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CIRCLE APPROXIMATION BY QUARTIC G^2 SPLINE USING ALTERNATION OF ERROR FUNCTION

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ABSTRACT. In this paper we present a method of circular arc approximation by quartic Bézier curve. Our quartic approximation method has a smaller error than previous quartic approximation methods due to the alternation of the error function of our quartic approximation. Our method yields a closed form of error so that subdivision algorithm is available, and curvature-continuous quartic spline under the subdivision of circular arc with equal-length until error is less than tolerance. We illustrate our method by some numerical examples.

1. INTRODUCTION

Circular arc and conic section have been widely used in CAD/CAM or Computer Vision. But these curves cannot be expressed by polynomial curve. Thus circle approximation and conic approximation by spline curve are important tasks in CAGD(Computer Aided Geometric Design) or Geometric Modeling. In recent twenty years a lot of methods of circle approximation and conic approximation by Bézier curve or spline have been presented.

The methods of Circle and Conic approximation by quadratic Bézier curve are simple and easy to calculate error. Mørken[13] presented the best approximation method of the circular arc by quadratic Bézier curve. Lee et al.[11] introduced some approximation methods of the circular arc by quadratic Bézier curves to obtain the offset approximation of planar spline curve using convolution of the quadratic approximant and the planar spline curve. Floater[6] showed that the quadratic approximant of conic section is curvature continuous under the subdivision of shoulder point of conic, and presented the upper bound of the Hausdorff distance between the conic section and the quadratic approximant.

The methods of circle and conic approximation by spline of odd degree are as follows. Dokken et al.[3] and Goldapp[8] proposed the best G^k cubic approximations of circular arc for $k = 0, 1, 2$. Floater[7] presented a great approximation method of conic section by spline of odd degree n having approximation order $2n$, which is the optimal approximation order. Fang[4, 5] gave G^k quintic approximation of circular arc and conic section for $k \geq 2$.

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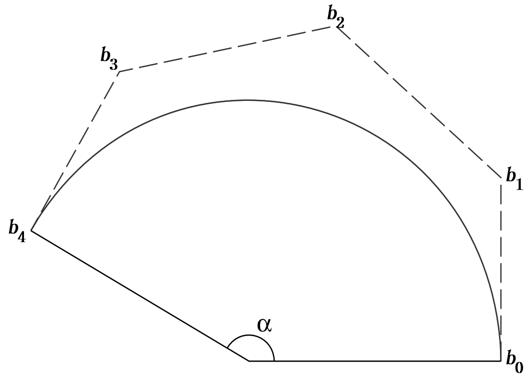


FIGURE 1. Unit circular arc (red color) $\mathbf{c}(\theta) = (\cos\theta, \sin\theta)$, $0 \le \theta \le \alpha$ and quartic Bézier approximation(blue color) $\mathbf{b}(t)$ with control points $\mathbf{b}_i$, $i = 0, \cdots, 4$. The dash-lines(blue) are control polygon.

Some methods of circle and conic approximation by quartic spline have been presented. Ahn and Kim[2] obtained G^k quartic and quintic Bézier approximations of circular arcs using error functions, $k \ge 2$. Kim and Ahn[10] presented another quartic approximation methods of circular arc, and Ahn[1] extended the circle approximation methods by quartic spline to conic approximation. Hur and Kim[9] proposed the best G^2 quartic and G^1 cubic approximation of circular arc. Liu et al.[12] gave circular arc approximation by quartic G^2 spline having smaller error than previous quartic approximation. In this paper we present a quartic G^2 spline approximation of circular arc having smaller error than previous quartic approximation using alternation of error function.

In §2, previous methods for circle approximation by quartic spline are introduced. In §3, our quartic approximation of the circular arc is presented and the closed form of the Hausdorff distance between the circular arc and the quartic approximation curve is obtained. We have the subdivision algorithm and some numerical examples in §4, and summary our results in §5.

2. PRELIMINARIES FOR QUARTIC APPROXIMATION OF THE CIRCULAR ARC

In this section we propose the quartic Bézier approximation of the circular arc with angle $0 < \alpha \le 2\pi$ and radius 1. The unit circular arc $\mathbf{c} : [0, \alpha] \to \mathbf{R}^2$ can be parametrized by

$$\mathbf{c}(\theta) := (\cos\theta, \sin\theta), \qquad 0 \le \theta \le \alpha,$$

as shown in Figure 1. Let $B_i^4(t)$ be the quartic Bernstein polynomial

$$B_i^4(t) = \frac{4!}{i!(4-i)!} t^i (1-t)^{4-i}.$$

The quartic Bézier approximation curve $\mathbf{b} : [0,1] \to \mathbf{R}^2$ of the circular arc $\mathbf{c}(\theta)$ is given by

$$\mathbf{b}(t) := (x(t), y(t)) := \sum_{i=0}^{4} \mathbf{b}_i B_i^4(t) \tag{2.1}$$

with its control points $\mathbf{b}_i := (x_i, y_i)$, $0 \le i \le 4$

$$\begin{aligned} &\mathbf{b}_0 = (\ 1, 0\ ), \quad \mathbf{b}_1 = (\ 1, u\ ), \quad \mathbf{b}_2 = v(\ \cos\tfrac{\alpha}{2}, \sin\tfrac{\alpha}{2}\ ) \\ &\mathbf{b}_3 = (\ \cos\alpha, \sin\alpha\ ) + u(\ \sin\alpha, -\cos\alpha\ ), \quad \mathbf{b}_4 = (\ \cos\alpha, \sin\alpha\ ) \end{aligned} \tag{2.2}$$

so that the quartic approximation $\mathbf{b}(t)$ is a G^1 endpoint interpolation of the circular arc $\mathbf{c}(\theta)$ for $u > 0$.

For the circular arc $\mathbf{c}$, the Hausdorff distance $d_H(\mathbf{b}, \mathbf{c})$ between two curves $\mathbf{c}$ and $\mathbf{b}$ is

$$d_H(\mathbf{b}, \mathbf{c}) = \max_{t\in[0,1]} |\sqrt{x^2(t) + y^2(t)} - 1|$$

Ahn and Kim[2] used the error function $\psi(t)$ by

$$\psi(t) := x^2(t) + y^2(t) - 1. \tag{2.3}$$

to calculate $d_H(\mathbf{b}, \mathbf{c}) = \sqrt{\|\psi(t)\|_\infty + 1} - 1$ for nonnegative ψ, where the uniform norm of ψ on $[0,1]$ is denoted by

$$\|\psi(t)\|_\infty := \max_{t\in[0,1]} |\psi(t)|.$$

If $\psi(t) \ge -1$ for all $t \in [0,1]$, then

$$d_H(\mathbf{b}, \mathbf{c}) = \max\{\sqrt{\max_{t\in[0,1]} \psi(t) + 1} - 1, 1 - \sqrt{1 - \min_{t\in[0,1]} \psi(t)}\}.$$

Equations (2.1)-(2.3) yields $\psi(t) = 4t^2(t-1)^2\zeta(t)$ where

$$\begin{aligned} \zeta(t) &= \left(-9v^2 - (24u\sin\frac{\alpha}{2} + 18\cos\frac{\alpha}{2})v - 16u^2\sin^2\frac{\alpha}{2} - 12u\sin\alpha - 9\cos^2\frac{\alpha}{2}\right)(t-\frac{1}{2})^4 \\ &+ \left(\frac{9}{2}v^2 - 9\cos\frac{\alpha}{2}v - 8u^2\sin^2\frac{\alpha}{2} - 8u\sin\alpha + 4 - \frac{17}{2}\cos^2\frac{\alpha}{2}\right)(t-\frac{1}{2})^2 \\ &- \frac{9}{16}v^2 - (\frac{3}{2}u\sin\frac{\alpha}{2} + \frac{15}{8}\cos\frac{\alpha}{2})v - u^2\sin^2\frac{\alpha}{2} - \frac{5}{4}u\sin\alpha + 4 - \frac{25}{16}\cos^2\frac{\alpha}{2}. \end{aligned} \tag{2.4}$$

As shown in Table 1, Ahn and Kim[2] proposed the approximations $\mathbf{b}_{u_3}$ whose error function ψ has quadruple-zero at both end points, $t = 0, 1$, and $\mathbf{b}_{\mu_2}$ triple-zero at both end points and double-zero at midpoint, $t = \frac{1}{2}$. Our approximation curve $\mathbf{b}(t)$ has contact with the circular arc $\mathbf{c}$ at the midpoint. Solving $\zeta(1/2) = 0$, we have two solutions

$$v_i = -\frac{5}{3}\cos(\alpha/2) - \frac{4}{3}u\sin(\alpha/2) + (-1)^i\frac{8}{3}$$

Quartic approximation	zeros of $\psi(t)$ at	$\lim_{\alpha\to 0}\frac{d_H(\mathbf{q},\mathbf{b})}{\alpha^8}$	$d_H(\mathbf{q},\mathbf{b})$, $\alpha=\pi/2$
$\mathbf{b}_{u_3}$ [2]	$0,0,0,0,1,1,1,1$	$\frac{17-12\sqrt{2}}{2^{15}}\approx 8.98\times 10^{-7}$	3.50×10^{-5}
$\mathbf{b}_{\mu_2}$ [2]	$0,0,0,\frac{1}{2},\frac{1}{2},,1,1,1$	$\frac{27(17-12\sqrt{2})}{2^{23}}\approx 9.47\times 10^{-8}$	3.55×10^{-6}
$\mathbf{b}$ [10]	$0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},1,1$	$\frac{(\sqrt{2}-1)^4}{2^{19}}\approx 5.61\times 10^{-8}$	2.03×10^{-6}
$\tilde{\mathbf{b}}$ [12]	$0,0,\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4},1,1$	$\frac{(1413+399\sqrt{57})(\sqrt{2}-1)^4}{2^{32}}\approx 3.03\times 10^{-8}$	1.11×10^{-6}
$\mathbf{b}_{u_{2,2}}$	$0,0,a,\frac{1}{2},\frac{1}{2},1-a,1,1$	$\frac{(2\sqrt{2}-3)^2}{2^7}f(b_1)\approx 2.07\times 10^{-8}$	7.60×10^{-7}

TABLE 1. The circular arc approximation by quartic Bézier approximations $\mathbf{b}_{u_3}$, $\mathbf{b}_{\mu_2}$, $\mathbf{b}$, and $\tilde{\mathbf{b}}$ are proposed by Ahn and Kim[2], Kim and Ahn[10], and Liu[12]. They have different zeros of error function $\psi(t)$. At the last line, $\mathbf{b}_{u_{2,2}}$ is presented by our method.

$i=1,2$. If $v=v_i$, then $\zeta(t)=4(t-1/2)^2\eta_i(t)$ where

$$\begin{aligned}\eta_1(t) &= (16(\cos\frac{\alpha}{2}+1))(2\sin^2\frac{\alpha}{4}u^2+2\sin\frac{\alpha}{2}u+1+\cos\frac{\alpha}{2})t(t-1) \qquad (2.5)\\ &\quad +4u^2-2u\sin\alpha-(\cos\frac{\alpha}{2}+1)(5\sin\frac{\alpha}{2}+3)\\ \eta_2(t) &= 2^6\sin^2\frac{\alpha}{4}(u\cos\frac{\alpha}{4}-\sin\frac{\alpha}{4})^2t(t-1)\\ &\quad +4u^2-2u\sin\alpha-4\sin^2\frac{\alpha}{4}(1-5\sin^2\frac{\alpha}{4}). \qquad (2.6)\end{aligned}$$

After choosing $v=v_i$ the quartic approximation $\mathbf{b}(t)$ is depend only on one parameter u. Kim and Ahn[10] proposed the G^2 quartic approximation $\mathbf{b}$ with $\eta_2(t)$ having double-zero at the midpoint, and Liu et al.[12] $\tilde{\mathbf{b}}$ with $\eta_2(t)$ having zero at $t=\frac{1}{4},\frac{3}{4}$.

If $u>0$, then the quartic Bézier curve $\mathbf{b}_u(t)$ is an G^1 endpoint interpolation of the circular arc $\mathbf{c}(\theta)$. Furthermore, by symmetry, $\mathbf{b}_u(t)$ have the same curvature at both end points $\mathbf{b}_u(0)$ and $\mathbf{b}_u(1)$. Hence if $u>0$, the method of quartic approximation $\mathbf{b}_u(t)$ of circular arc yields G^2 quartic spline under the subdivision of equal-length of circular arc until the error is less than tolerance. (Refer to [10, 7].)

3. G^2 QUARTIC SPLINE APPROXIMATION OF CIRCULAR WITH ALTERNATION OF ERROR FUNCTION

LEMMA 3.1. *The eighth-degree monic polynomial* $f(t)=t^2(t-1)^2(t-1/2)^2(t-a)(t-(1-a))$, $t\in[0,1]$ *satisfy*

$$||f||_\infty = \max_{t\in[0,1]} f(t) = -\min_{t\in[0,1]} f(t) \qquad (3.1)$$

if $a=\frac{1}{2}\pm\frac{1}{6}\sqrt{6-4\sqrt{3}+2\sqrt{6}\sqrt{\sqrt{3}-1}}$.

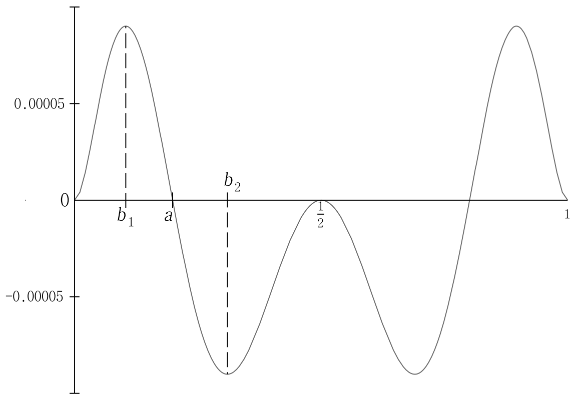


FIGURE 2. The eighth-degree monic polynomial $f(t) = t^2(t-1)^2(t-1/2)^2(t-a)(t-(1-a))$, for $a = \frac{1}{2} \pm \frac{1}{6}\sqrt{6-4\sqrt{3}+2\sqrt{6}\sqrt{\sqrt{3}-1}}$. $f(-b) = \max f(x) = -\min f(x) = ||f||_\infty$.

Proof. Solving the equation $f'(t) = 0$, the solutions are $t = (4 \pm \sqrt{12a^2-20a+9} \pm \sqrt{12a^2-4a+1})/8, 0, 1, 1/2$. Put $b_i = (4 - \sqrt{12a^2-20a+9} + (-1)^i\sqrt{12a^2-4a+1})/8$, $i = 1, 2$. The polynomial $f(t)$ has the maximum $f(b_1) = f(1-b_1)$ and the minimum $f(b_2) = f(1-b_2)$ on $[0, 1]$, if $0 < a < \frac{1}{2}$ or $\frac{1}{2} < a < 1$, as shown in Figure 2. Solving the equation $f(b_1) = -f(b_2)$ with respect to a, we obtain two real solution $\frac{1}{2} \pm \frac{1}{6}\sqrt{6-4\sqrt{3}+2\sqrt{6}\sqrt{\sqrt{3}-1}}$. Any of them satisfies Equation (3.1). □

Numerically, we have $a \approx 0.199$ and $||f||_\infty = f(b_1) \approx 9.01 \times 10^{-5}$, as shown in Figure 2.

Solving equation $\eta_i(a) = 0$ with respect to u, there are four solutions $u_{i,j}$, $i, j = 1, 2$. Since for $u = u_{1,j}$, $j = 1, 2$, the quadratic polynomial $\eta_1(t)$ is

$$\eta_1(t) = (16(\cos\frac{\alpha}{2}+1))(2\sin^2\frac{\alpha}{4}u^2 + 2\sin\frac{\alpha}{2}u + 1 + \cos\frac{\alpha}{2})(t-a)(t-(1-a))$$

and $\|\psi\|_\infty = 2^{10}f(b_1) + \mathcal{O}(\alpha^2)$, the approximation $\mathbf{b}_{u_{1,j}}$, $j = 1, 2$ has approximation order zero, and so $\mathbf{b}_{u_{1,j}}$ could not be a good approximation. Now, we consider two approximations $\mathbf{b}_{u_{2,j}}$, $j = 1$ and 2 and compare their errors. The two parameters are $u_{2,j} = (\xi_1 + (-1)^j\sqrt{\xi_2})/2\xi$ where

$$\begin{aligned} \xi &= 4a(a-1)\sin^2\frac{\alpha}{2} + 1 \\ \xi_1 &= \sin\frac{\alpha}{2}(\cos\frac{\alpha}{2} + 16a(a-1)\sin^2\frac{\alpha}{4}) \\ \xi_2 &= 16\sin^6\frac{\alpha}{4}(2-(2a-1)^2\sin^2\frac{\alpha}{4}). \end{aligned} \tag{3.2}$$

PROPOSITION 3.2. $\mathbf{b}_{u_{2,2}}$ *is better approximation than* $\mathbf{b}_{u_{2,1}}$.

Proof. If $u = u_{2,j}$, then $\eta_2(t) = \eta_{2,j}(t)$ is a quadratic polynomial with zero at $t = a, 1-a$. Thus

$$\eta_{2,j}(t) = 2^6 \sin^2\frac{\alpha}{4}(u_{2,j}\cos\frac{\alpha}{4} - \sin\frac{\alpha}{4})^2(t-a)(t-(1-a))$$

and we have

$$(u_{2,1}\cos\frac{\alpha}{4} - \sin\frac{\alpha}{4})^2 - (u_{2,2}\cos\frac{\alpha}{4} - \sin\frac{\alpha}{4})^2 = \frac{\sin\frac{\alpha}{2}\sin^2\frac{\alpha}{4}(2\cos^2\frac{\alpha}{4}+1)\sqrt{\xi_2}}{\xi^2} \geq 0.$$

Hence $\mathbf{b}_{u_{2,2}}$ is better approximation than $\mathbf{b}_{u_{2,1}}$. □

PROPOSITION 3.3. *For* $u = u_{2,2}$, *the error function is given by*

$$\psi(t) = 2^{10}\sin^2\frac{\alpha}{4}(u_{2,2}\cos\frac{\alpha}{4} - \sin\frac{\alpha}{4})^2 t^2(t-1)^2(t-\frac{1}{2})^2(t-a)(t-(1-a)) \quad (3.3)$$

and

$$\|\psi\|_\infty = 2^{10}\sin^2\frac{\alpha}{4}(u_{2,2}\cos\frac{\alpha}{4} - \sin\frac{\alpha}{4})^2 f(b_1) = \frac{(2\sqrt{3}-3)^2}{2^6}f(b_1)\alpha^8 + \mathcal{O}(\alpha^{10}). \quad (3.4)$$

Proof. Equation (3.3) follows from the equations $\psi(t) = 4t^2(t-1)^2\zeta(t)$, $\zeta(t) = 4(t-1/2)^2\eta_2(t)$, and $\eta_2(t) = 2^6\sin^2\frac{\alpha}{4}(u_{2,2}\cos\frac{\alpha}{4} - \sin\frac{\alpha}{4})^2(t-a)(t-(1-a))$. Since $u_{2,2}\cos\frac{\alpha}{4} - \sin\frac{\alpha}{4} = \frac{2\sqrt{2}-3}{2^6}\alpha^3 + \mathcal{O}(\alpha^5)$, we obtain Equation (3.4). □

We present a closed form of $d_H(\mathbf{q}, \mathbf{b})$ and show that the approximation order of $\mathbf{b}_{u_{2,2}}$ is eight, as follows.

PROPOSITION 3.4. *The Hausdorff distance between the quartic approximation* $\mathbf{b}$ *and the circular arc* $\mathbf{c}$ *is given by*

$$d_H(\mathbf{b}, \mathbf{c}) = 1 - \sqrt{1 - 2^{10}\sin^2\frac{\alpha}{4}(u_{2,2}\cos\frac{\alpha}{4} - \sin\frac{\alpha}{4})^2 f(b_1)} \quad (3.5)$$

and its asymptotic behavior is

$$d_H(\mathbf{b}, \mathbf{c}) = \frac{(2\sqrt{2}-3)^2}{2^7}f(b_1)\alpha^8 + \mathcal{O}(\alpha^{10}). \quad (3.6)$$

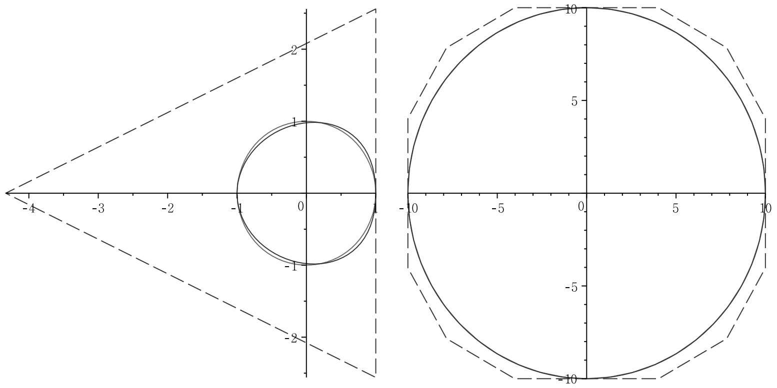
Proof. Since the range of $\psi(t)$ is $[-||\psi||_\infty, ||\psi||_\infty]$, that of $\sqrt{\psi(t)+1} - 1$ is $[\sqrt{1-||\psi||_\infty} - 1, \sqrt{1+||\psi||_\infty} - 1]$. Thus the Hausdorff distance $d_H(\mathbf{b}, \mathbf{c})$ between the circular arc $\mathbf{c}$ and the approximation $\mathbf{b}$ is

$$d_H(\mathbf{b}, \mathbf{c}) = \max\{|\sqrt{1-||\psi||_\infty} - 1|, |\sqrt{1+||\psi||_\infty} - 1|\}. \quad (3.7)$$

Since $|\sqrt{1-s} - 1| \geq |\sqrt{1+s} - 1|$ for any real number $0 < s < 1$, we have

$$d_H(\mathbf{b}, \mathbf{c}) = 1 - \sqrt{1 - 2^{10}\sin^2\frac{\alpha}{4}(u_{2,2}\cos\frac{\alpha}{4} - \sin\frac{\alpha}{4})^2 f(b_1)}. \quad (3.8)$$

Equation (3.6) follows from Equations (3.4)-(3.5). □



(a) (b)

FIGURE 3. (a) The unit circle(red) and our quartic Bézier approximation(blue). The dash-lines(blue) are control polygon. The Hausdorff distance is 4.72×10^{-2}. (b) The circle(red) of radius 10 and our quartic G^2 spline approximation(blue) using four segment of quartic Bézier curve. At each junction points the spline curve is curvature-continuous. The Hausdorff distance is 7.60×10^{-6}.

The Hausdorff distance $d_H(\mathbf{b}, \mathbf{c})$ between the unit circular arc $\mathbf{c}$ of angle α and our quartic approximation $\mathbf{b}_{u_{2,2}}$ in Equation (3.5) is now depend only on α. Thus we denote the Hausdorff distance $d_H(\mathbf{b}, \mathbf{c})$ by $\varepsilon(\alpha)$.

4. ALGORITHM AND NUMERICAL EXAMPLES

Using our circle approximation method by quartic Bézier curve, we present a subdivision algorithm for the quartic G^2 spline approximation of the circular arc within the given tolerance as follows.

Algorithm

1. Input: the radius r and angle ϕ of circular arc, and tolerance TOL.
2. Find the smallest positive integer k satisfying
$$r\varepsilon(\frac{\phi}{k}) < TOL.$$
3. Calculate the control points $\mathbf{b}_i$ of the quartic approximation $\mathbf{b}(t)$ for unit circle using Equations (2.1)-(3.2) for $\alpha = \frac{\phi}{k}$.
4. Output: the control points $T^j r\mathbf{b}_i$, $i = 0, \cdots, 4$, and $j = 0, \cdots, k-1$ of quartic spline approximation, where T is the rotation transformation of angle α.

Figure 3(a) shows the unit full circle(red color) and its quartic Bézier approximation(blue color) using our method. The quartic Bézier curve have the same curvature at the meet point of start-point and end-point. The control polygon is plotted by dash-lines. The Hausdorff distance between the circle and the quartic Bézier curve is 4.72×10^{-2}.

If the radius of circle is given by $r = 10$ and the error tolerance TOL is 10^{-5}, the algorithm yields $k = 4$ and the quartic G^2 spline as shown in Figure 3(b). The Hausdorff distance between the unit circle and quartic spline is $7.60 \times 10^{-6} < TOL$. The control points $\mathbf{b}_i$ of quartic Bézier approximation of unit circle is obtained from Equations (2.1)-(3.2) and $\alpha = \frac{\pi}{2}$. All control points of the quartic spline approximation are also obtained from $T^j r\mathbf{b}_i$, $j = 0, \cdots, 3$ using the rotation transformation T of angle $\frac{\pi}{2}$.

5. Comments and future work

In this paper we presented a method of circle approximation by quartic spline curve. Our circle approximation by quartic G^2 spline has some merits. Our method has smaller error than other previous methods of circle approximation by quartic Bézier curve. Also our method yields a closed form of error and the curvature-continuous quartic spline as previous methods. As a future work, we will extend the circle approximation by quartic spline to the conic approximation and surface approximation such as ellipsoid and torus.

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