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THE TRANSLATION THEOREM ON THE GENERALIZED ANALOGUE OF WIENER SPACE AND ITS APPLICATIONS

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Abstract. In this note, we prove the translation theorem for the generalized analogue of Wiener measure space and we show some properties of the generalized analogue of Wiener measure from it.

1. Preliminaries

Wiener proved the existence theorem of the measure m_w, based on the properties of Brownian motion, on the space $C_0[0,T]$, the space of all real-valued continuous functions on a closed bounded interval $[0,T]$ which vanish at 0 in 1923. This space $C_0[0,T]$ is called the Wiener space [8]. The theories for this space were studied extensively and applied to various mathematical subjects by too many mathematicians.

The theories of the generalized Wiener measure space $(C_0[0,T], m_{a,b})$, where $a : [0,T] \to \mathbb{R}$ is continuous and $b : [0,T] \to \mathbb{R}$ is strictly increasing continuous, were developed by S.J. Chang, D.M. Chung and D.L. Skoug in [2, 5]. The generalized Wiener measure space is a kind of generalization of the concrete Wiener measure space.

The author and Im introduced the definition and several properties of analogue of Wiener measure on the space $C[0,T]$ with the uniform topology, the space of all real-valued continuous function on $[0,T]$ in [6]. The analogue of Wiener measure space $(C[0,T], m_\varphi)$ is another generalization of the concrete Wiener measure space. In 2010, the author presented the definition and some properties of generalized analogue of Wiener measure space $(C[0,T], m_{a,b;\varphi})$ in [7]. This is the generalization of the concrete Wiener measure space, including both concepts of the

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generalized of the concrete Wiener measure space and the analogue of Wiener measure space.

In this note, we prove the translation theorem for the generalized analogue of Wiener measure space and we show some propositions from it.

Let T be a positive real number. Let $a, b : [0,T] \to \mathbb{R}$ be two continuous functions such that b is strictly increasing and $b(0) = 0$. Let φ be a probability Borel measure on $\mathbb{R}$. For $\vec{t} = (t_0, t_1, t_2, \cdots, t_n)$ with $0 = t_0 < t_1 < t_2 < \cdots < t_n \le T$, let $J_{\vec{t}} : C[0,T] \to \mathbb{R}^{n+1}$ be a function with $J_{\vec{t}}(x) = (x(t_0), x(t_1), \cdots, x(t_n))$ and let

$$\begin{aligned} &W_n(\vec{t}, \vec{u}; a, b) \\ &= \Big[\prod_{j=1}^{n} 2\pi \Delta_j b(t)\Big]^{-\frac{1}{2}} \exp\Big\{ -\sum_{j=1}^{n} \frac{(\Delta_j(\vec{u} - a(t)))^2}{2\Delta_j b(t)} \Big\} \end{aligned}$$

where $\Delta_j b(t) = b(t_j) - b(t_{j-1})$ and $\Delta_j(\vec{u} - a(t)) = u_j - a(t_j) - u_{j-1} + a(t_{j-1})$. For Borel subsets $B_0, B_1, \cdots, B_n$ of $\mathbb{R}$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)$ of $C[0,T]$ is called an interval. Let $\mathcal{I}$ be the set of all intervals. We let

$$\begin{aligned} &m_{a,b;\varphi}\Big(J_{\vec{t}}^{-1}\Big(\prod_{j=0}^{n} B_j\Big)\Big) \\ &= \int_{\prod_{j=0}^{n} B_j} W_n(\vec{t}, \vec{u}; a, b)\ d\Big(\varphi \times \prod_{j=1}^{n} m_L\Big)(u_0, (u_1, u_2, \cdots, u_n)) \end{aligned}$$

where m_L is the Lebesgue measure on $\mathbb{R}$.
From the Chapman-Kolmogorov equation in [4], we have the following theorem.

THEOREM 1.1. *$m_{a,b;\varphi}$ is well-defined on $\mathcal{I}$.*

By the change of variable formula, we obtain the following theorem.

THEOREM 1.2. *(Tame Function) Let $0 = t_0 < t_1 < t_2 < \cdots < t_n \le T$ and let $F(x) = f(x(t_0), x(t_1), \cdots, x(t_n))$, where $f : \mathbb{R}^{n+1} \to \mathbb{C}$ is a function. Then F is measurable if and only if f is Borel measurable and*

$$\begin{aligned} &\int_{C[0,T]} F(x)\ dm_{a,b;\varphi}(x) \\ &\overset{*}{=} \int_{\mathbb{R}^{n+1}} f(\vec{u}) W_n(\vec{t}, \vec{u}; a, b)\ d\Big(\varphi \times \prod_{j=1}^{n} m_L\Big)(u_0, (u_1, u_2, \cdots, u_n)), \end{aligned}$$

where $\overset{*}{=}$ is strong in the sense that if one side exists then the other side exists with equality.

EXAMPLE 1.3. (1) If $f(u) = u$ is φ-integrable then

$$\int_{C[0,T]} x(0)\ dm_{a,b;\varphi}(x) = \int_{\mathbb{R}} u_0 d\varphi(u_0).$$

(2) If $f(u) = u^2$ is φ-integrable and $0 \le t_1 < t_2 \le T$ then

$$\begin{aligned}&\int_{C[0,T]} x(t_1)x(t_2)\ dm_{a,b;\varphi}(x)\\ &= (a(t_2) - a(0))(a(t_2) - a(0)) + (a(t_2) + a(t_1) - 2a(0)) \int_{\mathbb{R}} u_0 d\varphi(u_0)\\ &\quad + \int_{\mathbb{R}} u_0^2 d\varphi(u_0) + b(t_1).\end{aligned}$$

If $f(u) = u^2$ is φ-integrable and $0 \le t \le T$ then

$$\begin{aligned}&\int_{C[0,T]} x(t)^2\ dm_{a,b;\varphi}(x)\\ &= (a(t) - a(0))^2 + 2(a(t) - a(0)) \int_{\mathbb{R}} u_0 d\varphi(u_0) + \int_{\mathbb{R}} u_0^2 d\varphi(u_0) + b(t).\end{aligned}$$

(3) Let $\mathcal{F}(\varphi)$ be the Fourier transform of φ, that is, $[\mathcal{F}(\varphi)](\xi) = \int_{\mathbb{R}} \exp\{i\xi\ x(t)\} d\varphi(u)$. Then for $0 \le t \le T$,

$$\begin{aligned}&\int_{C[0,T]} \exp\{i\xi x(t)\}\ dm_{a,b;\varphi}(x)\\ &= \exp\Big\{ - \frac{b(t)}{2}\xi^2 + i\xi(a(t) - a(0))\Big\}[\mathcal{F}(\varphi)](\xi).\end{aligned}$$

2. The translation theorem

It is well-known fact that there is no quasi-invariant probability measure on the infinite dimensional vector space [9]. So, there are no quasi-invariant probability measure on the concrete Wiener measure space $(C_0[0,T], m_w)$ or the generalized Wiener measure space $(C_0[0,T], m_{a,b})$ or the analogue of Wiener measure space $(C[0,T], m_\varphi)$.

In 1944, Cameron and Martin proved a translation theorem on the concrete Wiener space $(C_0[0,T], m_w)$ [1]. In 1996, S.J. Chang and D.M. Chung established a translation theorem on the generalized Wiener space $(C_0[0,T], m_{a,b})$[2]. In 2002, the author and Im showed a translation theorem on the analogue of Wiener space $(C[0,T], m_\varphi)$ [3].

In this section, we prove a translation theorem on the generalized analogue of Wiener space $(C[0,T], m_{a,b;\varphi})$.

THEOREM 2.1. *(The translation Theorem for the generalized analogue of Wiener measure) Let b be continuously differentiable with $b' > 0$ and let a be of bounded variation. Let h be in $C[0,T]$ and be of bounded variation. Let α be in $\mathbb{R}$, let $x_0(s) = \int_0^s h(u)dm_L(u) + \alpha$ for $0 \le s \le T$ and let $\varphi_\alpha(B) = \varphi(B+\alpha)$ for Borel sets B in $\mathbb{R}$. Then if F is $m_{a,b;\varphi}$-integrable then $F(x+x_0)$ is $m_{a,b;\varphi}$-integrable and*

$$\begin{aligned}
&\int_{C[0,T]} F(y+x_0)\exp\Big\{-\int_0^T \frac{h(t)}{b'(t)}dy(t)\Big\}\ dm_{a,b;\varphi}(y) \\
&\qquad\qquad \exp\Big\{-\frac{1}{2}\int_0^T \frac{h^2(t)}{b'(t)}dt\Big\}\exp\Big\{\int_0^T \frac{h(t)}{b'(t)}da(t)\Big\} \\
&= \int_{C[0,T]} F(y)\ dm_{a,b;\varphi_\alpha}(y).
\end{aligned}$$

Proof. Suppose F is bounded continuous and vanishes on $\{y \in C[0,T] \,|\, ||y||_\infty > M\}$ for some real number M. Then $F(y+x_0)$ is $m_{a,b;\varphi}$-integrable of y. For a natural number n and $0 = t_0 < t_1 < t_2 < \cdots < t_n \le T$, we consider two functions $P_n : C[0,T] \to C[0,T]$ and $G_n : \mathbb{R}^{n+1} \to \mathbb{R}$ such that

$$\big[P_n(y)\big](t) = \frac{b(t)-b(t_{j-1})}{b(t_j)-b(t_{j-1})}(y(t_j)-y(t_{j-1})) + y(t_{j-1})$$

for $t \in [t_{j-1}, t_j]$ and $G_n(y(t_0), y(t_1), \cdots, y(t_n)) = F(P_n(y))$. Then

$$\begin{aligned}
&\int_{C[0,T]} F(P_n(y+x_0))\exp\Big\{-\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \frac{h(t)}{b'(t)}d[P_n(y)](t)\Big\} \\
&\quad \exp\Big\{-\frac{1}{2}\sum_{j=1}^n \frac{(x_0(t_j)-x_0(t_{j-1}))^2}{\Delta_j b(t)}\Big\} \\
&\quad \exp\Big\{\sum_{j=1}^n \frac{(x_0(t_j)-x_0(t_{j-1}))(a(t_j)-a(t_{j-1}))}{\Delta_j b(t)}\Big\}\ dm_{a,b;\varphi}(y)
\end{aligned}$$

$$= \Big[\prod_{j=1}^{n} 2\pi\Delta_j b(t)\Big]^{-\frac{1}{2}} \int_{\mathbb{R}^{n+1}} G_n(u_0 + x_0(t_0), u_1 + x_0(t_1), \cdots, u_n$$

$$+x_0(t_n)) \exp\Big\{\sum_{j=1}^{n} \frac{(x_0(t_j) - x_0(t_{j-1}))(u_j - u_{j-1})}{\Delta_j b(t)}\Big\} \exp\Big\{-\frac{1}{2}$$

$$\sum_{j=1}^{n} \frac{(\Delta_j(\vec{u} - a(t))^2}{\Delta_j b(t)}\Big\}\, d\Big(\varphi \times \prod_{j=1}^{n} m_L\Big)(u_o, (u_1, u_2, \cdots, u_n))$$

$$= \Big[\prod_{j=1}^{n} 2\pi\Delta_j b(t)\Big]^{-\frac{1}{2}} \int_{\mathbb{R}^{n+1}} G_n(v_0, v_1, \cdots, v_n) \exp\Big\{-\frac{1}{2}\sum_{j=1}^{n}$$

$$\frac{(\Delta_j(\vec{v} - a(t))^2}{\Delta_j b(t)}\Big\} d\Big(\varphi_\alpha \times \prod_{j=1}^{n} m_L\Big)(v_0, (v_1, v_2, \cdots, v_n))$$

$$= \int_{C[0,T]} F(P_n(y))\ dm_{a,b;\varphi_\alpha}(y).$$

The first equality in above comes from Theorem 1.1. Putting $u_j + x_0(t_j) = v_j$ $(j = 0, 1, 2, \cdots, n)$, we have the second equality in above. By the elementary calculus, we obtain the last equality.

Taking $n \to \infty$ with $\max\{t_j - t_{j-1} | j = 1, 2, \cdots, n\} \to 0$ in the above equality,

$$\int_{C[0,T]} F(y + x_0) \exp\Big\{-\int_0^T \frac{h(t)}{b'(t)} dy(t)\Big\}\ dm_{a,b;\varphi}(y)$$

$$\exp\Big\{-\frac{1}{2}\int_0^T \frac{h^2(t)}{b'(t)} dt\Big\} \exp\Big\{\int_0^T \frac{h(t)}{b'(t)} da(t)\Big\}$$

$$= \int_{C[0,T]} F(y)\ dm_{a,b;\varphi_\alpha}(y).$$

When F is a non-negative, bounded continuous function, letting

$$M_n(u) = \begin{cases} 1 & \text{if} \quad 0 \le u \le n \\ n + 1 - u & \text{if} \quad n \le u \le n + 1 \\ 0 & \text{if} \quad n + 1 \le u \end{cases}$$

and $F_n(x) = F(x)M_n(||x_0||_\infty)$ for each natural number n, the equality in above holds for F_n. By the monotone convergence theorem, our equality holds for F. From the usual methods in the measure theory, our equality holds for $m_{a,b;\varphi}$-measurable function F. □

In the above theorem, letting $G(y) = F(y)\exp\{\int_0^T \frac{h(t)}{b'(t)}dy(t)\}$, we obtain the following theorem.

THEOREM 2.2. *Under the assumptions in Theorem2.1, if G is $m_{a,b;\varphi}$-integrable then*

$$\begin{aligned}&\int_{C[0,T]} G(y+x_0)\ dm_{a,b;\varphi}(y)\\&= \int_{C[0,T]} G(y)\exp\Big\{\int_0^T \frac{h(t)}{b'(t)}dy(t)\Big\}\ dm_{a,b;\varphi}(y)\\&\quad \exp\Big\{-\frac{1}{2}\int_0^T \frac{h^2(t)}{b'(t)}dt - \int_0^T \frac{h(t)}{b'(t)}da(t)\Big\}.\end{aligned}$$

Putting $F \equiv 1$ in Theorem 2.1, we have the following corollary

COROLLARY 2.3. *Under the assumptions in Theorem 2.1*

$$\begin{aligned}&\int_{C[0,T]} \exp\Big\{-\int_0^T \frac{h(t)}{b'(t)}dy(t)\Big\}\ dm_{a,b;\varphi}(y)\\&= \exp\Big\{\frac{1}{2}\int_0^T \frac{h^2(t)}{b'(t)}dt\Big\}\exp\Big\{\int_0^T \frac{h(t)}{b'(t)}da(t)\Big\}\end{aligned}$$

holds.

Replacing h by λh in the above corollary, by analytic extension theorem in the theory of complex analysis, we obtain the following corollaries.

COROLLARY 2.4. *Under the assumptions in Theorem 2.1, for all λ in $\mathbb{C}$,*

$$\begin{aligned}&\int_{C[0,T]} \exp\Big\{-\lambda\int_0^T \frac{h(t)}{b'(t)}dy(t)\Big\}\ dm_{a,b;\varphi}(y)\\&= \exp\Big\{\frac{1}{2}\lambda^2\int_0^T \frac{h^2(t)}{b'(t)}dt\Big\}\exp\Big\{\lambda\int_0^T \frac{h(t)}{b'(t)}da(t)\Big\}\end{aligned}$$

holds.

COROLLARY 2.5. *Under the assumptions in Theorem 2.1, consider a random variable $X : C[0,T] \to \mathbb{R}$ with $X(x) = \int_0^T \frac{h(t)}{b'(t)}dx(t)$. Then X has a normal distribution with the mean $\int_0^T \frac{h(t)}{b'(t)}da(t)$ and the variance $\int_0^T \frac{h^2(t)}{b'(t)}dt$.*

Proof. From Corollary 2.4, letting $\lambda = i\xi$, the Fourier transform of X is given by $[\mathcal{F}(X)](\xi) = \exp\{-\frac{\xi^2}{2}\int_0^T \frac{h^2(t)}{b'(t)}dt\}\exp\{i\xi\int_0^T \frac{h(t)}{b'(t)}da(t)\}$, as desired. $\square$

By the essentially similar method as in the proof of Theorem 29.7 in [10], we have the following corollary.

COROLLARY 2.6. *Under the assumptions in Theorem 2.1, let* $\{\frac{h_1(t)}{\sqrt{b'(t)}}, \frac{h_2(t)}{\sqrt{b'(t)}}, \cdots, \frac{h_n(t)}{\sqrt{b'(t)}}\}$ *be an orthogonal system such that each* h_j *is of bounded variation and for* $j = 1, 2, \cdots, n$, *let* $X_j(x) = \int_0^T \frac{h_j(t)}{b'(t)}dx(t)$. *Then* $X_1, X_2, \cdots, X_n$ *are independent, each* X_j *has the normal distribution with the mean* $\int_0^T \frac{h_j(t)}{b'(t)}da(t)$ *and the variance* $\int_0^T \frac{h^2(t)}{b'(t)}dt$.

Moreover, if $f : \mathbb{R}^n \to \mathbb{R}$ *is Borel measurable,*

$$\int_{C[0,T]} f(X_1(x), X_2(x), \cdots, X_n(x))\, dm_{a,b;\varphi}(x)$$

$$\stackrel{*}{=} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(u_1, u_2, \cdots, u_n) \exp\Big\{ -\frac{1}{2}\sum_{j=1}^n \frac{(u_j - \int_0^T \frac{h_j(t)}{b'(t)}da(t))^2}{\int_0^T \frac{h_j(t)^2}{b'(t)}dt}\Big\}$$

$$d\Big(\prod_{j=1}^n m_L\Big)(u_1, u_2, \cdots, u_n),$$

where $\stackrel{*}{=}$ *is strong in the sense that if one side exists then the other side exists with equality.*

REMARK 2.7. Let $\{\frac{h_j(t)}{\sqrt{b'(t)}} | j = 1, 2, \cdots\}$ be a complete orthonormal set in $L^2[0,T]$ such that each $\frac{h_n(t)}{\sqrt{b'(t)}}$ is of bounded variation. For f in $L^2[0,T]$ and x in $C[0,T]$, we let

$$\int_0^T f(t)\widehat{dx(t)} = \lim_{n\to\infty}\int_0^T \Big[\sum_{j=1}^n \Big(\int_0^T f(t)\frac{h_j(t)}{\sqrt{b'(t)}}dt\Big)\frac{h_j(s)}{\sqrt{b'(s)}}\Big]dx(s)$$

if the limit exists. $\int_0^T f(t)\widehat{dx(t)}$ is called the Paley-Wiener-Zygmund integral of f according to x. By the routine method in the theory of Wiener space, we can prove the following facts.

i) For f in $L^2[0,T]$, the Paley-Wiener-Zygmund integral $\int_0^T f(t)\widehat{dx(t)}$ exists for $dm_{a,b;\varphi}$-*a.e.* x in $C[0,T]$.

ii) The Paley-Wiener-Zygmund integral $\int_0^T f(t)\widehat{dx(t)}$ is essentially independent of the complete orthonormal set.

iii) If f is of bounded variation on $[0,T]$, then the Paley-Wiener-Zygmund integral $\int_0^T f(t)\widehat{dx(t)}$ equals the Riemann-Stieltjes integral $\int_0^T f(t)dx(t)$.

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