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A P-HIERARCHICAL ERROR ESTIMATOR FOR A FEM-BEM COUPLING OF AN EDDY CURRENT PROBLEM IN $\mathbb{R}^3$

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ABSTRACT. We extend a p-hierarchical decomposition of the second degree finite element space of Nédélec for tetrahedral meshes in three dimensions given in [1] to meshes with hexahedral elements, and derive p-hierarchical decompositions of the second degree finite element space of Raviart-Thomas in two dimensions for triangular and quadrilateral meshes. After having proved stability of these subspace decompositions and requiring certain saturation assumptions to hold, we construct a local a posteriori error estimator for fem and bem coupling of a time-harmonic electromagnetic eddy current problem in $\mathbb{R}^3$. We perform some numerical tests to underline reliability and efficiency of the estimator and test its usefulness in an adaptive refinement scheme.

1. INTRODUCTION

This paper is concerned with the construction of a reliable and efficient p-hierarchical based local a posteriori error estimator for a fem-bem coupling of a time-harmonic electromagnetic problem in $\mathbb{R}^3$.

The use of boundary elements for exterior problems in electromagnetics is not new, we mention the early work of MacCamy & Stephan [2, 3, 4, 5] and Nédélec [6, 7]. The coupling of fem and bem in electromagnetics has been pursued most notably by Bossavit [8], Costabel & Stephan [9], Nédélec et al [10, 11, 12, 13] and Hiptmair [14, 15]. In this paper we will be considering a field-based coupling formulation for an eddy current problem taken from [14]. The problem is discretized by edge elements inside the conductor and the exterior region is taken into account by means of a suitable boundary integral coupling.

Given a conductor and a monochromatic exciting current, the task in eddy current computations is to compute the resulting magnetic and electric fields, in the conductor as well as in

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the exterior domain. To this end, let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected open Lipschitz polyhedron with boundary $\Gamma = \partial\Omega$, and further set $\Omega_E = \mathbb{R}^3 \setminus \bar{\Omega}$. The domain Ω then represents the conductor with conductivity $\sigma \in L^\infty(\mathbb{R}^3)$, $\sigma_1 \geq \sigma(\mathbf{x}) \geq \sigma_0 > 0$ and magnetic permeability $\mu \in L^\infty(\mathbb{R}^3)$, $\mu_1 \geq \mu(\mathbf{x}) \geq \mu_0 > 0$ with positive constants σ_0, σ_1, μ_0, μ_1. In the exterior region Ω_E, which represents air, we set $\sigma \equiv 0$ and by scaling $\mu \equiv 1$. The elementwise regularity of the material parameters reflects the fact that Ω can consist of different conducting materials, i.e. the conductivity and permeability can jump from one material to another. We further assume a source current $\mathbf{J}_0 \in \mathbf{H}(\mathrm{div}, \mathbb{R}^3)$ with $\mathrm{supp}(\mathbf{J}_0) \subset \bar{\Omega}$. It follows that $\mathbf{J}_0 \cdot \mathbf{n} = 0$ on Γ (there is no flow of $\mathbf{J}_0$ through Γ), where $\mathbf{n}$ denotes the unit normal vector field on Γ, defined almost everywhere and pointing from Ω into Ω_E.

A mathematical model of the resulting time-harmonic eddy current problem for low frequencies (cf. Ammari, Buffa & Nédélec[16], MacCamy & Stephan[5]) consists of Maxwell's equations

$$\mathbf{curl}\,\mathbf{E} = -i\omega\mu\mathbf{H}, \quad \mathbf{curl}\,\mathbf{H} = \sigma\mathbf{E} + \mathbf{J}_0 \qquad \text{in } \mathbb{R}^3, \tag{1.1}$$

the Coulomb gauge $\mathrm{div}\,\mathbf{E} = 0$ in Ω_E together with the transmission conditions

$$[\mathbf{E} \times \mathbf{n}]_\Gamma = 0, \ [\mathbf{H} \times \mathbf{n}]_\Gamma = 0, \tag{1.2}$$

and the Silver-Müller radiation conditions

$$\mathbf{E}(\mathbf{x}) = \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right), \ \mathbf{H}(\mathbf{x}) = \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right) \text{uniformly for } |\mathbf{x}| \to \infty. \tag{1.3}$$

The equations in (1.1) are just the time-harmonic Maxwell equations with neglected displacement currents (formally setting $\omega\epsilon = 0$, where ϵ denotes the electric permittivity). This approximation is justified in view of low frequencies ω. Note that the second equation in (1.1) reduces to $\mathbf{curl}\,\mathbf{H} = 0$ in the exterior domain Ω_E. Therefore $\mathbf{E}$ cannot be uniquely determined in Ω_E and requires the Coulomb gauge condition. The transmission conditions (1.2) result from requiring $\mathbf{curl}\,\mathbf{E}$ and $\mathbf{curl}\,\mathbf{H}$ to be in $\mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$.

It must be stated that in spite of the Coulomb gauge, $\mathbf{E}$ is unique only up to harmonic Dirichlet vector fields in Ω_E (cf. [16]). But $\mathbf{H} := \frac{1}{i\omega\mu}\mathbf{curl}\,\mathbf{E}$ (by (1.1)) remains unique, so a scheme for determining the magnetic field, which is in fact the interesting quantity in most applications, can consist in first computing a solution $\mathbf{E}$ and then deriving $\mathbf{H}$ from $\mathbf{E}$. If in addition we require $\int_\Gamma \mathbf{E} \cdot \mathbf{n} = 0$, then the solution $\mathbf{E}$ is unique.

In [14], Hiptmair derives an $\mathbf{E}$-based coupling method for solving the problem (1.1)–(1.3) which is based on Costabel's symmetric coupling method [17] (see also [18]). It is this variational formulation that we will be working with. The unknowns of the coupled formulation in this paper are $\mathbf{u}$, the electrical field $\mathbf{E}$ in Ω, and $\boldsymbol{\lambda}$, the twisted tangential trace of the magnetic field on the transmission surface Γ. The natural Sobolev space for $\mathbf{u}$ is $\mathbf{H}(\mathbf{curl}, \Omega)$, the space of $\mathbf{L}^2$-fields in Ω with rotation in $\mathbf{L}^2(\Omega)$, and the space for $\boldsymbol{\lambda}$ turns out to be a trace space of $\mathbf{H}(\mathbf{curl}, \Omega)$. The discretization of $\mathbf{u}$ uses the lowest order $\mathbf{H}(\mathbf{curl}, \Omega)$-conforming finite element space of Nédélec [19]. We use the corresponding trace space for discretizing $\boldsymbol{\lambda}$, which is just a generalization of the lowest order finite element space of Raviart-Thomas on Γ. These spaces belong to the class of edge element spaces, as their degrees of freedom correspond to edges of the grid.

Let $(\mathbf{E}, \boldsymbol{\lambda})$ be the solution of the continuous problem for of the above mentioned fem-bem coupling formulations and let $(\mathbf{E}_h, \boldsymbol{\lambda}_h)$ be the solution of the discrete problem. Then we are interested in finding a reliable and efficient p-hierarchical error estimator for the Galerkin error $(\mathbf{E}-\mathbf{E}_h, \boldsymbol{\lambda}-\boldsymbol{\lambda}_h)$ in the energy norm. One of the main reasons such local a posteriori estimators are so valuable is their usefulness in adaptive mesh refinement schemes. For a residual type error estimator of the fem-bem coupling solution of (1.1)–(1.3) see [20].

Though hierarchical error estimators have long been in use for elliptic problems (cf. [21, 22] for h-hierarchical estimation) and even for fem-bem coupling problems [23], the investigation of their usefulness in electromagnetics has only begun recently. A p-hierarchical error estimator for an eddy current problem in three dimensions using tetrahedral Nédélec elements can be found in [1]. As usual, this estimator depends on a stable subspace decomposition of the higher order finite element space and requires a saturation assumption. We extend the results of that paper in two directions – first, we also consider hexahedral meshes, but more importantly we now deal with coupling formulations, i.e. we have additional boundary element terms. Thus, stable subspace decompositions of higher order Raviart-Thomas elements are needed as well. We derive these from the decompositions of the Nédélec elements by virtue of the twisted tangential trace mapping. In [24] we have applied the results of this paper to the fem-bem coupling of a time-harmonic scattering problem. This work can also be seen as an extension of [25, 26, 27, 28, 23], which deal with a posteriori error estimates for (non-electromagnetic) coupling problems. Concerning electromagnetics, other recent articles dealing with a posteriori error estimators for edge elements are [29, 30, 31, 32].

The paper is organized as follows: In Section 2 we present the coupling formulation for the eddy current problem and in Section 3 we discuss the Galerkin method. Section 4 defines the finite element space $\mathcal{ND}_k(\mathcal{T}_h)$ of first kind Nédélec elements on a mesh $\mathcal{T}_h$ in Ω and investigates $\mathcal{RT}_k(\mathcal{K}_h)$, the twisted tangential trace of $\mathcal{ND}_k(\mathcal{T}_h)$, defined on the trace mesh $\mathcal{K}_h$. Starting from the decomposition of $\mathcal{ND}_2(\mathcal{T}_h)$ for tetrahedra given in [1], we then construct stable p-hierarchical decompositions of $\mathcal{ND}_2(\mathcal{T}_h)$ for hexahedra and of $\mathcal{RT}_2(\mathcal{K}_h)$ for triangles and quadrilaterals in Section 5. In Section 6, we then apply the theory of the last section to find a local a posteriori error estimator for an eddy current fem-bem coupling formulation. Finally, the last section is devoted to numerically underlining the efficiency and reliability of a simplified form of the error estimator for the eddy current problem. We also test its usefulness in an adaptive mesh refinement scheme.

2. Coupling Formulation

In this paper, we assume Ω to be a simply connected polyhedron. Let us then denote the planar boundary faces by Γ_i, $i = 1, \dots, N_\Gamma$ such that $\partial\Omega = \Gamma = \bigcup_{i=1}^{N_\Gamma} \Gamma_i$.

The complex duality pairings in Ω and on Γ will be denoted by $(\cdot,\cdot)_\Omega$ and $\langle\cdot,\cdot\rangle_\Gamma$. We use the usual Sobolev spaces $H^s(\Omega)$ for scalar functions and $\mathbf{H}^s(\Omega)$ for vector fields of order $s \in \mathbb{R}$

(cf. Grisvard [33]). Furthermore we use the spaces

$$\mathbf{H}(\mathbf{curl},\Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) \,:\, \mathbf{curl}\,\mathbf{v} \in \mathbf{L}^2(\Omega)\},$$
$$\mathbf{H}(\mathrm{div},\Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) \,:\, \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\},$$

and the spaces of distributional tangential fields $\mathbf{H}_{\|}^{-1/2}(\Gamma)$ and $\mathbf{H}_{\perp}^{-1/2}(\Gamma)$ as introduced in [34] together with the trace spaces

$$\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) := \{\boldsymbol{\zeta} \in \mathbf{H}_{\|}^{-1/2}(\Gamma) \,:\, \mathrm{div}_\Gamma\,\boldsymbol{\zeta} \in H^{-1/2}(\Gamma)\},$$
$$\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0,\Gamma) := \{\boldsymbol{\zeta} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \,:\, \mathrm{div}_\Gamma\,\boldsymbol{\zeta} = 0, \boldsymbol{\zeta} \in H^{-1/2}(\Gamma)\},$$
$$\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) := \{\boldsymbol{\zeta} \in \mathbf{H}_{\perp}^{-1/2}(\Gamma) \,:\, \mathrm{curl}_\Gamma\,\boldsymbol{\zeta} \in H^{-1/2}(\Gamma)\},$$

with the surface divergence operator $\mathrm{div}_\Gamma\,\mathbf{u} := -\,\mathrm{curl}_\Gamma(\mathbf{u}\times\mathbf{n})$ and the surface curl operator $\mathrm{curl}_\Gamma\,\mathbf{u} := \mathbf{curl}\,\mathbf{u}\cdot\mathbf{n}$, see also [34, 35, 14]. We furthermore need the vectorial surface rotation for a scalar function ϕ defined by $\mathbf{curl}_\Gamma\,\phi := \gamma_t^\times(\mathbf{grad}\,\phi)$.

In the coupling formulation we will need integral operators to represent the exterior problem in (1.1)–(1.3). These operators are defined for $\mathbf{x}\in\Gamma$ as follows (for their properties see e.g. [14]).

$$\mathcal{V}(\boldsymbol{\lambda})(\mathbf{x}) := \gamma_D\mathbf{V}(\boldsymbol{\lambda})(\mathbf{x}) = \gamma_D\int_\Gamma \Phi(\mathbf{x},\mathbf{y})\boldsymbol{\lambda}(\mathbf{y})\,ds(\mathbf{y}),$$
$$\mathcal{K}(\boldsymbol{\lambda})(\mathbf{x}) := \gamma_D\mathbf{K}(\boldsymbol{\lambda})(\mathbf{x}) = \gamma_D\,\mathbf{curl}_{\mathbf{x}}\int_\Gamma \Phi(\mathbf{x},\mathbf{y})(\mathbf{n}\times\boldsymbol{\lambda})(\mathbf{y})\,ds(\mathbf{y}),$$
$$\tilde{\mathcal{K}}(\boldsymbol{\lambda})(\mathbf{x}) := \gamma_N\mathbf{V}(\boldsymbol{\lambda})(\mathbf{x}) = (\gamma_t^\times)\mathbf{K}(\boldsymbol{\lambda}\times\mathbf{n})(\mathbf{x}) = \gamma_N\int_\Gamma \Phi(\mathbf{x},\mathbf{y})\boldsymbol{\lambda}(\mathbf{y})\,ds(\mathbf{y}),$$
$$\mathcal{W}(\boldsymbol{\lambda})(\mathbf{x}) := \gamma_N\mathbf{K}(\boldsymbol{\lambda})(\mathbf{x}) = (\gamma_t^\times)\mathbf{W}(\boldsymbol{\lambda})(\mathbf{x}) = \gamma_N\,\mathbf{curl}_{\mathbf{x}}\int_\Gamma \Phi(\mathbf{x},\mathbf{y})(\mathbf{n}\times\boldsymbol{\lambda})(\mathbf{y})\,ds(\mathbf{y})$$

with Laplace kernel $\Phi(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$ and the limits γ_D and γ_N from Ω_E onto Γ of the traces $\gamma_D\mathbf{u} := \mathbf{n}\times(\mathbf{u}\times\mathbf{n}) =: \mathbf{u}_\Gamma$ and $\gamma_N\mathbf{u} := \gamma_t^\times(\mathbf{curl}\,\mathbf{u})$, where $\gamma_t^\times\mathbf{u} := \mathbf{u}\times\mathbf{n}$. Furthermore we need $\gamma_n\mathbf{u} := \mathbf{u}\cdot\mathbf{n}$.

After having collected the operators and spaces needed we formulate the coupled variational problem for the eddy current problem as ([14] and [18]):
Find $\mathbf{u}\in\mathbf{H}(\mathbf{curl},\Omega)$, $\boldsymbol{\lambda}\in\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0,\Gamma)$ *such that*

$$\begin{aligned}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_\Omega + i\omega(\sigma\mathbf{u},\mathbf{v})_\Omega - \langle\mathcal{W}\mathbf{u}_\Gamma,\mathbf{v}_\Gamma\rangle_\Gamma + \langle\tilde{\mathcal{K}}\boldsymbol{\lambda},\mathbf{v}_\Gamma\rangle_\Gamma &= -i\omega(\mathbf{J}_0,\mathbf{v})_\Omega,\\ \langle(\mathcal{I}-\mathcal{K})\mathbf{u}_\Gamma,\boldsymbol{\zeta}\rangle_\Gamma + \langle\mathcal{V}\boldsymbol{\lambda},\boldsymbol{\zeta}\rangle_\Gamma &= 0\end{aligned}\tag{2.1}$$

for all $\mathbf{v}\in\mathbf{H}(\mathbf{curl},\Omega), \boldsymbol{\zeta}\in\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0,\Gamma)$.

For brevity we write (2.1) as

$$\mathcal{A}(\mathbf{u},\boldsymbol{\lambda};\mathbf{v},\boldsymbol{\zeta}) = \mathcal{L}(\mathbf{v},\boldsymbol{\zeta}).$$

The above formulation is obtained by using Green's formula in Ω and a Stratton-Chu representation formula for $\mathbf{E}$ in Ω_E. The unknown $\mathbf{u}$ corresponds to $\mathbf{E}_{|\Omega}$, and the unknown $\boldsymbol{\lambda}$ on the boundary corresponds to $\gamma_N \mathbf{E} = -i\omega \mathbf{H}_{|\Omega_E} \times \mathbf{n}$, which can indeed be seen to be surface divergence free. Due to the transmission conditions there holds $\boldsymbol{\lambda} = \gamma_N \mathbf{u}$. Note that the formulation (2.1) is block skew-symmetric. As observed by Hiptmair [14], the sesquilinear form $\mathcal{A}$ is continuous and elliptic on $(\mathbf{H}(\mathbf{curl}, \Omega) \times \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma 0, \Gamma))^2$. Thus, the variational formulation (2.1) admits a unique solution. Setting $\mathbf{E}_{|\Omega} := \mathbf{u}$, $\mathbf{E}_{|\Omega_E} := \mathbf{curl}\, \mathbf{V}(\mathbf{n} \times \gamma_D \mathbf{E}) - \mathbf{V}(\boldsymbol{\lambda})$ with the single layer potential $\mathbf{V}$ with Laplace kernel and $\mathbf{H} := \frac{1}{i\omega\mu} \mathbf{curl}\, \mathbf{E}$ gives a solution to the original problem (1.1)–(1.3) (in which the quantity $\mathbf{H}$ is unique, as mentioned earlier).

3. The Galerkin Method

Let $\mathcal{T}_h$ be a regular triangulation (with tetrahedral or hexahedral elements) of Ω and $\mathcal{K}_h = \{T \cap \Gamma \,:\, T \in \mathcal{T}_h\}$ the induced triangulation on Γ. For the Galerkin method we use the finite element spaces suggested in [14] , namely the well known $\mathbf{H}(\mathbf{curl}, \Omega)$-conforming finite element space $\mathcal{ND}_1(\mathcal{T}_h)$ of first kind Nédélec elements of first order [19] for discretization of the unknown $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathcal{RT}_1^0(\mathcal{K}_h) := \{\boldsymbol{\lambda}_h \in \mathcal{RT}_1(\mathcal{K}_h),\, \operatorname{div}_\Gamma \boldsymbol{\lambda}_h = 0\}$ for the boundary unknown $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma 0, \Gamma)$, where $\mathcal{RT}_1(\mathcal{K}_h)$ denotes the lowest order $\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$-conforming finite element space of Raviart-Thomas, which can be obtained as the image of $\mathcal{ND}_1(\mathcal{T}_h)$ under the mapping $\gamma_t^\times$. Thus the Galerkin method reads:
Find $\mathbf{u}_h \in \mathcal{ND}_1(\mathcal{T}_h)$, $\boldsymbol{\lambda}_h \in \mathcal{RT}_1^0(\mathcal{K}_h)$ *such that*

$$
\begin{aligned}
(\mu^{-1}\, \mathbf{curl}\, \mathbf{u}_h, \mathbf{curl}\, \mathbf{v}_h)_\Omega &+ i\omega(\sigma \mathbf{u}_h, \mathbf{v}_h)_\Omega \\
&-\langle \mathcal{W}\gamma_D \mathbf{u}_h, \gamma_D \mathbf{v}_h\rangle_\Gamma + \langle \tilde{\mathcal{K}} \boldsymbol{\lambda}_h, \gamma_D \mathbf{v}_h \rangle_\Gamma = -i\omega(\mathbf{J}_0, \mathbf{v}_h)_\Omega, \\
&\quad \langle (I - \mathcal{K})\gamma_D \mathbf{u}_h, \boldsymbol{\zeta}_h\rangle_\Gamma + \langle \mathcal{V}\boldsymbol{\lambda}_h, \boldsymbol{\zeta}_h\rangle_\Gamma = 0
\end{aligned} \tag{3.1}
$$

for all $\mathbf{v}_h \in \mathcal{ND}_1(\mathcal{T}_h)$, $\boldsymbol{\zeta}_h \in \mathcal{RT}_1^0(\mathcal{K}_h)$.

Now the conformity of the discrete spaces and the strong ellipticity of $\mathcal{A}(\cdot,\cdot)$ imply that the Galerkin formulation (3.1) has a unique solution $(\mathbf{u}_h, \boldsymbol{\lambda}_h) \in \mathcal{ND}_1(\mathcal{T}_h) \times \mathcal{RT}_1^0(\mathcal{K}_h)$.

Next, we give an equivalent formulation of the above Galerkin method which is useful for the numerical implementation of the scheme. As Γ is simply connected, we have $\mathcal{RT}_1^0(\mathcal{K}_h) = \mathbf{curl}_\Gamma \mathcal{S}_1(\mathcal{K}_h)$, where $\mathcal{S}_1(\mathcal{K}_h)$ denotes the finite element space of scalar, continuous and piecewise linear functions [36]. Instead of seeking $\boldsymbol{\lambda}_h \in \mathcal{RT}_1^0(\mathcal{K}_h)$, we now seek a function $\varphi_h \in \tilde{\mathcal{S}}_1(\mathcal{K}_h) := \{\psi \in \mathcal{S}_1(\mathcal{K}_h),\, \int_\Gamma \psi\, ds(\mathbf{x}) = 0\}$ (and set $\boldsymbol{\lambda}_h := \mathbf{curl}_\Gamma \varphi_h$). We achieve this for a $\varphi_h \in \mathcal{S}_1(\mathcal{K}_h)$ by adding the equation $\mathcal{P}(\varphi_h, \tau_h) := (\int_\Gamma \varphi_h(\mathbf{x})\, ds(\mathbf{x}))\overline{(\int_\Gamma \tau_h(\mathbf{x})\, ds(\mathbf{x}))} = 0$ for all $\tau_h \in \mathcal{S}_1(\mathcal{K}_h)$ to our linear system. Note that the sesquilinear form $\mathcal{P}(\varphi, \tau)$ is positive semi-definite ($\mathcal{P}(\varphi, \varphi) = |\int_\Gamma \varphi(\mathbf{x})\, ds(\mathbf{x})|^2$), and that the corresponding matrix has rank 1. Thus the alternative Galerkin method reads: *Find* $\mathbf{u}_h \in \mathcal{ND}_1(\mathcal{T}_h)$, $\varphi_h \in \mathcal{S}_1(\mathcal{K}_h)$ *such that*

$$\begin{aligned} &(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h)_\Omega + i\omega(\sigma\mathbf{u}_h, \mathbf{v}_h)_\Omega \\ &\qquad -\langle \mathcal{W}\gamma_D\mathbf{u}_h, \gamma_D\mathbf{v}_h\rangle_\Gamma + \langle \tilde{\mathcal{K}}\,\mathbf{curl}_\Gamma\varphi_h, \gamma_D\mathbf{v}_h\rangle_\Gamma = -i\omega(\mathbf{J}_0, \mathbf{v}_h)_\Omega, \\ &\langle (I-\mathcal{K})\gamma_D\mathbf{u}_h, \mathbf{curl}_\Gamma\tau_h\rangle_\Gamma + \langle \mathcal{V}\,\mathbf{curl}_\Gamma\varphi_h, \mathbf{curl}_\Gamma\tau_h\rangle_\Gamma + \mathcal{P}(\varphi_h, \tau_h) = 0 \end{aligned} \tag{3.2}$$

for all $\mathbf{v}_h \in \mathcal{ND}_1(\mathcal{T}_h)$, $\tau_h \in \mathcal{S}_1(\mathcal{K}_h)$.

Now, again, the conformity of the discrete spaces and the strong ellipticity of $\mathcal{A}(\cdot,\cdot)$ imply that the Galerkin formulation (3.2) has a unique solution $(\mathbf{u}_h, \boldsymbol{\lambda}_h) \in \mathcal{ND}_1(\mathcal{T}_h) \times \mathbf{curl}_\Gamma\mathcal{S}_1(\mathcal{K}_h)$.

4. Finite element spaces

We consider meshes with tetrahedral elements and with parallelepiped elements (which we will just call hexahedral elements) in $\mathbb{R}^3$, and we simply name them *tetrahedral* or *hexahedral* meshes. Analogously, we speak of *triangular* and *quadrilateral* meshes in $\mathbb{R}^2$, although quadrilateral elements are understood to be parallelograms.

In [19], Nédélec defines a family of conforming finite elements for $\mathbf{H}(\mathbf{curl}, \Omega)$. For any element T of the regular tetrahedral mesh $\mathcal{T}_h$ define the local finite element space

$$\mathcal{ND}_k(T) := (\mathbb{P}_{k-1}(T))^3 + \{\mathbf{p} \in (\mathbb{P}_k(T))^3 :\ \mathbf{p}^\mathsf{T} \cdot \mathbf{x} = 0\} \subset (\mathbb{P}_k(T))^3,$$

inducing the global finite element space

$$\mathcal{ND}_k(\mathcal{T}_h) := \left\{\boldsymbol{\eta}_h \in \mathbf{H}(\mathbf{curl}, \Omega);\ \boldsymbol{\eta}_{h|T} \in \mathcal{ND}_k(T)\ \forall\, T \in \mathcal{T}_h\right\}.$$

$\mathbb{P}_k(T)$ denotes the space of polynomials of order k (a monomial is of order k on a tetrahedron when the sum of the exponents equals k). The local degrees of freedom are given by

(1) $\int_e \mathbf{u}\cdot\mathbf{t}\,q\,ds\ \ \forall q \in \mathbb{P}_{k-1}$, e edge of T,
(2) $\int_F (\mathbf{u}\times\mathbf{n})\cdot\mathbf{q}\,dS;\ \forall \mathbf{q} \in (\mathbb{P}_{k-2})^2$, F face of T,
(3) $\int_T \mathbf{u}\cdot\mathbf{q}\,d\mathbf{x}\ \ \forall \mathbf{q} \in (\mathbb{P}_{k-3})^3$.

This choice of degrees of freedom ensures tangential continuity and thus $\mathbf{H}(\mathbf{curl}, \Omega)$-conformity [19]. If an element T is the image of another element $\widehat{T}$ under the affine transformation

$$\mathbf{x} = \ell(\hat{\mathbf{x}}) := B\hat{\mathbf{x}} + \mathbf{d}, \quad B \in \mathcal{L}(\widehat{T}, \mathbb{R}^3),\ \mathbf{d} \in \mathbb{R}^3 \tag{4.1}$$

and $\{\hat{\mathbf{b}}_j,\ j = 1, \ldots, n_k\}$ is a basis of $\mathcal{ND}_k(\widehat{T})$, then a local basis on T is given by

$$\mathbf{b}_j(\mathbf{x}) = (B^\mathsf{T})^{-1}\,\hat{\mathbf{b}}_j(\hat{\mathbf{x}}), \quad j = 1, \ldots, n_k. \tag{4.2}$$

Global form functions are obtained by glueing together the local basis functions belonging to a common edge or face.

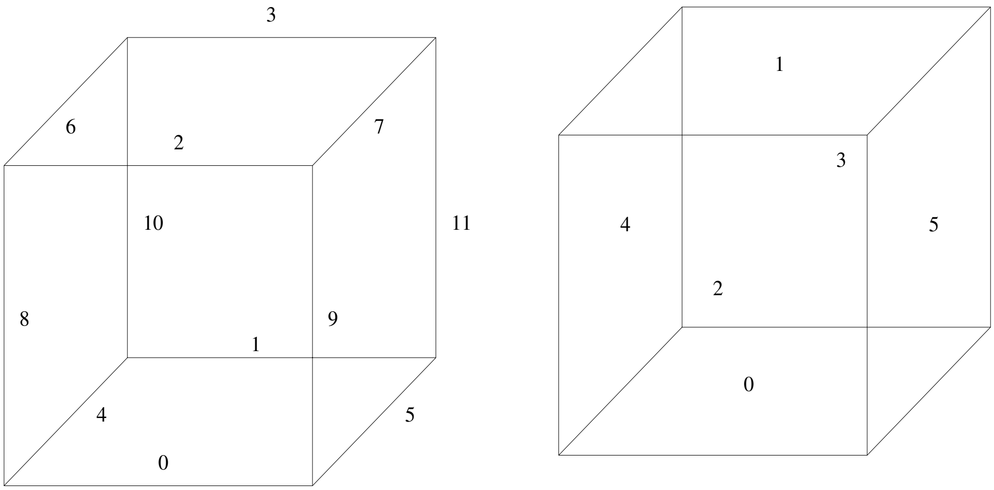

FIGURE 1. Numbering of the edges and faces on the unit cube $[-1,1]^3$.

The construction of Nédélec finite elements on hexahedral meshes is very similar. Let $\mathcal{T}_h$ be a regular hexahedral mesh on Ω. Define the local space

$$\mathcal{ND}_k(T) = \mathbb{Q}_{k-1,k,k}(T) \times \mathbb{Q}_{k,k-1,k}(T) \times \mathbb{Q}_{k,k,k-1}(T) \subset (\mathbb{Q}_{k,k,k}(T))^3.$$

Here $\mathbb{Q}_{k,l,m}(T)$ denotes the space of polynomials with maximum degree k in x, l in y and m in z. Then define the global finite element space $\mathcal{ND}_k(\mathcal{T}_h)$ as before. The degrees of freedom are now given by

(1) $\int_e \mathbf{u} \cdot \mathbf{t}\, q\, ds \;\; \forall q \in \mathbb{Q}_{k-1}$, e edge of T,

(2) $\int_F (\mathbf{u} \times \mathbf{n}) \cdot \mathbf{q}\, dS \;\; \forall \mathbf{q} \in \mathbb{Q}_{k-2,k-1} \times \mathbb{Q}_{k-1,k-2}$, F face of T,

(3) $\int_{\widehat{T}} \mathbf{u} \cdot \mathbf{q}\, d\mathbf{x} \;\; \forall \mathbf{q} \in \mathbb{Q}_{k-1,k-2,k-2} \times \mathbb{Q}_{k-2,k-1,k-2} \times \mathbb{Q}_{k-2,k-2,k-1}$.

For the lowest order $p = 1$ we get the following basis functions associated to the edges of the reference element, see Figure 1.

$$\begin{aligned}
\mathbf{b}^{(e_0)} &= \frac{1}{8}(1-y)(1-z)\mathbf{e}_x, & \mathbf{b}^{(e_1)} &= \frac{1}{8}(1+y)(1-z)\mathbf{e}_x, & \mathbf{b}^{(e_2)} &= \frac{1}{8}(1-y)(1+z)\mathbf{e}_x,\\
\mathbf{b}^{(e_3)} &= \frac{1}{8}(1+y)(1+z)\mathbf{e}_x, & \mathbf{b}^{(e_4)} &= \frac{1}{8}(1-x)(1-z)\mathbf{e}_y, & \mathbf{b}^{(e_5)} &= \frac{1}{8}(1+x)(1-z)\mathbf{e}_y,\\
\mathbf{b}^{(e_6)} &= \frac{1}{8}(1-x)(1+z)\mathbf{e}_y, & \mathbf{b}^{(e_7)} &= \frac{1}{8}(1+x)(1+z)\mathbf{e}_y, & \mathbf{b}^{(e_8)} &= \frac{1}{8}(1-x)(1-y)\mathbf{e}_z,\\
\mathbf{b}^{(e_9)} &= \frac{1}{8}(1+x)(1-y)\mathbf{e}_z, & \mathbf{b}^{(e_{10})} &= \frac{1}{8}(1-x)(1+y)\mathbf{e}_z, & \mathbf{b}^{(e_{11})} &= \frac{1}{8}(1+x)(1+y)\mathbf{e}_z.
\end{aligned}$$

We remark that the edge functions are constant on the edge which they are associated to.

Here are some examples of the 54 basis functions for the polynomial degree $p = 2$.

- There are two edge functions associated to the edge e_0 ($y = -1,\ z = -1$).

$$\mathbf{b}_1^{(e_0)} := \frac{1}{32}\ (3y+1)(y-1)(3z+1)(z-1)\mathbf{e}_x,$$
$$\mathbf{b}_2^{(e_0)} := \frac{3}{32}x(3y+1)(y-1)(3z+1)(z-1)\mathbf{e}_x.$$

The function $\mathbf{b}_1^{(e_0)}$ is constant on the edge e_0 with the value $\frac{1}{2}$. Furthermore, the tangential component vanishes everywhere except on the two faces which are adjacent to the edge.

- There are four face functions associated to the face F_0 ($z = -1$), two in each direction.

$$\mathbf{b}_1^{(F_0)} := \frac{3}{32}(1-y^2)(3z+1)(z-1)\mathbf{e}_x,\ \mathbf{b}_2^{(F_0)} := \frac{9}{32}x\,(1-y^2)(3z+1)(z-1)\mathbf{e}_x,$$
$$\mathbf{b}_3^{(F_0)} :=-\frac{3}{32}(1-x^2)(3z+1)(z-1)\mathbf{e}_y,\ \mathbf{b}_4^{(F_0)} :=-\frac{9}{32}y\,(1-x^2)(3z+1)(z-1)\mathbf{e}_y.$$

The tangential component of the face function is only non-zero on its associated face.

- There are six interior functions

$$\mathbf{b}_1^{(T)} := \frac{9}{32}(1-y^2)(1-z^2)\mathbf{e}_x,\ \mathbf{b}_2^{(T)} := \frac{27}{32}x\,(1-y^2)(1-z^2)\mathbf{e}_x,$$
$$\mathbf{b}_3^{(T)} := \frac{9}{32}(1-x^2)(1-z^2)\mathbf{e}_y,\ \mathbf{b}_4^{(T)} := \frac{27}{32}y\,(1-x^2)(1-z^2)\mathbf{e}_y,$$
$$\mathbf{b}_5^{(T)} := \frac{9}{32}(1-x^2)(1-y^2)\mathbf{e}_z,\ \mathbf{b}_6^{(T)} := \frac{27}{32}z\,(1-x^2)(1-y^2)\mathbf{e}_z.$$

The interior functions are zero on four faces and have a vanishing normal component on the other two faces.

$\mathcal{ND}_k$ is still invariant under the affine transformation (4.1) if we transform the basis functions using (4.2), and we obtain the global basis functions by glueing together the local basis functions as before.

For both mesh types, define $\Pi^{\mathcal{ND}_k(T)}\mathbf{u} \in \mathcal{ND}_k(T)$ as the unique interpolate of $\mathbf{u} \in (\mathcal{C}^\infty(\overline{T}))^3$ such that $\alpha(\mathbf{u} - \Pi^{\mathcal{ND}_k(T)}\mathbf{u}) = 0$ for all degrees of freedom α. We then have the approximation property [19, Theorem 2]:

Lemma 4.1. *For* $\mathbf{u} \in \mathbf{H}^{k+1}(T) \subset \mathbf{H}(\mathbf{curl}, T)$ ($k \in \mathbb{N}_0$) *and an element* T *(tetrahedral or hexahedral) with diameter* h_T *there holds*

$$\|\mathbf{u} - \Pi^{\mathcal{ND}_k(T)}\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},T)} \le c h_T^k \|\mathbf{u}\|_{\mathbf{H}^{k+1}(T)}$$

with a constant c *dependent only on* k *and the regularity of the element* T.

We now turn our attention to the discretization of the trace space $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. We assume Ω to be a polyhedron, so that Γ is piecewise plane. According to [34, 35], we know that the space $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is just the twisted tangential trace of $\mathbf{H}(\mathbf{curl}, \Omega)$. It is thus obvious to discretize $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ using the twisted tangential trace of the space of Nédélec elements. It is well known [37] that this yields the two dimensional $\mathbf{H}(\mathrm{div},\Omega)$-conforming space of Raviart-Thomas, i.e.

$$\gamma_t^\times : \mathcal{ND}_k(\mathcal{T}_h) \to \mathcal{RT}_k(\mathcal{K}_h). \tag{4.3}$$

Also, the degrees of freedom carry over [37], i.e. for an element $T \in \mathcal{T}_h$, a face K of T and $\mathbf{u} \in (\mathcal{C}^\infty(\overline{T}))^3$ we have the identity

$$\gamma_t^\times \Pi^{\mathcal{ND}_k(T)}\mathbf{u} = \Pi^{\mathcal{RT}_k(K)}\gamma_t^\times \mathbf{u}. \tag{4.4}$$

A definition of the Raviart-Thomas space $\mathcal{RT}_k$ can be found in [38, 19], but we will be content to define $\mathcal{RT}_k(\mathcal{K}_h)$ by the above characterization (4.3). The next lemma derives the quality of the approximation of $\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ by $\mathcal{RT}_k$ from the approximation property of $\mathcal{ND}_k$ given in Lemma 4.1.

Lemma 4.2. *For $\boldsymbol{\lambda} \in \mathbf{H}^{k+1/2}(K) \subset \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma, K)$ ($k \in \mathbb{N}_0$) and K face of the element T with diameter h there holds*

$$\|\boldsymbol{\lambda} - \Pi^{\mathcal{RT}_k(K)}\boldsymbol{\lambda}\|_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma,K)} \le ch^k\|\boldsymbol{\lambda}\|_{\mathbf{H}^{k+1/2}(K)}$$

with a constant c dependent only on k and the regularity of the element T.

Proof. The constant c appearing in this proof is always to be regarded as a generic constant. Let $\mathbf{u} \in \mathbf{H}^{k+1}(T) \subset \mathbf{H}(\mathbf{curl}, T)$ with $(\mathbf{u}\times\mathbf{n})_{|K} = \boldsymbol{\lambda}$. Continuity of $\gamma_t^\times$, (4.4) and Lemma 4.1 yield

$$\|\boldsymbol{\lambda} - \Pi^{\mathcal{RT}_k(K)}\boldsymbol{\lambda}\|_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma,K)} \le c\|\mathbf{u} - \Pi^{\mathcal{ND}_k(T)}\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},T)} \le ch^k\|\mathbf{u}\|_{\mathbf{H}^{k+1}(T)}. \tag{4.5}$$

Defining the $\mathbf{H}^{k+1/2}$-norm on K by

$$\|\boldsymbol{\phi}\|_{\mathbf{H}^{k+1/2}(K)} = \inf_{\boldsymbol{\Phi}\in\mathbf{H}^{k+1}(T), \boldsymbol{\Phi}_{|K}=\boldsymbol{\phi}} \|\boldsymbol{\Phi}\|_{\mathbf{H}^{k+1}(T)},$$

we obtain

$$\|\mathbf{n}\times\boldsymbol{\lambda}\|_{\mathbf{H}^{k+1/2}(K)} = \inf_{\substack{\mathbf{u}\in\mathbf{H}^{k+1}(T)\\ \mathbf{u}_{|K}=\boldsymbol{\lambda}\times\mathbf{n}}} \|\mathbf{u}\|_{\mathbf{H}^{k+1}(T)}.$$

Because of $\boldsymbol{\lambda}\cdot\mathbf{n} = 0$, every $\mathbf{u}$ with $\mathbf{u}_{|K} = \boldsymbol{\lambda}\times\mathbf{n}$ satisfies $(\mathbf{u}\times\mathbf{n})_{|K} = \boldsymbol{\lambda}$, so that (4.5) leads to

$$\|\boldsymbol{\lambda} - \Pi^{\mathcal{RT}_k(K)}\boldsymbol{\lambda}\|_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma,K)} \le ch^k\|\mathbf{n}\times\boldsymbol{\lambda}\|_{\mathbf{H}^{k+1/2}(K)} \le ch^k\|\boldsymbol{\lambda}\|_{\mathbf{H}^{k+1/2}(K)}.$$

□

5. TWO-LEVEL DECOMPOSITIONS AND p-HIERARCHICAL ERROR ESTIMATORS

We seek stable two-level decompositions of the finite element spaces introduced in the last section for ultimately constructing hierarchical error estimators.

In [1], the authors consider a p-hierarchical two-level decomposition of $\mathcal{ND}_2(\mathcal{T}_h)$ for tetrahedral grids and describe the construction of a hierarchical error estimator. Here, we will extend this result to hexahedral grids and then use the trace mapping (4.3) to derive $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$-stable decompositions of $\mathcal{RT}_2(\mathcal{K}_h)$, again producing hierarchical error estimators.

Here and in the rest of the paper, the symbol $\lesssim$ signifies "$\leq$ up to a multiplicative constant". Such constants are always assumed to be independent of the mesh width h (if present in the context). The symbol $\simeq$ means "$\lesssim$ and $\gtrsim$".

5.1. **Decomposition of $\mathcal{ND}_2(\mathcal{T}_h)$.** Let $\mathcal{T}_h$ be a regular grid on Ω with mesh width h, and denote with M the number of edges, with N the number of faces and with L the number of elements. Further, let $\mathcal{S}_k$ denote the finite element space of scalar, continuous and piecewise polynomial functions of order k, and let $\widetilde{\mathcal{S}}_k := \mathcal{S}_k \setminus \mathcal{S}_{k-1}$ (the hierarchical surplus). The dimension of $\mathcal{S}_k(T)$ is $\dim \mathcal{S}_k(T) = \frac{1}{6}(k+1)(k+2)(k+3)$ for a tetrahedron T and $\dim \mathcal{S}_k(T) = (k+1)^3$ for a hexahedron T.

For tetrahedral grids, the decomposition given in [29, 1] reads

$$\mathcal{ND}_2(\mathcal{T}_h) = \mathcal{ND}_1(\mathcal{T}_h) \oplus \mathbf{grad}\, \widetilde{\mathcal{S}}_2(\mathcal{T}_h) \oplus \widetilde{\mathcal{ND}}_2^{\perp}(\mathcal{T}_h) \tag{5.1}$$

where

$$\widetilde{\mathcal{ND}}_2^{\perp}(\mathcal{T}_h) := \{\mathbf{u}_h \in \mathcal{ND}_2(\mathcal{T}_h) : \int_e \mathbf{u}_h \cdot \mathbf{t} q \, ds = 0,\ \forall q \in \mathbb{P}_1,\ e \text{ edge of } \mathcal{T}_h\},$$

i. e. $\widetilde{\mathcal{ND}}_2^{\perp}(\mathcal{T}_h)$ is spanned by face functions only.
Counting degrees of freedom on element T, one sees that (5.1) is a direct sum: the dimension of $\mathcal{ND}_1(T)$ equals the number of edges, i.e. six, and the dimension of $\widetilde{\mathcal{S}}_2(T)$ is $10-4=6$ (again equal to the number of edges). We write $\mathbf{grad}\, \widetilde{\mathcal{S}}_2(\mathcal{T}_h) = \mathrm{span}\{\mathbf{grad}\, \phi^{(e_1)}, \dots, \mathbf{grad}\, \phi^{(e_M)}\}$. The space $\mathcal{ND}_2(T)$ has dimension 20, corresponding to two basis functions per edge and two per face of T. The basis functions on the faces span the space $\widetilde{\mathcal{ND}}_2^{\perp}(T)$, which thus has dimension eight. Accordingly, for a tetrahedral mesh we write

$$\widetilde{\mathcal{ND}}_2^{\perp}(\mathcal{T}_h) = \mathrm{span}\{\mathbf{b}_1^{(F_1)}, \mathbf{b}_2^{(F_1)}, \dots, \mathbf{b}_1^{(F_N)}, \mathbf{b}_2^{(F_N)}\}$$

for the space spanned by the face-orientated basis functions of $\mathcal{ND}_2(\mathcal{T}_h)$. The decomposition (5.1) can then be written as:

$$\mathcal{ND}_2(\mathcal{T}_h) = \mathcal{ND}_1(\mathcal{T}_h) \oplus \sum_{i=1}^{M} \mathrm{span}\{\mathbf{grad}\, \phi^{(e_i)}\} \oplus \sum_{j=1}^{N} \mathrm{span}\{\mathbf{b}_1^{(F_j)}, \mathbf{b}_2^{(F_j)}\}. \tag{5.2}$$

This construction cannot be extended offhand to the hexahedral case, for the decomposition defined in (5.1) is then no longer a direct sum. Counting degrees of freedom, we see that

$\mathbf{grad}\,\widetilde{\mathcal{S}}_2(T)$ and $\widetilde{\mathcal{ND}}_2^{\perp}(T)$ overlap: the dimension of $\mathcal{ND}_1(T)$ equals the number of edges, i.e. 12, the dimension of $\widetilde{\mathcal{S}}_2(T)$ is equal to $27-8=19$ (corresponding to one function per edge, one per face and one inner function), and the dimension of $\widetilde{\mathcal{ND}}_2^{\perp}(T)$ is 30 (four functions per face and six inner functions). But the dimension of $\mathcal{ND}_2(T)$ is 54, so that there must hold $\dim(\mathbf{grad}\,\widetilde{\mathcal{S}}_2(T)\cap\widetilde{\mathcal{ND}}_2^{\perp}(T))=7$. Hence, if we are to find a direct decomposition of $\mathcal{ND}_2(T)$ for hexahedra, we must determine 7 functions to eliminate from $\mathbf{grad}\,\widetilde{\mathcal{S}}_2(T)\cap\widetilde{\mathcal{ND}}_2^{\perp}(T)$. Let us write

$$\widetilde{\mathcal{S}}_2(T)=\mathrm{span}\{\phi^{(e_0)},\dots,\phi^{(e_{11})},\phi^{(F_0)},\dots,\phi^{(F_5)},\phi^{(T)}\}$$

with edge based functions $\phi^{(e_i)}$, face based functions $\phi^{(F_i)}$ and bubble function $\phi^{(T)}$. Furthermore with face based functions $\mathbf{b}_i^{(F_j)}$ and suitable 'bubble' functions $\mathbf{b}_i^{(T)}$ we can write

$$\widetilde{\mathcal{ND}}_2^{\perp}(T)=\mathrm{span}\{\mathbf{b}_1^{(F_0)},\dots,\mathbf{b}_4^{(F_0)},\dots,\mathbf{b}_1^{(F_5)},\dots,\mathbf{b}_4^{(F_5)},\mathbf{b}_1^{(T)},\dots,\mathbf{b}_6^{(T)}\}.$$

By explicitly computing the basis functions of $\widetilde{\mathcal{ND}}_2^{\perp}(T)$ for the reference element $T=[-1,1]^3$ (according to the degrees of freedom given earlier), one ascertains that the face functions of $\mathbf{grad}\,\widetilde{\mathcal{S}}_2(T)$ can be described by functions of $\widetilde{\mathcal{ND}}_2^{\perp}(T)$, for example there holds

$$\mathbf{grad}\,\phi^{(F_0)}=\mathbf{grad}(1-x^2)(1-y^2)(1-z)=-\frac{32}{9}(\mathbf{b}_2^{(F_0)}-\mathbf{b}_4^{(F_0)}+\mathbf{b}_2^{(T)}+\mathbf{b}_4^{(T)}+\mathbf{b}_5^{(T)}),$$

and similarly for the other $\mathbf{grad}\,\phi^{(F_j)}$. There further holds

$$\mathbf{grad}\,\phi^{(T)}=\mathbf{grad}(1-x^2)(1-y^2)(1-z^2)=-\frac{64}{27}(\mathbf{b}_2^{(T)}+\mathbf{b}_4^{(T)}+\mathbf{b}_6^{(T)}).$$

With this information, there are now many ways to exchange the spaces $\mathbf{grad}\,\widetilde{\mathcal{S}}_2(T)$ and $\widetilde{\mathcal{ND}}_2^{\perp}(T)$ by reduced spaces $\mathbf{grad}\,\widetilde{\mathcal{S}}_2^{-}(T)$ and $\widetilde{\mathcal{ND}}_2^{\perp,-}(T)$ to obtain a direct sum. We propose:

(1) leave $\widetilde{\mathcal{S}}_2(T)$ as it is,
(2) substitute $\mathbf{b}_2^{(F_j)}+\mathbf{b}_4^{(F_j)}$ for the face functions $\mathbf{b}_2^{(F_j)}$, $\mathbf{b}_4^{(F_j)}$ ($j=0,\dots,5$) and further substitute $\mathbf{b}_2^{(T)}-\mathbf{b}_4^{(T)}$ and $\mathbf{b}_4^{(T)}-\mathbf{b}_6^{(T)}$ for the interior functions $\mathbf{b}_2^{(T)}$, $\mathbf{b}_4^{(T)}$, $\mathbf{b}_6^{(T)}$; this changes $\widetilde{\mathcal{ND}}_2^{\perp}(T)$ to $\widetilde{\mathcal{ND}}_2^{\perp,-}(T)$.

We then obtain the global space

$$\begin{aligned}\widetilde{\mathcal{ND}}_2^{\perp,-}(\mathcal{T}_h):=\mathrm{span}\{&\mathbf{b}_1^{(F_j)},\mathbf{b}_3^{(F_j)},\mathbf{b}_2^{(F_j)}+\mathbf{b}_4^{(F_j)},\\ &\mathbf{b}_1^{(T_k)},\mathbf{b}_3^{(T_k)},\mathbf{b}_5^{(T_k)},\mathbf{b}_2^{(T_k)}-\mathbf{b}_4^{(T_k)},\mathbf{b}_4^{(T_k)}-\mathbf{b}_6^{(T_k)},\\ &j=1,\dots,N,\ k=1,\dots,L\}\end{aligned}$$

and the direct decomposition

$$\mathcal{ND}_2(\mathcal{T}_h)=\mathcal{ND}_1(\mathcal{T}_h)\oplus\mathbf{grad}\,\widetilde{\mathcal{S}}_2(\mathcal{T}_h)\oplus\widetilde{\mathcal{ND}}_2^{\perp,-}(\mathcal{T}_h)$$

for hexahedral grids, which can be broken down to:

$$\mathcal{ND}_2(\mathcal{T}_h) = \mathcal{ND}_1(\mathcal{T}_h) \oplus \sum_{i=1}^{M} \mathrm{span}\{\mathbf{grad}\,\phi^{(e_i)}\}$$
$$\oplus \sum_{j=1}^{N} \Big(\mathrm{span}\{\mathbf{grad}\,\phi^{(F_j)}\} \oplus \mathrm{span}\{\mathbf{b}_1^{(F_j)}, \mathbf{b}_3^{(F_j)}, \mathbf{b}_2^{(F_j)} + \mathbf{b}_4^{(F_j)}\}\Big)$$
$$\oplus \sum_{k=1}^{L} \Big(\mathrm{span}\{\mathbf{grad}\,\phi^{(T_k)}\} \oplus \mathrm{span}\{\mathbf{b}_1^{(T_k)}, \mathbf{b}_3^{(T_k)}, \mathbf{b}_5^{(T_k)}, \mathbf{b}_2^{(T_k)} - \mathbf{b}_4^{(T_k)}, \mathbf{b}_4^{(T_k)} - \mathbf{b}_6^{(T_k)}\}\Big)\,. \tag{5.3}$$

In what follows the stability of the decompositions (5.2) and (5.3) is crucial for the derivation of hierarchical error indicators. To this aim, we define for tetrahedra the subspace projections

$$P_1 : \mathcal{ND}_2(\mathcal{T}_h) \to \mathcal{ND}_1(\mathcal{T}_h),$$
$$P^{(F)} : \mathcal{ND}_2(\mathcal{T}_h) \to \mathrm{span}\{\mathbf{b}_1^{(F)}, \mathbf{b}_2^{(F)}\},$$
$$R^{(e)} : \mathcal{ND}_2(\mathcal{T}_h) \to \mathrm{span}\{\mathbf{grad}\,\phi^{(e)}\},$$

and for hexahedra the projections

$$\tilde{P}_1 : \mathcal{ND}_2(\mathcal{T}_h) \to \mathcal{ND}_1(\mathcal{T}_h),$$
$$\tilde{P}^{(F)} : \mathcal{ND}_2(\mathcal{T}_h) \to \mathrm{span}\{\mathbf{b}_1^{(F)}, \mathbf{b}_3^{(F)}, \mathbf{b}_2^{(F)} + \mathbf{b}_4^{(F)}\},$$
$$\tilde{P}^{(T)} : \mathcal{ND}_2(\mathcal{T}_h) \to \mathrm{span}\{\mathbf{b}_1^{(T_k)}, \mathbf{b}_3^{(T_k)}, \mathbf{b}_5^{(T_k)}, \mathbf{b}_2^{(T_k)} - \mathbf{b}_4^{(T_k)}, \mathbf{b}_4^{(T_k)} - \mathbf{b}_6^{(T_k)}\},$$
$$\tilde{R}^{(e)} : \mathcal{ND}_2(\mathcal{T}_h) \to \mathrm{span}\{\mathbf{grad}\,\phi^{(e)}\},$$
$$\tilde{R}^{(F)} : \mathcal{ND}_2(\mathcal{T}_h) \to \mathrm{span}\{\mathbf{grad}\,\phi^{(F)}\},$$
$$\tilde{R}^{(T)} : \mathcal{ND}_2(\mathcal{T}_h) \to \mathrm{span}\{\mathbf{grad}\,\phi^{(T)}\},$$

so that for $\mathbf{u}_2 \in \mathcal{ND}_2(\mathcal{T}_h)$ the decompositions (5.2) and (5.3) can be written as

$$\mathbf{u}_2 = P_1\mathbf{u}_2 + \sum_{i=1}^{M} R^{(e_i)}\mathbf{u}_2 + \sum_{j=1}^{N} P^{(F_j)}\mathbf{u}_2 \tag{5.4}$$

and

$$\mathbf{u}_2 = \tilde{P}_1\mathbf{u}_2 + \sum_{i=1}^{M} \tilde{R}^{(e_i)}\mathbf{u}_2 + \sum_{j=1}^{N} \Big(\tilde{R}^{(F_j)}\mathbf{u}_2 + \tilde{P}^{(F_j)}\mathbf{u}_2\Big) + \sum_{k=1}^{L} \Big(\tilde{R}^{(T_k)}\mathbf{u}_2 + \tilde{P}^{(T_k)}\mathbf{u}_2\Big)\,. \tag{5.5}$$

The following lemma is standard (see [39] for details). It is used in the proof of Lemma 5.2 which postulates the stability of the decompositions.

Lemma 5.1. *Let $T \in \mathcal{T}_h$ be an element with diameter h_T and $\widehat{T}$ be the reference element. Let $\hat{\mathbf{q}} : \widehat{T} \to \mathbb{R}^3$, $\mathbf{q} : T \to \mathbb{R}^3$.*

(1) *Then with the* $\mathbf{H}(\mathbf{curl})$*-conforming transformation (cf.* (4.1) *and* (4.2)*)*

$$\mathbf{q}(\mathbf{x}) = (B^{\mathsf{T}})^{-1}\,\hat{\mathbf{q}}(\ell^{-1}(\mathbf{x})) \tag{5.6}$$

there holds

$$\|\mathbf{q}\|_{\mathbf{L}^2(T)} \sim h^{1/2}\|\hat{\mathbf{q}}\|_{\mathbf{L}^2(\widehat{T})}, \tag{5.7}$$

(2) *Then with the* $\mathbf{H}(\mathrm{div})$*-conforming transformation*

$$\mathbf{q}(\mathbf{x}) = \frac{1}{\det B} B\,\hat{\mathbf{q}}(\ell^{-1}(\mathbf{x})) \tag{5.8}$$

there holds

$$\|\mathbf{q}\|_{\mathbf{L}^2(T)} \sim h^{-1/2}\|\hat{\mathbf{q}}\|_{\mathbf{L}^2(\widehat{T})}, \tag{5.9}$$

The next lemma states the stability result. For the sake of clarity, we will denote the $\mathbf{H}(\mathbf{curl},\Omega)$-norm simply by $\|\,\|$.

Lemma 5.2. *The decompositions* (5.2) *resp.* (5.3) *are stable with respect to the* $\mathbf{H}(\mathbf{curl},\Omega)$*-norm, i.e. for all* $\mathbf{u}_2 \in \mathcal{ND}_2(\mathcal{T}_h)$ *there holds*

$$\|\mathbf{u}_2\|^2 \simeq \|P_1\mathbf{u}_2\|^2 + \sum_{i=1}^{M}\|R^{(e_i)}\mathbf{u}_2\|^2 + \sum_{j=1}^{N}\|P^{(F_j)}\mathbf{u}_2\|^2 \tag{5.10}$$

resp.

$$\begin{aligned}\|\mathbf{u}_2\|^2 \simeq \|\tilde{P}_1\mathbf{u}_2\|^2 &+ \sum_{i=1}^{M}\|\tilde{R}^{(e_i)}\mathbf{u}_2\|^2 + \sum_{j=1}^{N}\left(\|\tilde{R}^{(F_j)}\mathbf{u}_2\|^2 + \|\tilde{P}^{(F_j)}\mathbf{u}_2\|^2\right)\\ &+ \sum_{k=1}^{L}\left(\|\tilde{R}^{(T_k)}\mathbf{u}_2\|^2 + \|\tilde{P}^{(T_k)}\mathbf{u}_2\|^2\right).\end{aligned} \tag{5.11}$$

Proof. First let us consider the case of hexahedra, i.e. (5.11). First we observe that due to the uniqueness of the decomposition (5.3) the mapping $|\!|\!|\cdot|\!|\!|$ is a norm where $|\!|\!|\cdot|\!|\!|$ is defined by

$$\begin{aligned}|\!|\!|\mathbf{u}_2|\!|\!|^2_{\mathbf{L}^2(\Omega)} :=& \|\tilde{P}_1\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} + \sum_{i=1}^{M}\|\tilde{R}^{(e_i)}\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} + \sum_{j=1}^{N}\left(\|\tilde{R}^{(F_j)}\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} + \|\tilde{P}^{(F_j)}\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)}\right)\\ &+ \sum_{k=1}^{L}\left(\|\tilde{R}^{(T_k)}\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} + \|\tilde{P}^{(T_k)}\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)}\right) =: \sum_{T\in\mathcal{T}_h}\sum_{P_T}\|P_T\mathbf{u}_2\|^2_{\mathbf{L}^2(T)}\end{aligned}$$

Since the L^2-Norm is local we conclude with (5.7) that there holds

$$|\!|\!|\mathbf{u}_2|\!|\!|^2_{\mathbf{L}^2(\Omega)} = \sum_{T\in\mathcal{T}_h}|\!|\!|\mathbf{u}_2|\!|\!|^2_{\mathbf{L}^2(T)} = \sum_{T\in\mathcal{T}_h}\sum_{P_T}\|P_T\mathbf{u}_2\|^2_{\mathbf{L}^2(T)} \sim \sum_{T\in\mathcal{T}_h} h_T \sum_{P_T}\|\widehat{P_T\hat{\mathbf{u}}_2}\|^2_{\mathbf{L}^2(\widehat{T})}, \tag{5.12}$$

where $\hat{\mathbf{v}}(\hat{\mathbf{x}}) = B^{\mathsf{T}}\mathbf{v}(\mathbf{x})$ is the transformation of $\mathbf{v}$ to the reference element $\widehat{T}$, and P_T denotes a projection operator that is related to the element T. The constant in the equivalence relation depends only on the shape regularity of the mesh. Furthermore there holds for $\hat{\mathbf{u}}_2 \in \mathcal{ND}_2(\widehat{T})$

$$|\!|\!|\hat{\mathbf{u}}_2|\!|\!|_{\mathbf{L}^2(\widehat{T})} = \sum_{P_{\hat{T}}} \|P_{\hat{T}}\hat{\mathbf{u}}_2\|_{\mathbf{L}^2(\widehat{T})} \sim \|\hat{\mathbf{u}}_2\|_{\mathbf{L}^2(\widehat{T})},$$

since all norms are equivalent on a finite dimensional space and the number of projection operators on an element is bounded. Here the constant in the equivalence relation depends only on the decomposition on $\widehat{T}$. With (5.12) and (5.7) we obtain

$$|\!|\!|\mathbf{u}_2|\!|\!|^2_{\mathbf{L}^2(\Omega)} \sim \sum_{T\in\mathcal{T}_h} h_T |\!|\!|\hat{\mathbf{u}}_2|\!|\!|^2_{\mathbf{L}^2(\widehat{T})} \sim \sum_{T\in\mathcal{T}_h} \|\mathbf{u}_2\|^2_{\mathbf{L}^2(T)} = \|\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)}.$$

Now it is left to show that there holds

$$|\!|\!|\mathbf{curl}\,\mathbf{u}_2|\!|\!|_{\mathbf{L}^2(\Omega)} := \sum_P \|\mathbf{curl}\,P\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} \sim \|\mathbf{curl}\,\mathbf{u}_2\|_{\mathbf{L}^2(\Omega)}.$$

This follows with the same arguments as above, when we use relation (5.9) for the transformation to the reference element since $\mathbf{curl}\,\mathbf{u}_2 \in \mathcal{RT}_2(\mathcal{T}_h)$ for $\mathbf{u}_2 \in \mathcal{ND}_2(\mathcal{T}_h)$. Note that also the following decomposition is unique:

$$\begin{aligned}\mathbf{curl}\,\mathbf{u}_2 = \mathbf{curl}\,\tilde{P}_1\mathbf{u}_2 + \sum_{i=1}^{M} \mathbf{curl}\,\tilde{R}^{(e_i)}\mathbf{u}_2 &+ \sum_{j=1}^{N}\left(\mathbf{curl}\,\tilde{R}^{(F_j)}\mathbf{u}_2 + \mathbf{curl}\,\tilde{P}^{(F_j)}\mathbf{u}_2\right)\\ &+ \sum_{k=1}^{L}\left(\mathbf{curl}\,\tilde{R}^{(T_k)}\mathbf{u}_2 + \mathbf{curl}\,\tilde{P}^{(T_k)}\mathbf{u}_2\right).\end{aligned}$$

This can be seen as follows.

Let $\mathbf{curl}\,\mathbf{u}_2 = 0$. Then we have $\mathbf{curl}\,\tilde{P}_1\mathbf{u}_2 = \mathbf{curl}\,\mathbf{\Pi}^{\mathcal{ND}_1}\mathbf{u}_2 = \mathbf{\Pi}^{\mathcal{RT}_1}\,\mathbf{curl}\,\mathbf{u}_2 = 0$, furthermore there holds $\mathbf{curl}\,\tilde{R}^{(e_i)}\mathbf{u}_2 = \mathbf{curl}\,\tilde{R}^{(F_j)}\mathbf{u}_2 = \mathbf{curl}\,\tilde{R}^{(T_k)}\mathbf{u}_2 = 0$ due to $\mathbf{curl}\,\mathbf{grad} \equiv 0$. The decomposition (5.5) yields $\mathbf{curl}\left(\sum_{j=1}^{N}\tilde{P}^{(F_j)}\mathbf{u}_2 + \sum_{k=1}^{L}\tilde{P}^{(T_k)}\mathbf{u}_2\right) = 0$. Therefore there exists $\psi_2 \in \widetilde{\mathcal{S}}_2(\mathcal{T}_h)$ with $\sum_{j=1}^{N}\tilde{P}^{(F_j)}\mathbf{u}_2 + \sum_{k=1}^{L}\tilde{P}^{(T_k)}\mathbf{u}_2 = \mathbf{grad}\,\psi_2$. Now $\mathbf{grad}\,\psi_2 = 0$ since the sum (5.3) is direct, hence $\tilde{P}^{(F_j)}\mathbf{u}_2 = 0$ for all j and $\tilde{P}^{(T_k)}\mathbf{u}_2 = 0$ for all k. Especially there holds $\mathbf{curl}\,\tilde{P}^{(F_j)}\mathbf{u}_2 = 0$ for all j and $\mathbf{curl}\,\tilde{P}^{(T_k)}\mathbf{u}_2 = 0$ for all k. Thus $\mathbf{curl}\,\mathbf{u}_2 = 0$ implies $\mathbf{curl}\,P\mathbf{u}_2 = 0$ for all projections P. Altogether there holds, independently of the meshsize h,

$$|\!|\!|\mathbf{u}_2|\!|\!|_{\mathbf{L}^2(\Omega)} \sim \|\mathbf{u}_2\|_{\mathbf{L}^2(\Omega)}, \quad |\!|\!|\mathbf{curl}\,\mathbf{u}_2|\!|\!|_{\mathbf{L}^2(\Omega)} \sim \|\mathbf{curl}\,\mathbf{u}_2\|_{\mathbf{L}^2(\Omega)}.$$

This gives the assertion of the lemma in case of a hexahedral mesh for the $\mathbf{H}(\mathbf{curl}, \Omega)$-norm and the equivalent energy norm.
Similar arguments apply to the case of tetrahedral grids (see [1, Lemma 3 and Lemma 4]).

□

Now let

$$a(\mathbf{u},\mathbf{v}) := (\alpha\,\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_\Omega + (\beta\mathbf{u},\mathbf{v})_\Omega \tag{5.13}$$

($\alpha,\beta \in \mathbb{C}\setminus\{0\}$, $\frac{\alpha}{\beta} \notin \mathbb{R}_{<0}$) be such that $a(\cdot,\cdot)$ is continuous on $(\mathbf{H}(\mathbf{curl},\Omega))^2$ and satisfies there a Gårdings inequality. Then define

$$\|\mathbf{v}\|_{\mathfrak{E}} := \|\mathbf{v}\|_{\mathfrak{E}(\Omega)} := |a(\mathbf{v},\mathbf{v})|^{1/2},$$

the energy norm induced by a, equivalent to the $\mathbf{H}(\mathbf{curl},\Omega)$-norm. Thus the stability proven above holds for the energy norm as well. We now arrive at the construction of a hierarchical error estimator for the Galerkin method to the variational problem

Find $\mathbf{u} \in \mathbf{H}(\mathbf{curl},\Omega)$ *such that*

$$a(\mathbf{u},\mathbf{v}) = f(\mathbf{v}) \tag{5.14}$$

for all $\mathbf{v} \in \mathbf{H}(\mathbf{curl},\Omega)$ *for a given right hand side* $f \in \mathbf{H}(\mathbf{curl},\Omega)'$.

Denote with $\mathbf{u}_h$ and $\mathbf{u}_2$ the solutions of the Galerkin formulations in $\mathcal{ND}_1(\mathcal{T}_h)$ resp. in $\mathcal{ND}_2(\mathcal{T}_h)$. A crucial requirement now needed is the *saturation assumption*: There exists a sequence $(\delta_h)_h$ with $\delta_h \le \delta < 1$ such that

$$\|\mathbf{u}-\mathbf{u}_2\|_{\mathfrak{E}} \le \delta_h \|\mathbf{u}-\mathbf{u}_h\|_{\mathfrak{E}}. \tag{5.15}$$

One infers that the error $\|\mathbf{u}-\mathbf{u}_h\|$ is equivalent to $\|\mathbf{u}_2-\mathbf{u}_h\|$:

Lemma 5.3. *If the saturation assumption* (5.15) *holds, one has*

$$\|\mathbf{e}_2\|_{\mathfrak{E}} \le \|\mathbf{u}-\mathbf{u}_h\|_{\mathfrak{E}} \le \frac{1}{1-\delta}\|\mathbf{e}_2\|_{\mathfrak{E}}$$

with the error term $\mathbf{e}_2 := \mathbf{u}_2 - \mathbf{u}_h$.

Proof. See [1, Lemma 1] or [21, Equation (4.13)]. □

One thus seeks an estimate of $\|\mathbf{e}_2\|_{\mathfrak{E}}$, preferably of local type. Note that $\mathbf{e}_2 = \mathbf{u}_2 - \mathbf{u}_h$ satisfies the defect equation

$$a(\mathbf{e}_2,\boldsymbol{\eta}) = r(\boldsymbol{\eta}) := f(\boldsymbol{\eta}) - a(\mathbf{u}_h,\boldsymbol{\eta}) \quad \forall \boldsymbol{\eta} \in \mathcal{ND}_2(\mathcal{T}_h). \tag{5.16}$$

Now let $\tilde{a}$ be the decoupled sesquilinear form defined on $\mathcal{ND}_2(\mathcal{T}_h) \times \mathcal{ND}_2(\mathcal{T}_h)$ through the sesquilinear form a and the decompositions (5.4) resp. (5.5), i.e.

$$\tilde{a}(\mathbf{u}_2,\mathbf{v}_2) := a(P_1\mathbf{u}_2,P_1\mathbf{v}_2) + \sum_{i=1}^{M} a(R^{(e_i)}\mathbf{u}_2,R^{(e_i)}\mathbf{v}_2) + \sum_{j=1}^{N} a(P^{(F_j)}\mathbf{u}_2,P^{(F_j)}\mathbf{v}_2) \tag{5.17}$$

for tetrahedra and

$$\begin{aligned}\tilde{a}(\mathbf{u}_2, \mathbf{v}_2) :=& a(\tilde{P}_1\mathbf{u}_2, \tilde{P}_1\mathbf{v}_2) + \sum_{i=1}^{M} a(\tilde{R}^{(e_i)}\mathbf{u}_2, \tilde{R}^{(e_i)}\mathbf{v}_2) \\ &+ \sum_{j=1}^{N} \Big(a(\tilde{R}^{(F_j)}\mathbf{u}_2, \tilde{R}^{(F_j)}\mathbf{v}_2) + a(\tilde{P}^{(F_j)}\mathbf{u}_2, \tilde{P}^{(F_j)}\mathbf{v}_2) \Big) \\ &+ \sum_{k=1}^{L} \Big(a(\tilde{R}^{(T_k)}\mathbf{u}_2, \tilde{R}^{(T_k)}\mathbf{v}_2) + a(\tilde{P}^{(T_k)}\mathbf{u}_2, \tilde{P}^{(T_k)}\mathbf{v}_2) \Big)\end{aligned} \tag{5.18}$$

for hexahedra. Lemma 5.2 states that $\tilde{a}$ is equivalent to a, i.e. there holds $\tilde{a}(\mathbf{u}_2, \mathbf{u}_2) \simeq a(\mathbf{u}_2, \mathbf{u}_2)$. Hence, $\tilde{a}$ is continuous on $(\mathbf{H}(\mathbf{curl}, \Omega))^2$ and satisfies there a Gårdings inequality. Now define the error term $\tilde{\mathbf{e}}_2 \in \mathcal{ND}_2(\mathcal{T}_h)$ by

$$\tilde{a}(\tilde{\mathbf{e}}_2, \boldsymbol{\eta}) = a(\mathbf{e}_2, \boldsymbol{\eta}) = r(\boldsymbol{\eta}) := f(\boldsymbol{\eta}) - a(\mathbf{u}_h, \boldsymbol{\eta}) \quad \forall \boldsymbol{\eta} \in \mathcal{ND}_2(\mathcal{T}_h). \tag{5.19}$$

We expect $\tilde{\mathbf{e}}_2$ to be a good approximation of $\mathbf{e}_2 \in \mathcal{ND}_2(\mathcal{T}_h)$, and indeed there holds

Lemma 5.4. *For* $\tilde{\mathbf{e}}_2$ *defined by* (5.19) *there holds*

$$\|\tilde{\mathbf{e}}_2\|_{\mathfrak{E}} \simeq \|\mathbf{e}_2\|_{\mathfrak{E}}.$$

Proof. See [1, Lemma 2]. □

Now let P be a projection operator from (5.4) resp. (5.5) and let $V_P \subset \mathcal{ND}_2(\mathcal{T}_h)$ be the corresponding subspace. Because of the decoupling character of the sesquilinear form $\tilde{a}$, the defect equation (5.19) can be solved locally, yielding the defect equations

$$a(P\tilde{\mathbf{e}}_2, \boldsymbol{\eta}) = r(\boldsymbol{\eta}) := f(\boldsymbol{\eta}) - a(\mathbf{u}_h, \boldsymbol{\eta}) \quad \forall \boldsymbol{\eta} \in V_P \subset \mathcal{ND}_2(\mathcal{T}_h). \tag{5.20}$$

In particular, for $V_P = \mathcal{ND}_1(\mathcal{T}_h)$ there holds the localized equation

$$a(P_1\tilde{\mathbf{e}}_2, \boldsymbol{\eta}) = f(\boldsymbol{\eta}) - a(\mathbf{u}_h, \boldsymbol{\eta}) = 0 \quad \forall \boldsymbol{\eta} \in \mathcal{ND}_1(\mathcal{T}_h).$$

Thus $P_1\tilde{\mathbf{e}}_2 = 0$, so that $\tilde{\mathbf{e}}_2$ actually lies in the hierarchical surplus $\widetilde{\mathcal{ND}}_2(\mathcal{T}_h) := (Id - \Pi^{\mathcal{ND}_1})\mathcal{ND}_2(\mathcal{T}_h)$. Regarding the other subspaces for the tetrahedral case, (5.20) yields the one-dimensional problems

For $i = 1, \ldots, M$ *find* $\psi^{(e_i)} \in \operatorname{span}\{\phi^{(e_i)}\}$ *such that*

$$(\beta\, \mathbf{grad}\, \psi^{(e_i)}, \mathbf{grad}\, \phi^{(e_i)})_\Omega = r(\mathbf{grad}\, \phi^{(e_i)}) \tag{5.21}$$

and the two-dimensional problems

For $j = 1, \ldots, N$ *find* $\Psi^{(F_j)} \in \operatorname{span}\{\mathbf{b}_1^{(F_j)}, \mathbf{b}_2^{(F_j)}\}$ *such that*

$$a(\Psi^{(F_j)}, \mathbf{b}_2^{\perp}) = r(\mathbf{b}_2^{\perp}) \quad \forall \mathbf{b}_2^{\perp} \in \operatorname{span}\{\mathbf{b}_1^{(F_j)}, \mathbf{b}_2^{(F_j)}\}. \tag{5.22}$$

There holds $\mathbf{grad}\,\psi^{(e)} = R^{(e)}\tilde{\mathbf{e}}_2$ and $\Psi^{(F)} = P^{(F)}\tilde{\mathbf{e}}_2$. We now define

$$\Theta^{(e)} := \|\mathbf{grad}\,\psi^{(e)}\|_{\mathfrak{E}},$$
$$\Theta^{(F)} := \|\Psi^{(F)}\|_{\mathfrak{E}}$$

and obtain by virtue of the stability result (5.10):

Proposition 5.5 (Theorem 1 of [1]). *If the saturation assumption* (5.15) *is satisfied, then on a tetrahedral grid there holds*

$$\eta \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{\mathfrak{E}} \lesssim \frac{1}{1-\delta}\eta$$

with the local a posteriori error estimator

$$\eta^2 := \sum_{i=1}^{M} \left(\Theta^{(e_i)}\right)^2 + \sum_{j=1}^{N} \left(\Theta^{(F_j)}\right)^2 .$$

Proof. The assertion follows from Lemmas 5.3 and 5.4. It is $\eta = \|\tilde{\mathbf{e}}_2\|_{\mathfrak{E}}$. □

Now let the edges of a tetrahedral element T be numbered by the indices $i = 0, \dots, 5$, and let the sides be numbered by $j = 0, \dots, 3$. Then the local contribution on T is given by

$$\eta_T^2 = \sum_{i=0}^{5} \frac{1}{k_i} \left(\Theta^{(e_i)}\right)^2 + \frac{1}{2}\sum_{j=0}^{3} \left(\Theta^{(F_j)}\right)^2 ,$$

where k_i denotes the number of elements sharing the edge with index i. An adaptive mesh refining algorithm would now consist of computing the local error estimators η_T for every element T of $\mathcal{T}_h$. The element is refined if this value exceeds a certain limit (usually depending on the mean value or the maximum of the η_T's, according to the chosen strategy). Of course additional refinement must be performed to maintain mesh regularity.

One last simplification arises from the fact that the defect problem for $\mathbf{grad}\,\phi^{(e_i)}$ is one-dimensional: Simple computations yield

$$\Theta^{(e)} = \frac{|f(\mathbf{grad}\,\phi^{(e)}) - a(\mathbf{u}_h, \mathbf{grad}\,\phi^{(e)})|}{\|\mathbf{grad}\,\phi^{(e)}\|_{\mathfrak{E}}}.$$

For the two-dimensional problems we can write

$$\Theta^{(F)} = \|\Psi^{(F)}\|_{\mathfrak{E}} = \|\kappa_1 \mathbf{b}_1^{(F)} + \kappa_2 \mathbf{b}_2^{(F)}\|_{\mathfrak{E}},$$

where $(\kappa_1, \kappa_2)^{\mathsf{T}}$ are solutions to the LSE

$$\begin{pmatrix} a(\mathbf{b}_1^{(F)}, \mathbf{b}_1^{(F)}) & a(\mathbf{b}_2^{(F)}, \mathbf{b}_1^{(F)}) \\ a(\mathbf{b}_1^{(F)}, \mathbf{b}_2^{(F)}) & a(\mathbf{b}_2^{(F)}, \mathbf{b}_2^{(F)}) \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} = \begin{pmatrix} f(\mathbf{b}_1^{(F)}) - a(\mathbf{u}_h, \mathbf{b}_1^{(F)}) \\ f(\mathbf{b}_2^{(F)}) - a(\mathbf{u}_h, \mathbf{b}_2^{(F)}) \end{pmatrix}.$$

For hexahedra this procedure yields the following theorem; the various Θ's are defined thereafter.

Proposition 5.6. *If the saturation assumption* (5.15) *is satisfied, then on a hexahedral grid there holds*

$$\eta \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{\mathfrak{E}} \lesssim \frac{1}{1-\delta}\eta$$

with the local a posteriori error estimator

$$\eta^2 := \sum_{i=1}^{M} \left(\Theta^{(e_i)}\right)^2 + \sum_{j=1}^{N} \left(\left(\Theta_1^{(F_j)}\right)^2 + \left(\Theta_2^{(F_j)}\right)^2\right) + \sum_{k=1}^{L} \left(\left(\Theta_1^{(T_k)}\right)^2 + \left(\Theta_2^{(T_k)}\right)^2\right).$$

The local contributions on an element T are

$$\eta_T^2 := \frac{1}{4}\sum_{i=0}^{11} \left(\Theta^{(e_i)}\right)^2 + \frac{1}{2}\sum_{j=0}^{5} \left(\left(\Theta_1^{(F_j)}\right)^2 + \left(\Theta_2^{(F_j)}\right)^2\right) + \left(\Theta_1^{(T)}\right)^2 + \left(\Theta_2^{(T)}\right)^2.$$

We have

$$\begin{aligned}
\Theta^{(e)} &:= \frac{|f(\mathbf{grad}\,\phi^{(e)}) - a(\mathbf{u}_h, \mathbf{grad}\,\phi^{(e)})|}{\|\mathbf{grad}\,\phi^{(e)}\|_{\mathfrak{E}}},\\
\Theta_1^{(F)} &:= \frac{|f(\mathbf{grad}\,\phi^{(F)}) - a(\mathbf{u}_h, \mathbf{grad}\,\phi^{(F)})|}{\|\mathbf{grad}\,\phi^{(F)}\|_{\mathfrak{E}}},\\
\Theta_1^{(T)} &:= \frac{|f(\mathbf{grad}\,\phi^{(T)}) - a(\mathbf{u}_h, \mathbf{grad}\,\phi^{(T)})|}{\|\mathbf{grad}\,\phi^{(T)}\|_{\mathfrak{E}}},\\
\Theta_2^{(F)} &:= \|\kappa_1 \mathbf{b}_1^{(F)} + \kappa_2 \widetilde{\mathbf{b}}_2^{(F)} + \kappa_3 \mathbf{b}_3^{(F)}\|_{\mathfrak{E}},
\end{aligned}$$

where $\widetilde{\mathbf{b}}_2^{(F)} := \mathbf{b}_2^{(F)} + \mathbf{b}_4^{(F)}$ and $(\kappa_1, \kappa_2, \kappa_3)^\mathsf{T}$ is the solution of the algebraic system

$$\begin{pmatrix} a(\mathbf{b}_1^{(F)}, \mathbf{b}_1^{(F)}) & a(\widetilde{\mathbf{b}}_2^{(F)}, \mathbf{b}_1^{(F)}) & a(\mathbf{b}_3^{(F)}, \mathbf{b}_1^{(F)}) \\ a(\mathbf{b}_1^{(F)}, \widetilde{\mathbf{b}}_2^{(F)}) & a(\widetilde{\mathbf{b}}_2^{(F)}, \widetilde{\mathbf{b}}_2^{(F)}) & a(\mathbf{b}_3^{(F)}, \widetilde{\mathbf{b}}_2^{(F)}) \\ a(\mathbf{b}_1^{(F)}, \mathbf{b}_3^{(F)}) & a(\widetilde{\mathbf{b}}_2^{(F)}, \mathbf{b}_3^{(F)}) & a(\mathbf{b}_3^{(F)}, \mathbf{b}_3^{(F)}) \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{pmatrix} = \begin{pmatrix} f(\mathbf{b}_1^{(F)}) - a(\mathbf{u}_h, \mathbf{b}_1^{(F)}) \\ f(\widetilde{\mathbf{b}}_2^{(F)}) - a(\mathbf{u}_h, \widetilde{\mathbf{b}}_2^{(F)}) \\ f(\mathbf{b}_3^{(F)}) - a(\mathbf{u}_h, \mathbf{b}_3^{(F)}) \end{pmatrix},$$

and

$$\Theta_2^{(T)} := \|\sum_{\ell=1}^{5} \kappa_\ell \widetilde{\mathbf{b}}_\ell^{(T)}\|_{\mathfrak{E}},$$

where $\widetilde{\mathbf{b}}_1^{(T)} := \mathbf{b}_1^{(T)}$, $\widetilde{\mathbf{b}}_2^{(T)} := \mathbf{b}_2^{(T)} - \mathbf{b}_4^{(T)}$, $\widetilde{\mathbf{b}}_3^{(T)} := \mathbf{b}_3^{(T)}$, $\widetilde{\mathbf{b}}_4^{(T)} := \mathbf{b}_4^{(T)} - \mathbf{b}_6^{(T)}$, $\widetilde{\mathbf{b}}_5^{(T)} := \mathbf{b}_5^{(T)}$ and $(\kappa_1, \dots, \kappa_5)^\mathsf{T}$ is the solution of the algebraic system

$$\big(a(\widetilde{\mathbf{b}}_k^{(T)}, \widetilde{\mathbf{b}}_\ell^{(T)})\big)_{k,\ell=1,\dots,5} \big(\kappa_\ell\big)_{\ell=1,\dots,5} = \big(f(\widetilde{\mathbf{b}}_k^{(T)}) - a(\mathbf{u}_h, \widetilde{\mathbf{b}}_k^{(T)})\big)_{k=1,\dots,5}.$$

5.2. **Decomposition of** $\mathcal{RT}_2(\mathcal{K}_h)$**.** We now turn our attention to the trace space $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. We aim to find a $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$-stable decomposition of $\mathcal{RT}_2(\mathcal{K}_h)$ using the results of the last section. Let m denote the number of edges and n the number of elements in $\mathcal{K}_h$, the triangular or quadrilateral trace mesh of $\mathcal{T}_h$. We apply the trace mapping (4.3) to decomposition (5.1) for tetrahedra and obtain the decomposition

$$\mathcal{RT}_2(\mathcal{K}_h) = \mathcal{RT}_1(\mathcal{K}_h) \oplus \mathbf{curl}_\Gamma\, \widetilde{\mathcal{S}}_2(\mathcal{K}_h) \oplus \widetilde{\mathcal{RT}}_2^{\perp}(\mathcal{K}_h) \tag{5.23}$$

for triangles, where

$$\widetilde{\mathcal{RT}}_2^{\perp}(\mathcal{K}_h) := \{\boldsymbol{\lambda}_h \in \mathcal{RT}_2(\mathcal{K}_h) : \int_e \boldsymbol{\lambda}_h \cdot \mathbf{n}\, q\, ds = 0, \forall q \in \mathbb{P}_1,\, e \text{ side of } \mathcal{K}_h\}$$

and $\widetilde{\mathcal{S}}_k(\mathcal{K}_h) := \mathcal{S}_k(\mathcal{K}_h) \setminus \mathcal{S}_{k-1}(\mathcal{K}_h)$. Here $\mathcal{S}_k(\mathcal{K}_h)$ is the space of piecewise polynomials in two dimensions of degree k.

For $K \in \mathcal{K}_h$ there holds $|\mathcal{S}_k(K)| = \frac{1}{2}(k+1)(k+2)$, and the dimension of $\mathcal{RT}_2(K)$ is $|\mathcal{RT}_2(K)| = 8$, corresponding to two basis functions per side and two inner functions. If $K \in \mathcal{K}_h$ is the face of the element $T \in \mathcal{T}_h$, then its three sides are three edges of T, so that the three basis functions spanning $\mathcal{RT}_1(K)$ are the images of the the three basis functions of $\mathcal{ND}_1(T)$ corresponding to those edges under the mapping $\gamma_t^\times$. The three basis functions of $\widetilde{\mathcal{S}}_2(K)$ are the images of the three basis functions of $\widetilde{\mathcal{S}}_2(T)$ corresponding to those edges and the two basis functions spanning $\widetilde{\mathcal{RT}}_2^{\perp}(K)$ are the images of the two basis functions of $\widetilde{\mathcal{ND}}_1^{\perp}(T)$ corresponding to the face K. Counting the basis functions yields that (5.23) is a direct sum. We write $\widetilde{\mathcal{S}}_2(\mathcal{K}_h) = \mathrm{span}\{\varphi^{(e_1)}, \ldots, \varphi^{(e_m)}\}$ and $\widetilde{\mathcal{RT}}_2^{\perp}(\mathcal{K}_h) = \mathrm{span}\{\boldsymbol{\lambda}_1^{(K_1)}, \boldsymbol{\lambda}_2^{(K_1)}, \ldots, \boldsymbol{\lambda}_1^{(K_n)}, \boldsymbol{\lambda}_2^{(K_n)}\}$. Localization as before yields

$$\mathcal{RT}_2(\mathcal{K}_h) = \mathcal{RT}_1(\mathcal{K}_h) \oplus \sum_{i=1}^{m} \mathrm{span}\{\mathbf{curl}_\Gamma\, \varphi^{(e_i)}\} \oplus \sum_{j=1}^{n} \mathrm{span}\{\boldsymbol{\lambda}_1^{(K_j)}, \boldsymbol{\lambda}_2^{(K_j)}\}. \tag{5.24}$$

For the trace mesh of a hexahedral grid we obtain the decomposition

$$\mathcal{RT}_2(\mathcal{K}_h) = \mathcal{RT}_1(\mathcal{K}_h) \oplus \mathbf{curl}_\Gamma\, \widetilde{\mathcal{S}}_2(\mathcal{K}_h) \oplus \widetilde{\mathcal{RT}}_2^{\perp,-}(\mathcal{K}_h), \tag{5.25}$$

with $\mathbf{curl}_\Gamma\, \widetilde{\mathcal{S}}_2(K) = \{\mathbf{curl}_\Gamma\, \varphi^{(e_0)}, \ldots, \mathbf{curl}_\Gamma\, \varphi^{(e_3)}, \mathbf{curl}_\Gamma\, \varphi^{(K)}\}$ (with a suitable bubble function $\varphi^{(K)}$) and $\widetilde{\mathcal{RT}}_2^{\perp,-}(K) = \{\boldsymbol{\lambda}_1^{(K)}, \boldsymbol{\lambda}_3^{(K)}, \boldsymbol{\lambda}_2^{(K)} + \boldsymbol{\lambda}_4^{(K)}\}$ where $\boldsymbol{\lambda}_i^{(K)}$ $(i = 1, \ldots, 4)$ are the images of the basis functions $\mathbf{b}_i^{(F)}$ in $\widetilde{\mathcal{ND}}_2^{\perp}(T)$ corresponding to the face K. Again, (5.25) constitutes a direct sum (there holds $\dim \mathcal{RT}_2(K) = 12$ for quadrilateral elements), and its localization reads

$$\mathcal{RT}_2(\mathcal{K}_h) = \mathcal{RT}_1(\mathcal{K}_h) \oplus \sum_{i=1}^{m} \operatorname{span}\{\mathbf{curl}_\Gamma\, \varphi^{(e_i)}\}$$
$$\oplus \sum_{j=1}^{n} \Big(\operatorname{span}\{\mathbf{curl}_\Gamma\, \varphi^{(K_j)}\} \oplus \operatorname{span}\{\boldsymbol{\lambda}_1^{(K_j)}, \boldsymbol{\lambda}_3^{(K_j)}, \boldsymbol{\lambda}_2^{(K_j)} + \boldsymbol{\lambda}_4^{(K_j)}\} \Big). \quad (5.26)$$

The task at issue is to show the stability of (5.24) resp. (5.26). To this aim, define for the tetrahedral case the projection operators

$$p_1 : \mathcal{RT}_2(\mathcal{K}_h) \to \mathcal{RT}_1(\mathcal{K}_h),$$
$$p^{(K)} : \mathcal{RT}_2(\mathcal{K}_h) \to \operatorname{span}\{\boldsymbol{\lambda}_1^{(K)}, \boldsymbol{\lambda}_2^{(K)}\},$$
$$r^{(e)} : \mathcal{RT}_2(\mathcal{K}_h) \to \operatorname{span}\{\mathbf{curl}_\Gamma\, \varphi^{(e)}\}$$

and for quadrilaterals the projections

$$\tilde{p}_1 : \mathcal{RT}_2(\mathcal{K}_h) \to \mathcal{RT}_1(\mathcal{K}_h),$$
$$\tilde{p}^{(K)} : \mathcal{RT}_2(\mathcal{K}_h) \to \operatorname{span}\{\boldsymbol{\lambda}_1^{(K)}, \boldsymbol{\lambda}_3^{(K)}, \boldsymbol{\lambda}_2^{(K)} + \boldsymbol{\lambda}_4^{(K)}\},$$
$$\tilde{r}^{(e)} : \mathcal{RT}_2(\mathcal{K}_h) \to \operatorname{span}\{\mathbf{curl}_\Gamma\, \varphi^{(e)}\},$$
$$\tilde{r}^{(K)} : \mathcal{RT}_2(\mathcal{K}_h) \to \operatorname{span}\{\mathbf{curl}_\Gamma\, \varphi^{(K)}\},$$

so that the decompositions (5.24) resp. (5.26) can then be written as

$$\boldsymbol{\lambda}_2 = p_1\boldsymbol{\lambda}_2 + \sum_{i=1}^{m} r^{(e_i)}\boldsymbol{\lambda}_2 + \sum_{j=1}^{n} p^{(K_j)}\boldsymbol{\lambda}_2 \quad (5.27)$$

resp.

$$\boldsymbol{\lambda}_2 = \tilde{p}_1\boldsymbol{\lambda}_2 + \sum_{i=1}^{m} \tilde{r}^{(e_i)}\boldsymbol{\lambda}_2 + \sum_{j=1}^{n} \left(\tilde{r}^{(K_j)}\boldsymbol{\lambda}_2 + \tilde{p}^{(K_j)}\boldsymbol{\lambda}_2 \right). \quad (5.28)$$

Now the stability of these $\mathcal{RT}_2$-decompositions can be proven via the stability of the $\mathcal{ND}_2$-decompositions, as we will show in the following lemma. For the sake of clarity, we will denote the $\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$-norm simply by $\|\cdot\|$ in the statement of the lemma.

Lemma 5.7. *Under the assumption that there exists a continuous extension from $\mathcal{RT}_p(\mathcal{K}_h)$ to $\mathcal{ND}_p(\mathcal{T}_h)$ which also preserves the basis functions, the decompositions* (5.24) *resp.* (5.26) *are stable with respect to the $\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$-norm, i.e. for all $\boldsymbol{\lambda}_2 \in \mathcal{RT}_2(\mathcal{K}_h)$ there holds*

$$\|\boldsymbol{\lambda}_2\|^2 \simeq \|p_1\boldsymbol{\lambda}_2\|^2 + \sum_{i=1}^{m} \|p^{(e_i)}\boldsymbol{\lambda}_2\|^2 + \sum_{j=1}^{n} \|p^{(K_j)}\boldsymbol{\lambda}_2\|^2 \quad (5.29)$$

resp.

$$\|\boldsymbol{\lambda}_2\|^2 \simeq \|\tilde{p}_1\boldsymbol{\lambda}_2\|^2 + \sum_{i=1}^{m}\|\tilde{p}^{(e_i)}\boldsymbol{\lambda}_2\|^2 + \sum_{j=1}^{n}\left(\|\tilde{r}^{(K_j)}\boldsymbol{\lambda}_2\|^2 + \|\tilde{p}^{(K_j)}\boldsymbol{\lambda}_2\|^2\right). \tag{5.30}$$

Remark 1. There exists a continuous extension operator from $\mathcal{RT}_p(\mathcal{K}_h)$ to $\mathcal{ND}_p(\mathcal{T}_h)$, see Alonso & Valli [40]. But this is only valid for the whole spaces $\mathcal{ND}_p(\mathcal{T}_h)$ and $\mathcal{RT}_p(\mathcal{K}_h)$. Although, we know that for every basis functions $\boldsymbol{\phi} \in \mathcal{RT}_p(\mathcal{K}_h)$ there exists a basis function $\mathbf{b} \in \mathcal{ND}_p(\mathcal{T}_h)$ with $\gamma_t^{\times}(\mathbf{b}) = \boldsymbol{\phi}$ it is not clear if the estimate

$$\|\mathbf{b}\|_{\mathbf{H}}(\mathbf{curl}, \Omega) \leq C\,\|\boldsymbol{\phi}\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}$$

is independent of the mesh size h.

Proof. [of Lemma 5.7] We take an arbitrary $\boldsymbol{\lambda}_2 \in \mathcal{RT}_2(\mathcal{K}_h)$. We decompose $\boldsymbol{\lambda}_2$ according to (5.27) resp. (5.28) into $\boldsymbol{\lambda}_2 = \sum_{i=0}^{r}\boldsymbol{\lambda}_{2,i}$ (where $r = m+n$ for a triangular mesh and $r = m+2n$ for a quadrilateral mesh). From [40] we know that there exists a $\mathbf{u}_2 \in \mathcal{ND}_2(\mathcal{K}_h)$ with $\gamma_t^{\times}\mathbf{u}_2 = \boldsymbol{\lambda}_2$ and $\|\mathbf{u}_2\|_{\mathbf{H}(\mathbf{curl},\Omega)} \lesssim \|\boldsymbol{\lambda}_2\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}$. Thus, $\mathbf{u}_2$ owns a stable decomposition according to Lemma 5.2 $\mathbf{u}_2 = \sum_{j=0}^{K}\mathbf{u}_{2,j}$ with $K = M+2N+2L$. We now assume that for every $\boldsymbol{\lambda}_{2,i}$ there exists a $\mathbf{u}_{2,j}$ of the decomposition with $\gamma_t^{\times}\mathbf{u}_{2,j} = \boldsymbol{\lambda}_{2,i}$ and $\|\mathbf{u}_{2,j}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \lesssim \|\boldsymbol{\lambda}_{2,j}\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}$. Using the continuity of $\gamma_t^{\times}$ we then obtain the equivalences

$$\|\mathbf{u}_2\|_{\mathbf{H}(\mathbf{curl},\Omega)} \simeq \|\boldsymbol{\lambda}_2\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)},$$
$$\|\mathbf{u}_{2,i}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \simeq \|\boldsymbol{\lambda}_{2,i}\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)},\ i = 1,\dots,r.$$

This, together with the $\mathcal{ND}_2$-stability $\sum_{i=0}^{r}\|\mathbf{u}_{2,i}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \simeq \|\mathbf{u}_2\|_{\mathbf{H}(\mathbf{curl},\Omega)}$ in Lemma 5.2 proves the statement of the lemma.

□

Now let $\mathcal{V}$ (V) denote the vectorial (scalar) single layer potential operator for the Laplace equation defined for vector (scalar) functions $\boldsymbol{\lambda}$ (λ) by

$$\mathcal{V}(\boldsymbol{\lambda})(\mathbf{x}) := \gamma_t \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\boldsymbol{\lambda}(\mathbf{x})\,dS(\mathbf{y}), \qquad \mathbf{x} \in \Gamma,$$
$$V(\boldsymbol{\lambda})(\mathbf{x}) := \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\lambda(\mathbf{x})\,dS(\mathbf{y}), \qquad \mathbf{x} \in \Gamma,$$

with the Laplace-kernel $\Phi(\mathbf{x},\mathbf{y}) := \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$, then we can define on $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ a continuous sesquilinear form b which satisfies a Gårdings inequality by

$$b(\boldsymbol{\lambda},\mathbf{w}) = \alpha\langle V\,\mathrm{div}_\Gamma\,\boldsymbol{\lambda}, \mathrm{div}_\Gamma\,\mathbf{w}\rangle_\Gamma + \beta\langle \mathcal{V}\boldsymbol{\lambda}, \mathbf{w}\rangle_\Gamma \tag{5.31}$$

with $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $\frac{\alpha}{\beta} \notin \mathbb{R}_{<0}$. We will consider

$$\|\boldsymbol{\lambda}\|_{\mathfrak{e}} := |b(\boldsymbol{\lambda}, \boldsymbol{\lambda})|^{1/2},$$

the energy norm induced by b, equivalent to the $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$-norm. We now search a p-hierarchical error estimator for the Galerkin method using Raviart-Thomas elements for the problem

Find $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *such that*

$$b(\boldsymbol{\lambda}, \mathbf{w}) = g(\mathbf{w}) \tag{5.32}$$

for all $\mathbf{w} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

with a right-hand side $g \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)'$. Having proven the stability estimates (5.29) and (5.30), the principal work has already been done. We now must simply proceed analogously to the construction of the error estimator for Nédélec elements.

Let $\boldsymbol{\lambda}_h$ and $\boldsymbol{\lambda}_2$ denote the solutions to the Galerkin formulations in $\mathcal{RT}_1(\mathcal{K}_h)$ and $\mathcal{RT}_2(\mathcal{K}_h)$. We again require the *saturation assumption*

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_2\|_{\mathfrak{e}} \leq \delta_h \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\mathfrak{e}} \tag{5.33}$$

to hold with $\delta_h \leq \delta < 1$. Exactly as before in Lemma 5.3 we then have:

Lemma 5.8. *If the saturation assumption* (5.33) *holds, one has*

$$\|\boldsymbol{\varepsilon}_2\|_{\mathfrak{e}} \leq \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\mathfrak{e}} \leq \frac{1}{1-\delta}\|\boldsymbol{\varepsilon}_2\|_{\mathfrak{e}}$$

with the error term $\boldsymbol{\varepsilon}_2 := \boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_h$.

We now define a decoupled sesquilinear form $\tilde{b}$ on $\mathcal{RT}_2(\mathcal{K}_h) \times \mathcal{RT}_2(\mathcal{K}_h)$ according to the decompositions (5.27) and (5.28) via

$$\tilde{b}(\boldsymbol{\lambda}_2, \mathbf{w}_1) := b(p_1\boldsymbol{\lambda}_2, p_1\mathbf{w}_1) + \sum_{i=1}^{m} b(r^{(e_i)}\boldsymbol{\lambda}_2, r^{(e_i)}\mathbf{w}_1) + \sum_{j=1}^{n} b(p^{(K_j)}\boldsymbol{\lambda}_2, p^{(K_j)}\mathbf{w}_1) \tag{5.34}$$

for triangles and

$$\begin{aligned}\tilde{b}(\boldsymbol{\lambda}_2, \mathbf{w}_1) := b(\tilde{p}_1\boldsymbol{\lambda}_2, \tilde{p}_1\mathbf{w}_1) + \sum_{i=1}^{m} b(\tilde{r}^{(e_i)}\boldsymbol{\lambda}_2, \tilde{r}^{(e_i)}\mathbf{w}_1)\\ + \sum_{j=1}^{n}\Big(b(\tilde{r}^{(K_j)}\boldsymbol{\lambda}_2, \tilde{r}^{(K_j)}\mathbf{w}_1) + b(\tilde{p}^{(K_j)}\boldsymbol{\lambda}_2, \tilde{p}^{(K_j)}\mathbf{w}_1)\Big)\end{aligned} \tag{5.35}$$

for quadrilaterals. Thanks to Lemma 5.7, $\tilde{b}$ is equivalent to b and thus continuous on $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and satisfies a Gårdings inequality. Define the error term $\tilde{\boldsymbol{\varepsilon}}_2 \in \mathcal{RT}_2(\mathcal{K}_h)$ by

$$\tilde{b}(\tilde{\boldsymbol{\varepsilon}}_2, \boldsymbol{\zeta}) = b(\boldsymbol{\varepsilon}_2, \boldsymbol{\zeta}) = g(\boldsymbol{\eta}) - b(\boldsymbol{\lambda}_h, \boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in \mathcal{RT}_2(\mathcal{K}_h). \tag{5.36}$$

Just as before in Lemma 5.4 there now holds

Lemma 5.9. *For* $\tilde{\varepsilon}_2$ *defined by* (5.36) *there holds*

$$\|\tilde{\varepsilon}_2\|_{\mathfrak{e}} \simeq \|\varepsilon_2\|_{\mathfrak{e}}.$$

Setting $\eta = \|\varepsilon\|_{\mathfrak{e}}$, the last two lemmas immediately give the following two theorems:

Proposition 5.10. *If the saturation assumption* (5.33) *is satisfied, then on a triangular grid there holds*

$$\eta \lesssim \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\mathfrak{e}} \lesssim \frac{1}{1-\delta}\eta$$

with the local a posteriori estimator

$$\eta^2 := \sum_{i=1}^{m} \left(\vartheta^{(e_i)}\right)^2 + \sum_{j=1}^{n} \left(\vartheta^{(K_j)}\right)^2.$$

The local contribution on a triangle K (with sides corresponding to the indices $i = 0, 1, 2$) is

$$\eta_K^2 := \frac{1}{2}\sum_{i=0}^{2} \left(\vartheta^{(e_i)}\right)^2 + \left(\vartheta^{(K)}\right)^2$$

with

$$\vartheta^{(e)} := \frac{|g(\mathbf{curl}_\Gamma\, \varphi^{(e)}) - b(\boldsymbol{\lambda}_h, \mathbf{curl}_\Gamma\, \varphi^{(e)})|}{\|\mathbf{curl}_\Gamma\, \varphi^{(e)}\|_{\mathfrak{e}}},$$

$$\vartheta^{(K)} := \|\kappa_1 \boldsymbol{\lambda}_1^{(K)} + \kappa_2 \boldsymbol{\lambda}_2^{(K)}\|_{\mathfrak{e}},$$

where $(\kappa_1, \kappa_2)^\mathsf{T}$ is the solution of the LSE

$$\begin{pmatrix} b(\boldsymbol{\lambda}_1^{(K)}, \boldsymbol{\lambda}_1^{(K)}) & b(\boldsymbol{\lambda}_2^{(K)}, \boldsymbol{\lambda}_1^{(K)}) \\ b(\boldsymbol{\lambda}_1^{(K)}, \boldsymbol{\lambda}_2^{(K)}) & b(\boldsymbol{\lambda}_2^{(K)}, \boldsymbol{\lambda}_2^{(K)}) \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} = \begin{pmatrix} g(\boldsymbol{\lambda}_1^{(K)}) - b(\boldsymbol{\lambda}_h, \boldsymbol{\lambda}_1^{(K)}) \\ g(\boldsymbol{\lambda}_2^{(K)}) - b(\boldsymbol{\lambda}_h, \boldsymbol{\lambda}_2^{(K)}) \end{pmatrix}.$$

The analogous statement for the quadrilateral case reads:

Proposition 5.11. *If the saturation assumption* (5.33) *is satisfied, then on a quadrilateral grid there holds*

$$\eta \lesssim \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\mathfrak{e}} \lesssim \frac{1}{1-\delta}\eta$$

with the local a posteriori error estimator

$$\eta^2 := \sum_{i=1}^{m} \left(\vartheta^{(e_i)}\right)^2 + \sum_{j=1}^{n} \left(\left(\vartheta_1^{(K_j)}\right)^2 + \left(\vartheta_2^{(K_j)}\right)^2 \right).$$

Here, the local contribution on an element K (whose sides correspond to the indices $i = 1, 2, 3, 4$) is

$$\eta_K^2 := \frac{1}{2}\sum_{i=1}^{4} \left(\vartheta^{(e_i)}\right)^2 + \left(\vartheta_1^{(K)}\right)^2 + \left(\vartheta_2^{(K)}\right)^2,$$

with

$$\vartheta^{(e)} := \frac{|g(\mathbf{curl}_\Gamma\,\varphi^{(e)}) - b(\boldsymbol{\lambda}_h, \mathbf{curl}_\Gamma\,\varphi^{(e)})|}{\|\mathbf{curl}_\Gamma\,\varphi^{(e)}\|_{\mathfrak{e}}},$$
$$\vartheta_1^{(K)} := \frac{|g(\mathbf{curl}_\Gamma\,\varphi^{(K)}) - b(\boldsymbol{\lambda}_h, \mathbf{curl}_\Gamma\,\varphi^{(K)})|}{\|\mathbf{curl}_\Gamma\,\varphi^{(K)}\|_{\mathfrak{e}}},$$
$$\vartheta_2^{(K)} := \|\kappa_1\boldsymbol{\lambda}_1^{(K)} + \kappa_2\widetilde{\boldsymbol{\lambda}}_2^{(K)} + \kappa_3\boldsymbol{\lambda}_3^{(K)}\|_{\mathfrak{e}},$$

where $\widetilde{\boldsymbol{\lambda}}_2^{(K)} := \boldsymbol{\lambda}_2^{(K)} + \boldsymbol{\lambda}_4^{(K)}$ and $(\kappa_1, \kappa_2, \kappa_3)^\mathsf{T}$ is the solution of the algebraic system

$$\begin{pmatrix} b(\boldsymbol{\lambda}_1^{(K)}, \boldsymbol{\lambda}_1^{(K)}) & b(\widetilde{\boldsymbol{\lambda}}_2^{(K)}, \boldsymbol{\lambda}_1^{(K)}) & b(\boldsymbol{\lambda}_3^{(K)}, \boldsymbol{\lambda}_1^{(K)}) \\ b(\boldsymbol{\lambda}_1^{(K)}, \widetilde{\boldsymbol{\lambda}}_2^{(K)}) & b(\widetilde{\boldsymbol{\lambda}}_2^{(K)}, \widetilde{\boldsymbol{\lambda}}_2^{(K)}) & b(\boldsymbol{\lambda}_3^{(K)}, \widetilde{\boldsymbol{\lambda}}_2^{(K)}) \\ b(\boldsymbol{\lambda}_1^{(K)}, \boldsymbol{\lambda}_1^{(K)}) & b(\widetilde{\boldsymbol{\lambda}}_2^{(K)}, \boldsymbol{\lambda}_1^{(K)}) & b(\boldsymbol{\lambda}_3^{(K)}, \boldsymbol{\lambda}_1^{(K)}) \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_2 \end{pmatrix}$$
$$= \begin{pmatrix} g(\boldsymbol{\lambda}_1^{(K)}) - b(\boldsymbol{\lambda}_h, \boldsymbol{\lambda}_1^{(K)}) \\ g(\widetilde{\boldsymbol{\lambda}}_2^{(K)}) - b(\boldsymbol{\lambda}_h, \widetilde{\boldsymbol{\lambda}}_2^{(K)}) \\ g(\boldsymbol{\lambda}_3^{(K)}) - b(\boldsymbol{\lambda}_h, \boldsymbol{\lambda}_3^{(K)}) \end{pmatrix}. \quad (5.37)$$

6. Application to the coupling formulation

We will now apply the theory of the last section to the symmetric eddy current formulation coupling finite elements in a bounded domain and boundary elements on the boundary for the homogeneous exterior domain as described in Section 3. To derive a p-hierarchical error estimator for the Galerkin method (3.1), let $\mathcal{X} := \mathbf{H}(\mathbf{curl}, \Omega) \times \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma\, 0, \Gamma)$ denote the continuous space of the variational formulation, $\mathcal{X}_h := \mathcal{ND}_1(\mathcal{T}_h) \times \mathbf{curl}_\Gamma\tilde{\mathcal{S}}_1(\mathcal{K}_h)$ the finite element space of the Galerkin formulation and $\mathcal{X}_2 := \mathcal{ND}_2(\mathcal{T}_h) \times \mathbf{curl}_\Gamma\tilde{\mathcal{S}}_2(\mathcal{K}_h)$ the higher order finite element space, and let

$$\begin{aligned} \mathcal{A}(\mathbf{u}, \boldsymbol{\lambda}; \mathbf{v}, \boldsymbol{\zeta}) := (\mu^{-1}\,\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_\Omega + i\omega(\sigma\mathbf{u}, \mathbf{v})_\Omega - \langle \mathcal{W}\mathbf{u}_\Gamma, \mathbf{v}_\Gamma\rangle_\Gamma \\ + \langle \tilde{\mathcal{K}}\boldsymbol{\lambda}, \mathbf{v}_\Gamma\rangle_\Gamma + \langle (I - \mathcal{K})\mathbf{u}_\Gamma, \boldsymbol{\zeta}\rangle_\Gamma + \langle \mathcal{V}\boldsymbol{\lambda}, \boldsymbol{\zeta}\rangle_\Gamma \end{aligned} \quad (6.1)$$

be the sesquilinear form on $\mathcal{X} \times \mathcal{X}$ from (2.1) and $\mathcal{L}(\mathbf{v}, \boldsymbol{\zeta}) \in \mathcal{X}'$ the right hand side given by (2.1); there holds $\mathcal{L}(0, \boldsymbol{\zeta}) = 0$. Theorem 7.1 in [14] implies that the energy norm induced by $\mathcal{A}$ is equivalent to the natural norm $\|\cdot\|_{\mathcal{X}}$ on $\mathcal{X}$. Let us define on $\mathcal{X} \times \mathcal{X}$ the sesquilinear form

$$\mathcal{Q}(\mathbf{u}, \boldsymbol{\lambda}; \mathbf{v}, \boldsymbol{\zeta}) := a(\mathbf{u}, \mathbf{v}) + b(\boldsymbol{\lambda}, \boldsymbol{\zeta})$$

with $a(\mathbf{u}, \mathbf{v}) := (\mu^{-1}\,\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_\Omega + i\omega(\sigma\mathbf{u}, \mathbf{v})_\Omega$ and $b(\boldsymbol{\lambda}, \boldsymbol{\zeta}) := \langle \mathcal{V}\boldsymbol{\lambda}, \boldsymbol{\zeta}\rangle_\Gamma$ and the energy norms

$$\|\mathbf{v}\|_{\mathfrak{E}}^2 := |a(\mathbf{v}, \mathbf{v})|, \qquad \|\boldsymbol{\zeta}\|_{\mathfrak{e}}^2 := |b(\boldsymbol{\zeta}, \boldsymbol{\zeta})|$$

on $\mathbf{H}(\mathbf{curl}, \Omega)$ resp. $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma\, 0, \Gamma)$. Note that the sesquilinear form a corresponds to the a from (5.13) with $\alpha = \mu^{-1}$ and $\beta = i\omega\sigma$ and the sesquilinear form b corresponds to b from

(5.31) with $\beta = 1$. The $\alpha\langle V \operatorname{div}_\Gamma \boldsymbol{\lambda}, \operatorname{div}_\Gamma \boldsymbol{\zeta}\rangle_\Gamma$-part from (5.31) does not appear here, as we are dealing with divergence-free functions on Γ. We further define the "decoupled" sesquilinear forms

$$\widetilde{\mathcal{Q}}(\mathbf{u}, \boldsymbol{\lambda}; \mathbf{v}, \boldsymbol{\zeta}) = \tilde{a}(\mathbf{u}, \mathbf{v}) + \tilde{b}(\boldsymbol{\lambda}, \boldsymbol{\zeta})$$

with $\tilde{a}$ from (5.17) resp. (5.18) and $\tilde{b}$ from (5.34) resp. (5.35).

Now let $(\mathbf{u}, \boldsymbol{\lambda}) \in \mathcal{X}$ be the solution of (2.1), $(\mathbf{u}_h, \boldsymbol{\lambda}_h) \in \mathcal{X}_h$ the Galerkin solution of (3.1) and $(\mathbf{u}_2, \boldsymbol{\lambda}_2) \in \mathcal{X}_2$ the Galerkin solution on the higher order finite element space. As before, M denotes the number of edges in $\mathcal{T}_h$, $m < M$ the number of edges in $\mathcal{K}_h$ (those on Γ), N the number of faces in $\mathcal{T}_h$, $n < N$ the number of faces in $\mathcal{K}_h$ (those on Γ) and L the number of elements in $\mathcal{T}_h$. We proceed as before:

Define the error terms $(\mathbf{e}_2, \boldsymbol{\varepsilon}_2) \in \mathcal{X}_2$ by

$$\mathcal{Q}(\mathbf{e}_2, \boldsymbol{\varepsilon}_2; \mathbf{v}, \boldsymbol{\zeta}) = \mathcal{L}(\mathbf{v}, \boldsymbol{\zeta}) - \mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; \mathbf{v}, \boldsymbol{\zeta}) \quad \forall (\mathbf{v}, \boldsymbol{\zeta}) \in \mathcal{X}_2$$

and $(\tilde{\mathbf{e}}_2, \tilde{\boldsymbol{\varepsilon}}_2) \in \mathcal{X}_2$ by

$$\widetilde{\mathcal{Q}}(\tilde{\mathbf{e}}_2, \tilde{\boldsymbol{\varepsilon}}_2; \mathbf{v}, \boldsymbol{\zeta}) = \mathcal{Q}(\mathbf{e}_2, \boldsymbol{\varepsilon}_2; \mathbf{v}, \boldsymbol{\zeta}) \quad \forall (\mathbf{v}, \boldsymbol{\zeta}) \in \mathcal{X}_2.$$

Using the notation of Section 5, define for tetrahedral grids the quantities

$$\begin{aligned} \Theta^{(e_i)} &:= \|P^{(e_i)}\tilde{\mathbf{e}}_2\|_{\mathfrak{E}}, \quad i = 1, \dots, M, \\ \Theta^{(F_j)} &:= \|R^{(F_j)}\tilde{\mathbf{e}}_2\|_{\mathfrak{E}} \quad j = 1, \dots, N, \\ \vartheta^{(e_i)} &:= \|r^{(e_i)}\tilde{\boldsymbol{\varepsilon}}_2\|_{\mathfrak{e}} \quad i = 1, \dots, m. \end{aligned}$$

There then holds (again using notation from Section 5)

$$\begin{aligned} \Theta^{(e)} &= \frac{|\mathcal{L}(\mathbf{grad}\,\phi^{(e)}, 0) - \mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; \mathbf{grad}\,\phi^{(e)}, 0)|}{\|\mathbf{grad}\,\phi^{(e)}\|_{\mathfrak{E}}}, \\ \Theta^{(F)} &= \|\kappa_1 \mathbf{b}_1^{(F)} + \kappa_2 \mathbf{b}_3^{(F)}\|_{\mathfrak{E}}, \end{aligned}$$

where $(\kappa_1, \kappa_2)^{\mathsf{T}}$ is the solution of the LSE

$$\begin{pmatrix} a(\mathbf{b}_1^{(F)}, \mathbf{b}_1^{(F)}) & a(\mathbf{b}_2^{(F)}, \mathbf{b}_1^{(F)}) \\ a(\mathbf{b}_1^{(F)}, \mathbf{b}_2^{(F)}) & a(\mathbf{b}_2^{(F)}, \mathbf{b}_2^{(F)}) \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} = \begin{pmatrix} \mathcal{L}(\mathbf{b}_1^{(F)}, 0) - \mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; \mathbf{b}_1^{(F)}, 0) \\ \mathcal{L}(\mathbf{b}_2^{(F)}, 0) - \mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; \mathbf{b}_2^{(F)}, 0) \end{pmatrix},$$

and further

$$\vartheta^{(e)} = \frac{|\mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h, ; 0, \mathbf{curl}_\Gamma \varphi^{(e)})|}{\|\mathbf{curl}_\Gamma \varphi^{(e)}\|_{\mathfrak{e}}}.$$

The quantities $\vartheta^{(F_j)} := \|p^{(F_j)}\tilde{\boldsymbol{\varepsilon}}_2\|_{\mathfrak{e}}$ do not appear here, as $\tilde{\boldsymbol{\varepsilon}}_2 \in \mathbf{curl}_\Gamma \tilde{\mathcal{S}}_2$ (i.e. $p^{(F_j)}\tilde{\boldsymbol{\varepsilon}}_2 = 0$).

As usual, we now require the saturation assumption

$$\|(\mathbf{u} - \mathbf{u}_2, \boldsymbol{\lambda} - \boldsymbol{\lambda}_2)\|_{\mathcal{X}} \le \delta_h \|(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h)\|_{\mathcal{X}} \tag{6.2}$$

with a $\delta_h \le \delta < 1$. There holds

Theorem 6.1. *If the saturation assumption* (6.2) *is satisfied, then on a tetrahedral grid there holds*

$$\eta \lesssim \|(\mathbf{u}-\mathbf{u}_h, \boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\|_{\mathcal{X}} \lesssim \frac{1}{1-\delta}\eta$$

with the local a posteriori estimator

$$\eta^2 := \sum_{i=1}^{M}\left(\Theta^{(e_i)}\right)^2 + \sum_{j=1}^{N}\left(\Theta^{(F_j)}\right)^2 + \sum_{i=1}^{m}\left(\vartheta^{(e_i)}\right)^2.$$

Proof. From the continuity and coercitivity of $\mathcal{A}$ we have

$$\begin{aligned}
\|(\mathbf{u}_2-\mathbf{u}_h, \boldsymbol{\lambda}_2-\boldsymbol{\lambda}_h)\|_{\mathcal{X}}^2 &\lesssim \mathcal{A}(\mathbf{u}_2-\mathbf{u}_h, \boldsymbol{\lambda}_2-\boldsymbol{\lambda}_h; \mathbf{u}_2-\mathbf{u}_h, \boldsymbol{\lambda}_2-\boldsymbol{\lambda}_h)\\
&= \mathcal{L}(\mathbf{u}_2-\mathbf{u}_h, \boldsymbol{\lambda}_2-\boldsymbol{\lambda}_h) - \mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; \mathbf{u}_2-\mathbf{u}_h, \boldsymbol{\lambda}_2-\boldsymbol{\lambda}_h)\\
&= \mathcal{Q}(\mathbf{e}_2, \boldsymbol{\varepsilon}_2; \mathbf{u}_2-\mathbf{u}_h, \boldsymbol{\lambda}_2-\boldsymbol{\lambda}_h)\\
&\lesssim \|(\mathbf{e}_2, \boldsymbol{\varepsilon}_2)\|_{\mathcal{X}}\|(\mathbf{u}_2-\mathbf{u}_h, \boldsymbol{\lambda}_2-\boldsymbol{\lambda}_h)\|_{\mathcal{X}}.
\end{aligned}$$

Hence there holds $\|(\mathbf{u}_2-\mathbf{u}_h, \boldsymbol{\lambda}_2-\boldsymbol{\lambda}_h)\|_{\mathcal{X}} \lesssim \|(\mathbf{e}_2, \boldsymbol{\varepsilon}_2)\|_{\mathcal{X}}$. We obtain the reverse inequality in the following way:

$$\begin{aligned}
\|(\mathbf{e}_2, \boldsymbol{\varepsilon}_2)\|_{\mathcal{X}}^2 &= \mathcal{Q}(\mathbf{e}_2, \boldsymbol{\varepsilon}_2; \mathbf{e}_2, \boldsymbol{\varepsilon}_2)\\
&= \mathcal{L}(\mathbf{e}_2, \boldsymbol{\varepsilon}_2) - \mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; \mathbf{e}_2, \boldsymbol{\varepsilon}_2)\\
&= \mathcal{A}(\mathbf{u}_2-\mathbf{u}_h, \boldsymbol{\lambda}_2-\boldsymbol{\lambda}_h; \mathbf{e}_2, \boldsymbol{\varepsilon}_2)\\
&\lesssim \|(\mathbf{u}_2-\mathbf{u}_h, \boldsymbol{\lambda}_2-\boldsymbol{\lambda}_h)\|_{\mathcal{X}}\|(\mathbf{e}_2, \boldsymbol{\varepsilon}_2)\|_{\mathcal{X}},
\end{aligned}$$

i.e. $\|(\mathbf{e}_2, \boldsymbol{\varepsilon}_2)\|_{\mathcal{X}} \lesssim \|(\mathbf{u}_2-\mathbf{u}_h, \boldsymbol{\lambda}_2-\boldsymbol{\lambda}_h)\|_{\mathcal{X}}$, and so we have proven the equivalence

$$\|(\mathbf{e}_2, \boldsymbol{\varepsilon}_2)\|_{\mathcal{X}} \simeq \|(\mathbf{u}_2-\mathbf{u}_h, \boldsymbol{\lambda}_2-\boldsymbol{\lambda}_h)\|_{\mathcal{X}}.$$

Now Lemmas 5.4 and 5.9 yield $\eta := \|(\tilde{\mathbf{e}}_2, \tilde{\boldsymbol{\varepsilon}}_2)\|_{\mathcal{X}} \simeq \|(\mathbf{e}_2, \boldsymbol{\varepsilon}_2)\|_{\mathcal{X}}$, and we have thus proven the statement of the theorem. □

The same procedure for hexahedral grids yields the quantities

$$\begin{aligned}
\Theta^{(e)} &:= \frac{|\mathcal{L}(\mathbf{grad}\,\phi^{(e)}, 0) - \mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; \mathbf{grad}\,\phi^{(e)}, 0)|}{\|\mathbf{grad}\,\phi^{(e)}\|_{\mathfrak{E}}},\\
\Theta_1^{(F)} &:= \frac{|\mathcal{L}(\mathbf{grad}\,\phi^{(F)}, 0) - \mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; \mathbf{grad}\,\phi^{(F)}, 0)|}{\|\mathbf{grad}\,\phi^{(F)}\|_{\mathfrak{E}}},\\
\Theta_1^{(T)} &:= \frac{|\mathcal{L}(\mathbf{grad}\,\phi^{(T)}, 0) - \mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; \mathbf{grad}\,\phi^{(T)}, 0)|}{\|\mathbf{grad}\,\phi^{(T)}\|_{\mathfrak{E}}},\\
\Theta_2^{(F)} &:= \|\kappa_1\mathbf{b}_1^{(F)} + \kappa_2\widetilde{\mathbf{b}}_2^{(F)} + \kappa_3\mathbf{b}_3^{(F)}\|_{\mathfrak{E}},
\end{aligned}$$

where $\widetilde{\mathbf{b}}_2^{(F)} := \mathbf{b}_2^{(F)} + \mathbf{b}_4^{(F)}$ and $(\kappa_1, \kappa_2, \kappa_3)^\mathsf{T}$ is the solution of the LSE

$$\begin{pmatrix} a(\mathbf{b}_1^{(F)}, \mathbf{b}_1^{(F)}) & a(\widetilde{\mathbf{b}}_2^{(F)}, \mathbf{b}_1^{(F)}) & a(\mathbf{b}_3^{(F)}, \mathbf{b}_1^{(F)}) \\ a(\mathbf{b}_1^{(F)}, \widetilde{\mathbf{b}}_2^{(F)}) & a(\widetilde{\mathbf{b}}_2^{(F)}, \widetilde{\mathbf{b}}_2^{(F)}) & a(\mathbf{b}_3^{(F)}, \widetilde{\mathbf{b}}_2^{(F)}) \\ a(\mathbf{b}_1^{(F)}, \mathbf{b}_3^{(F)}) & a(\widetilde{\mathbf{b}}_2^{(F)}, \mathbf{b}_3^{(F)}) & a(\mathbf{b}_3^{(F)}, \mathbf{b}_3^{(F)}) \end{pmatrix} \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{pmatrix}$$
$$= \begin{pmatrix} \mathcal{L}(\mathbf{b}_1^{(F)}, 0) - \mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; \mathbf{b}_1^{(F)}, 0) \\ \mathcal{L}(\widetilde{\mathbf{b}}_2^{(F)}, 0) - \mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; \widetilde{\mathbf{b}}_2^{(F)}, 0) \\ \mathcal{L}(\mathbf{b}_3^{(F)}, 0) - \mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; \mathbf{b}_3^{(F)}, 0) \end{pmatrix},$$

and further

$$\Theta_2^{(T)} := \|\sum_{\ell=1}^{5} \kappa_\ell \widetilde{\mathbf{b}}_\ell^{(T)}\|_{\mathfrak{E}},$$

where $\widetilde{\mathbf{b}}_1^{(T)} := \mathbf{b}_1^{(T)}$, $\widetilde{\mathbf{b}}_2^{(T)} := \mathbf{b}_2^{(T)} - \mathbf{b}_4^{(T)}$, $\widetilde{\mathbf{b}}_3^{(T)} := \mathbf{b}_3^{(T)}$, $\widetilde{\mathbf{b}}_4^{(T)} := \mathbf{b}_4^{(T)} - \mathbf{b}_6^{(T)}$, $\widetilde{\mathbf{b}}_5^{(T)} := \mathbf{b}_5^{(T)}$, and $(\kappa_1, \ldots, \kappa_5)^\mathsf{T}$ is the solution of the algebraic system

$$\big(a(\widetilde{\mathbf{b}}_k^{(T)}, \widetilde{\mathbf{b}}_\ell^{(T)})\big)_{k,\ell=1,\ldots,5} \big(\kappa_\ell\big)_{\ell=1,\ldots,5} = \big(\mathcal{L}(\widetilde{\mathbf{b}}_k^{(T)}, 0) - \mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; \widetilde{\mathbf{b}}_k^{(T)}, 0)\big)_{k=1,\ldots,5},$$

and further

$$\vartheta^{(e)} := \frac{|\mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; 0, \mathbf{curl}_\Gamma \varphi^{(e)})|}{\|\mathbf{curl}_\Gamma \varphi^{(e)}\|_{\mathfrak{e}}},$$
$$\vartheta^{(F)} := \frac{|\mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; 0, \mathbf{curl}_\Gamma \varphi^{(F)})|}{\|\mathbf{curl}_\Gamma \varphi^{(F)}\|_{\mathfrak{e}}}.$$

The corresponding theorem reads (the proof is the same as for the last theorem):

Theorem 6.2. *If the saturation assumption* (6.2) *is satisfied, then on a hexahedral grid there holds*

$$\eta \lesssim \|(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h)\|_{\mathcal{X}} \lesssim \frac{1}{1-\delta}\eta$$

with the local a posteriori estimator

$$\eta^2 := \sum_{i=1}^{M} \left(\Theta^{(e_i)}\right)^2 + \sum_{j=1}^{N} \left(\left(\Theta_1^{(F_j)}\right)^2 + \left(\Theta_2^{(F_j)}\right)^2\right) + \sum_{k=1}^{L} \left(\Theta_1^{(T_k)}\right)^2 + \left(\Theta_2^{(T_k)}\right)^2$$
$$+ \sum_{i=1}^{m} \left(\vartheta^{(e_i)}\right)^2 + \sum_{j=1}^{n} \left(\vartheta^{(F_j)}\right)^2.$$

7. Numerical experiments

We perform some numerical tests on hexahedral meshes to see if the error estimator gives a reliable and efficient estimate of the Galerkin error and to test its usefulness for an adaptive refinement scheme.

Let $\mathcal{X} := \mathbf{H}(\mathbf{curl}, \Omega) \times \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$ and $\mathcal{X}_h := \mathcal{ND}_1(\mathcal{T}_h) \times \mathbf{curl}_\Gamma \mathcal{S}_1(\mathcal{K}_h)$ the finite element space as described above. Furthermore, we denote by $\mathcal{X}_2 := \mathcal{ND}_2(\mathcal{T}_h) \times \mathbf{curl}_\Gamma \mathcal{S}_1(\mathcal{K}_h)$ the higher order finite element space. Here, we just consider a mesh of hexahedrons.

We define the energy norms on $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$ by

$$\|\mathbf{v}\|_{\mathfrak{E}}^2 := \left|(\mu^{-1}\, \mathbf{curl}\, \mathbf{u}, \mathbf{curl}\, \mathbf{u})_\Omega + i\omega(\sigma\mathbf{u}, \mathbf{u})_\Omega\right|, \qquad \|\boldsymbol{\lambda}\|_{\mathfrak{e}}^2 := |\langle \mathcal{V}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_\Gamma|.$$

In the first experiment we compute the solution to the Galerkin system as given in (3.1) with $\Omega = (-1,1)^3$, $\Gamma = \partial\Omega$, on a series of uniform hexahedral meshes, obtained by dividing each edge of Ω into n equal parts. On grid n we thus have a meshwidth of $h = \frac{2}{n}$. We then compare the energy norm $\sqrt{\|\mathbf{u} - \mathbf{u}_h\|_{\mathfrak{E}}^2 + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\mathfrak{e}}^2}$ of the Galerkin error with the value of the error estimator. In the second example we will use the error estimator to perform adaptive mesh refinements. In the tables, n will denote the mesh number (as defined above) and $N_{\mathbf{u}}$ and $N_{\boldsymbol{\lambda}}$ the number of degrees of freedom for the fem resp. the bem variable. The choice of Ω in both examples is only for simplicity; note that our above analysis is not restricted to convex domains Ω. All computations were performed using the program package `maiprogs` [41] (for further details see [20]).

Example 1. We choose the exact solution

$$\mathbf{u}(\mathbf{x}) = \mathbf{curl}(\mathbf{G}\boldsymbol{\rho})(\mathbf{x}) := \mathbf{curl} \int_\Omega \frac{1}{\|\mathbf{x} - \mathbf{y}\|} \boldsymbol{\rho}(\mathbf{y})\, d\mathbf{y}$$

with the density function $\boldsymbol{\rho}(\mathbf{x}) = \rho(\mathbf{x}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, where

$$\rho(\mathbf{x}) = \left((1 - x_1^2)(1 - x_2^2)(1 - x_3^2)\right)^2 x_1 x_2 x_3.$$

We compute the Galerkin method for $n = 1, \ldots, 13$ with hexahedral elements. In Figure 2 one sees that the error indicator η behaves nearly the same as the error in energy norm, the effectivity indices $q = \frac{\eta}{e}$, calculated in Table 1, are nearly constant.

Example 2. We now use the error estimator to construct an adaptive mesh. We use hexahedral elements without hanging nodes (with the drawback that the resulting mesh is no longer form-regular). Our geometry remains the cube $\Omega = (-1,1)^3$. We set $\mu = 1$ in Ω and choose a discontinuous σ, namely

$$\sigma = \begin{cases} 0.1, & \frac{1}{3} < x_1, x_2, x_3 < 1 \\ 1, & \text{else} \end{cases}.$$

For our right hand side in (2.1) we choose the function $\mathbf{J}_0 = (1,1,1)$ in Ω and $\mathbf{J}_0 = 0$ in Ω_E. Note that also in this case (2.1) holds. We start by computing the Galerkin solution for the

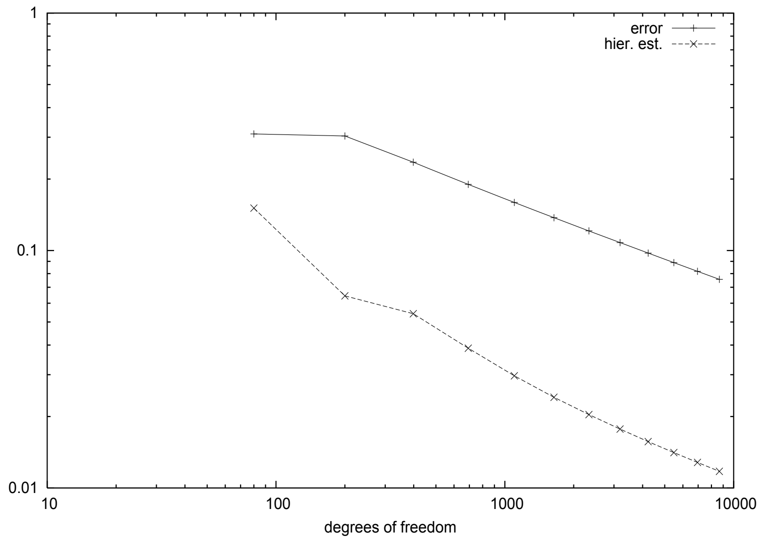

FIGURE 2. Energy norm e of the Galerkin error and the 2-level hierarchical error estimator η of Example 1.

n	h	DOF	e	η	$q = \frac{\eta}{e}$
2	1	80	0.30987	0.15081	0.4867
3	0.667	200	0.30369	0.06440	0.2121
4	0.5	398	0.23548	0.05420	0.2302
5	0.4	692	0.18994	0.03879	0.2042
6	0.333	1100	0.15938	0.02969	0.1863
7	0.143	1640	0.13748	0.02410	0.1753
8	0.25	2330	0.12095	0.02037	0.1684
9	0.222	3188	0.10800	0.01770	0.1639
10	0.2	4232	0.09755	0.01568	0.1607
11	0.091	5480	0.08894	0.01409	0.1584
12	0.083	6950	0.08172	0.01281	0.1568
13	0.077	8660	0.07558	0.01175	0.1555

TABLE 1. Energy norm e of the Galerkin error, the 2-level hierarchical error estimator η and the effectivity indices $q = \frac{\eta}{e}$ of Example 1.

uniform mesh with $n = 3$. The refinement algorithm then proceeds by first refining the 10% of

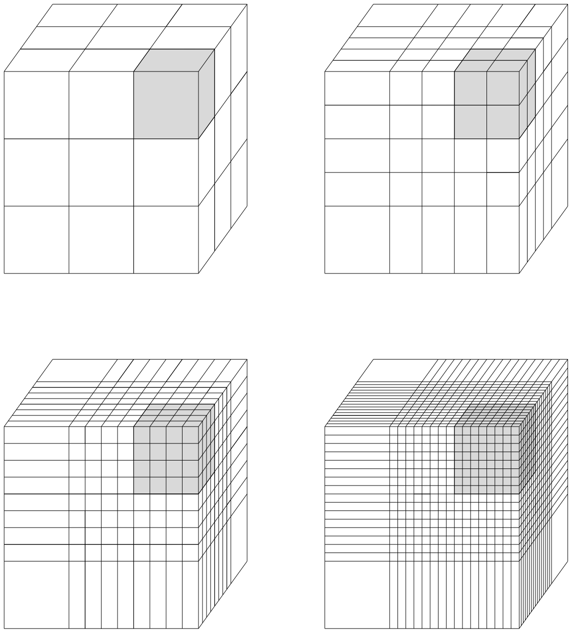

FIGURE 3. The adaptive meshes for Example 2 using the error estimator. It is $\sigma = 0.1$ on the grey cube and $\sigma = 1$ on the rest of the domain.

the elements on which the local contributions of the hierarchical error estimator are the largest and by then further refining in order to eliminate hanging nodes, since our algorithm yet cannot handle hanging nodes for 2nd order finite elements. We expect the algorithm to refine the mesh near the σ-discontinuity interface between $\Omega^{(1)} = (\frac{1}{3}, 1)^3$ and $\Omega^{(0)} = \Omega \setminus \Omega^{(1)}$, and especially close to the vertex $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Figure 3 shows adaptively generated meshes. Figure 4 shows that the adaptive refinement gives an improvement compared to uniform meshes. We expect even faster convergence when hanging nodes are allowed which avoid unnecessary refinement.

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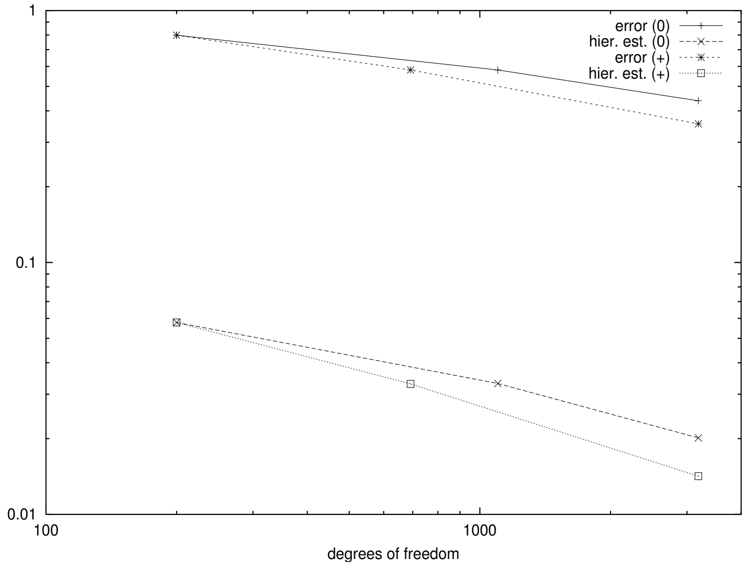

FIGURE 4. Energy norm e of the Galerkin error and the error estimator η for Example 2. 0 indicates uniform refinement, $+$ indicates adaptive refinement.

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