widetilde{v} \geq \dfrac{nm\widetilde{u}\widetilde{v}}{1 + k_1\widetilde{u} + k_2\widetilde{v}} - d\widetilde{v}, & \\ \widehat{v}_t - D_v \Delta \widehat{v} \leq \dfrac{nm\widehat{u}\widehat{v}}{1 + k_1\widehat{u} + k_2\widehat{v}} - d\widehat{v} & \text{in } \Omega \times (0, T], \\ \dfrac{\partial \widehat{u}}{\partial \nu} \leq 0 \leq \dfrac{\partial \widetilde{u}}{\partial \nu},\ \dfrac{\partial \widehat{v}}{\partial \nu} \leq 0 \leq \dfrac{\partial \widetilde{v}}{\partial \nu} & \text{on } \partial\Omega \times (0, T], \\ \widehat{u}(x,t) \leq u_\theta(x) \leq \widetilde{u}(x,t), & \\ \widehat{v}(x,t) \leq v_\theta(x) \leq \widetilde{v}(x,t) & \text{in } \Omega \times [-\tau, 0]. \end{cases}$$

Now we have the existence theorem as follows.

Theorem 2.2. *Let $u, v \in C(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\Omega \times (0, T))$ with $T \leq +\infty$ and (u, v) be a solution of (1.4). Then, for any nonnegative nontrivial initial functions, the system (1.4) has a unique global solution (u^*, v^*) such that $(0, 0) \leq (u^*, v^*) \leq (M_1, M_2)$ in $\overline{\Omega} \times [0, \infty)$, where*

$$M_1 = \max\Big\{\|u_\theta(x)\|_\infty, \frac{be^{-d_i\tau}}{a}\Big\},$$
$$M_2 = \max\Big\{\|v_\theta(x)\|_\infty, \frac{M_1(nm - dk_1) - d}{dk_2}\Big\}.$$

Proof. We show that the solution of (1.4) is uniformly bounded. We construct the upper bound of the solution. For any τ with $0 < \tau < T$ if $max_{\overline{\Omega}\times[0,\tau]} u(x,t) > \|u_\theta(x)\|_\infty$, $-\tau \leq \theta \leq 0$, then there exists $(x_0, t_0) \in \Omega \times (0, T]$ such that $u(x_0, t_0) = max_{\overline{\Omega}\times[0,\tau]} u(x,t)$. Observe that

$$\begin{aligned} u_t(x_0,t_0) - D_u\Delta u(x_0,t_0) &= ru(x_0, t_0 - \tau) - \frac{r}{K}u^2(x_0,t_0) \\ &\quad - \frac{mu(x_0,t_0)v(x_0,t_0)}{1 + k_1u(x_0,t_0) + k_2v(x_0,t_0)} \\ &\leq ru(x_0,t_0) - \frac{r}{K}u^2(x_0,t_0). \end{aligned}$$

Thus we have that $0 \le ru(1-\frac{1}{K}u)$, and so $u \le K = \frac{be^{-d_i\tau}}{a}$. Therefore $0 \le u \le M_1 = max\{\|u_\theta(x)\|_\infty, \frac{be^{-d_i\tau}}{a}\}$. Since $\tau(<T)$ is arbitrary, we have $0 \le u \le M_1$ in $\overline{\Omega} \times [0,T)$.

Similarly, if $max_{\overline{\Omega}\times[0,\tau]} v(x,t) > \|v_\theta(x)\|_\infty$, $-\tau \le \theta \le 0$, then there exists $(x_1,t_1) \in \Omega \times (0,T]$ such that $v(x_1,t_1) = max_{\overline{\Omega}\times[0,\tau]} v(x,t)$. Because the following holds:

$$v(x_1,t_1) \le \frac{u(x_1,t_1)(nm-dk_1)-d}{dk_2} \le \frac{M_1(nm-dk_1)-d}{dk_2},$$

we have $0 \le v \le M_2 = max\{\|v_\theta(x)\|_\infty, \frac{M_1(nm-dk_1)-d}{dk_2}\}$.

Let $\widehat{\mathbf{c}} = (\widehat{c}_1,\widehat{c}_2) = (0,0)$ and $\widetilde{\mathbf{c}} = (\widetilde{c}_1,\widetilde{c}_2) = (M_1,M_2)$ be a pair of nonnegative constant vectors. Then these constant vectors satisfy $\widehat{\mathbf{c}} \le \widetilde{\mathbf{c}}$ and (2.1) since

$$\begin{cases} r\widetilde{c}_1 - r\dfrac{\widetilde{c}_1^2}{K} - \dfrac{m\widetilde{c}_1\widehat{c}_2}{1+k_1\widetilde{c}_1+k_2\widehat{c}_2} \le 0 \le r\widehat{c}_1 - r\dfrac{\widehat{c}_1^2}{K} - \dfrac{m\widehat{c}_1\widetilde{c}_2}{1+k_1\widehat{c}_1+k_2\widetilde{c}_2}, \\ \dfrac{nm\widetilde{c}_1\widetilde{c}_2}{1+k_1\widetilde{c}_1+k_2\widetilde{c}_2} - d\widetilde{c}_2 \le 0 \le \dfrac{nm\widehat{c}_1\widehat{c}_2}{1+k_1\widehat{c}_1+k_2\widehat{c}_2} - d\widehat{c}_2. \end{cases}$$

Therefore $\widetilde{\mathbf{c}}$, $\widehat{\mathbf{c}}$ are coupled upper and lower solutions of (1.4).

The reaction functions in (1.4) satisfy the Lipschitz condition in invariant rectangle. Let

$$f_1(\mathbf{u},\mathbf{u}_\tau) = ru_\tau - \frac{r}{K}u^2 - \frac{muv}{1+k_1u+k_2v}, \quad f_2(\mathbf{u}) = \frac{nmuv}{1+k_1u+k_2v} - dv,$$

where $\mathbf{u} = (u,v)$, $\mathbf{v} = (u_1,v_1)$, $\mathbf{u}_\tau = (u(x,t-\tau), v(x,t-\tau)) = (u_\tau, v_\tau)$ and $\mathbf{v}_\tau = (u_1(x,t-\tau), v_1(x,t-\tau)) = (u_{1\tau}, v_{1\tau})$. It is easy to see that

$$\begin{aligned} &f_1(\mathbf{u},\mathbf{u}_\tau) - f_1(\mathbf{v},\mathbf{v}_\tau) \\ &= r(u_\tau - u_{1\tau}) - \frac{r}{K}(u^2-u_1^2) - m\Big(\frac{uv}{1+k_1u+k_2v} - \frac{u_1v_1}{1+k_1u_1+k_2v_1}\Big) \\ &= r(u_\tau - u_{1\tau}) - \frac{r}{K}(u+u_1)(u-u_1) \\ &\quad - m\Big[\frac{v(u-u_1)+u_1(v-v_1)+k_1uu_1(v-v_1)+k_2vv_1(u-u_1)}{(1+k_1u+k_2v)(1+k_1u_1+k_2v_1)}\Big] \\ &= r(u_\tau - u_{1\tau}) \\ &\quad - \Big[\frac{r}{K}(u+u_1) + \frac{mv(1+k_2v_1)}{(1+k_1u+k_2v)(1+k_1u_1+k_2v_1)}\Big](u-u_1) \\ &\quad - \Big[\frac{mu_1(1+k_1u)}{(1+k_1u+k_2v)(1+k_1u_1+k_2v_1)}\Big](v-v_1). \end{aligned}$$

Thus, we get

$$|f_1(\mathbf{u},\mathbf{u}_\tau)-f_1(\mathbf{v},\mathbf{v}_\tau)| \le r|u_\tau-u_{1\tau}|+A|u-u_1|+B|v-v_1| \\ \le M(|\mathbf{u}_\tau-\mathbf{v}_\tau|+|\mathbf{u}-\mathbf{v}|),$$

where $M=max\{r,A,B\}$, $|\frac{r}{K}(u+u_1)+\frac{mv(1+k_2v_1)}{(1+k_1u+k_2v)(1+k_1u_1+k_2v_1)}|\le A$, and $|\frac{mu_1(1+k_1u)}{(1+k_1u+k_2v)(1+k_1u_1+k_2v_1)}|\le B$. This shows that f_1 satisfy the Lipschitz condition in $<\widehat{\mathbf{u}},\widetilde{\mathbf{u}}>$.

Similarly, we have

$$\begin{aligned} & f_2(\mathbf{u})-f_2(\mathbf{v}) \\ &= \frac{nmuv}{1+k_1u+k_2v}-dv-\Big(\frac{nmu_1v_1}{1+k_1u_1+k_2v_1}-dv_1\Big) \\ &= nm\Big[\frac{v(u-u_1)+u_1(v-v_1)+k_1uu_1(v-v_1)+k_2vv_1(u-u_1)}{(1+k_1u+k_2v)(1+k_1u_1+k_2v_1)}\Big] \\ &\quad -d(v-v_1) \\ &= \Big[\frac{nmv(1+k_2v_1)}{(1+k_1u+k_2v)(1+k_1u_1+k_2v_1)}\Big](u-u_1) \\ &\quad +\Big[\frac{nmu_1(1+k_1u)}{(1+k_1u+k_2v)(1+k_1u_1+k_2v_1)}-d\Big](v-v_1). \end{aligned}$$

Thus, we get

$$|f_2(\mathbf{u})-f_2(\mathbf{v})| \le C|u-u_1|+D|v-v_1| \\ \le M'|\mathbf{u}-\mathbf{v}|,$$

where $|\frac{nmv(1+k_2v_1)}{(1+k_1u+k_2v)(1+k_1u_1+k_2v_1)}|\le C$, $|\frac{nmu_1(1+k_1u)}{(1+k_1u+k_2v)(1+k_1u_1+k_2v_1)}-d|\le D$ and $M'=max\{C,D\}$. This shows that f_2 satisfy the Lipschitz condition in $<\widehat{\mathbf{u}},\widetilde{\mathbf{u}}>$.

Therefore, the system (1.4) has a unique global solution (u^*,v^*) such that $(\widehat{c}_1,\widehat{c}_2)\le(u^*,v^*)\le(\widetilde{c}_1,\widetilde{c}_2)$ in $\overline{\Omega}\times[0,\infty)$ whenever $(\widehat{c}_1,\widehat{c}_2)\le(u_\theta(x),v_\theta(x))\le(\widetilde{c}_1,\widetilde{c}_2)$, $-\tau\le\theta\le 0$, by Theorem 2.2 in [4]. □

Next, we give the persistence of solutions of model (1.4). To achieve the goal, we use the following lemma which is Lemma 3.1 in [1].

LEMMA 2.3. *Let* $u\in C(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\Omega\times(0,\infty))$, *$u$ be a nonnegative nontrivial solution of the scalar problem:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - D\Delta u = Bu(x,t-\tau) \pm A_1 u(x,t) - A_2 u^2(x,t) & in\ \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = 0 & on\ \partial\Omega \times (0,\infty), \\ u(x,t) = \phi(x,t) \geq 0 & in\ \Omega \times [-\tau, 0], \end{cases}$$

with $A_1 \geq 0, B, A_2, \tau > 0$. *Then we have*
(i) if $B \pm A_1 > 0$, *then* $u \to (B \pm A_1)/A_2$ *as* $t \to +\infty$ *uniformly on* $\overline{\Omega}$,
(ii) if $B \pm A_1 < 0$, *then* $u \to 0$ *as* $t \to +\infty$ *uniformly on* $\overline{\Omega}$.

We obtain the long-term behavior of any nonnegative solutions (u, v) of (1.4) as $t \to \infty$ for all $x \in \Omega$.

THEOREM 2.4. *Suppose that* $\frac{nmK}{1+k_1K} > d$, *then the nonnegative solution* (u, v) *of (1.4) satisfies*

$$\limsup_{t\to\infty}(u(x,t), v(x,t)) \leq \Big(K, \frac{(nm - dk_1)K - d}{dk_2}\Big) \quad in\ \overline{\Omega}.$$

Proof. First of all, $\limsup_{t\to\infty} u(x,t) \leq K$ in $\overline{\Omega}$ follows from comparison argument for the parabolic problems and Lemma 2.3 since $ru_\tau - \frac{r}{K}u^2 - \frac{muv}{1+k_1u+k_2v} \leq ru_\tau - \frac{r}{K}u^2$ in $\Omega \times [0,\infty)$. Thus, for an arbitrary positive constant ϵ, there exists $T_1 \in (0,\infty)$ such that $u(x,t) \leq K + \epsilon$ in $\overline{\Omega} \times [T_1, \infty)$. By using this result and comparison argument for the parabolic problems, for an arbitrary positive ϵ, there exists $T_2 \in [T_1, \infty)$ such that $v(x,t) \leq \frac{(nm-dk_1)K-d}{dk_2} + \epsilon^*$ in $\overline{\Omega} \times [T_2, \infty)$, where $\epsilon^* = \epsilon(\frac{nm-dk_1}{dk_2}) + \epsilon$, since

$$\begin{aligned} \frac{nmuv}{1+k_1u+k_2v} - dv &\leq \frac{nm(K+\epsilon)v}{1+k_1(K+\epsilon)+k_2v} - dv \\ &= \frac{v[(nm-dk_1)K - d + \epsilon(nm-dk_1) - dk_2v]}{1+k_1(K+\epsilon)+k_2v} \end{aligned}$$

in $\Omega \times [T_1, \infty)$. Therefore, by the arbitrariness of ϵ, we obtain the desired result. □

THEOREM 2.5. *Suppose that* $1 > \frac{m}{rk_2}$ *and* $\frac{nmA}{k_1A+1} > d$, *then the nonnegative solution* (u, v) *of (1.4) satisfies*

$$\liminf_{t\to\infty}(u(x,t), v(x,t)) \geq \Big(A, \frac{(nm - dk_1)A - d}{dk_2}\Big) \quad in\ \overline{\Omega},$$

where $A := K(1 - \frac{m}{rk_2})$.

Proof. The comparison argument and Lemma 2.3 yield the desired result. For an arbitrary small positive constant ϵ, there exists $T_1 \in (0,\infty)$ such that $u(x,t) \geq K(1-\frac{m}{rk_2}) - \epsilon = A - \epsilon$ in $\overline{\Omega} \times [T_1,\infty)$, since $ru_\tau - \frac{r}{K}u^2 - \frac{muv}{1+k_1u+k_2v} \geq ru_\tau - \frac{r}{K}u^2 - \frac{m}{k_2}u$ in $\Omega \times [0,\infty)$. By using this fact, for an arbitry small positive constant ϵ, there exists $T_2 \in [T_1,\infty)$ such that $v(x,t) \geq \frac{(nm-dk_1)A-d}{dk_2} - \epsilon^*$ in $\overline{\Omega} \times [T_2,\infty)$, where $\epsilon^* = \epsilon(\frac{nm-dk_1}{dk_2}) + \epsilon$, since

$$\begin{aligned}\frac{nmuv}{1+k_1u+k_2v} - dv &\geq \frac{nm(A-\epsilon)v}{1+k_1(A-\epsilon)+k_2v} - dv \\ &= \frac{v[(nm-dk_1)A-d-\epsilon(nm-dk_1)-dk_2v]}{1+k_1(A-\epsilon)+k_2v}\end{aligned}$$

in $\Omega \times [T_1,\infty)$. Therefore, by the arbitrariness of ϵ, we obtain the desired result. □

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