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Interval-Valued Fuzzy Congruences on a Semigroup

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Abstract

We introduce the concept of interval-valued fuzzy congruences on a semigroup S and we obtain some important results: First, for any interval-valued fuzzy congruence R on a group G, the interval-valued congruence class R_e is an interval-valued fuzzy normal subgroup of G. Second, for any interval-valued fuzzy congruence R on a groupoid S, we show that a binary operation $*$ an S/R is well-defined and also we obtain some results related to additional conditions for S. Also we improve that for any two interval-valued fuzzy congruences R and Q on a semigroup S such that $R \subset Q$, there exists a unique semigroup homomorphism $g : S/R \to S/G$.

Keywords: Interval-valued fuzzy set, Interval-valued fuzzy (normal) subgroup, Interval-valued fuzzy congruence

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1. Introduction

As a generalization of fuzzy sets introduced by Zadeh [1], Zadeh [2] also suggested the concept of interval-valued fuzzy sets. After that time, Biswas [3] applied it to group theory, and Gorzalczany [4] introduced a method of inference in approximate reasoning by using interval-valued fuzzy sets. Moreover, Mondal and Samanta [5] introduced the concept of interval-valued fuzzy topology and investigated some of it's properties. In particular, Roy and Biswas [6] introduced the notion of interval-valued fuzzy relations and studied some of it's properties. Recently, Jun et al. [7] investigated strong semi-openness and strong semi-continuity in interval-valued fuzzy topology. Moreover, Min [8] studied characterizations for interval-valued fuzzy m-semicontinuous mappings, Min and Kim [9, 10] investigated interval-valued fuzzy m*-continuity and m*-open mappings. Hur et al. [11] studied interval-valued fuzzy relations in the sense of a lattice theory. Also, Choi et al. [12] introduced the concept of interval-valued smooth topological spaces and investigated some of it's properties.

On the other hand, Cheong and Hur [13], and Lee et al. [14] studied interval-valued fuzzy ideals/(generalized)bi-ideals in a semigroup. In particular, Kim and Hur [15] investigated interval-valued fuzzy quasi-ideals in a semigroup. Kang [16], Kang and Hur [17] applied the notion of interval-valued fuzzy sets to algebra. Jang et al. [18] investigated interval-valued fuzzy normal subgroups.

In this paper, we introduce the concept of interval-valued fuzzy congruences on a semigroup S and we obtain some important results:

(i) For any interval-valued fuzzy congruence R on a group G, the interval-valued congruence class R_e is an interval-valued fuzzy normal subgroup of G (Proposition 3.11).

(ii) For any interval-valued fuzzy congruence R on a groupoid S, we show that a binary operation $*$ an S/R is well-defined (Proposition 3.20) and also we obtain some results related to additional conditions for S (Theorem 3.21, Corollaries 3.21-1, 3.21-2, and 3.21-3). Also we improve that for any two interval-valued fuzzy congruences R and Q on a semigroup S such that $R \subset Q$, there exists a unique semigroup homomorphism $g : S/R \to S/G$ (Theorem 4.3).

2. Preliminaries

In this section, we list some concepts and well-known results which are needed in later sections.

Let $D(I)$ be the set of all closed subintervals of the unit interval $[0, 1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denoted , $\mathbf{0} = [0, 0]$, $\mathbf{1} = [1, 1]$, and $\boldsymbol{a} = [a, a]$ for every $a \in (0, 1)$, We also note that

(i) $(\forall M, N \in D(I))\ (M = N \Leftrightarrow M^L = N^L, M^U = N^U)$,

(ii) $(\forall M, N \in D(I))\ (M = N \leq M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the *complement* of M, denoted by M^C, is defined by $M^C = 1 - M = [1 - M^U, 1 - M^L]$([7, 14]).

Definition 2.1 [4, 10, 14]. A mapping $A : X \to D(I)$ is called an *interval-valued fuzzy set*(*IVFS*) in X, denoted by $A = [A^L, A^U]$, if $A^L, A^L \in I^X$ such that $A^L \leq A^U$, i.e., $A^L(x) \leq A^U(x)$ for each $x \in X$, where $A^L(x)$[resp $A^U(x)$] is called the *lower*[resp *upper*] *end point of x to A*. For any $[a, b] \in D(I)$, the interval-valued fuzzy A in X defined by $A(x) = [A^L(x), A^U(x)] = [a, b]$ for each $x \in X$ is denoted by $\widetilde{[a, b]}$ and if $a = b$, then the IVFS $\widetilde{[a, b]}$ is denoted by simply $\widetilde{a}$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in X, respectively.

We will denote the set of all IVFSs in X as $D(I)^X$. It is clear that set $A = [A, A] \in D(I)^X$ for each $A \in I^X$.

Definition 2.2 [14]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then

(i) $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.

(iii) $A^C = [1 - A^U, 1 - A^L]$.

(iv) $A \cup B = [A^L \vee B^L, A^U \vee B^U]$.

(iv)′ $\bigcup_{\alpha \in \Gamma} A_\alpha = [\bigvee_{\alpha \in \Gamma} A^L_\alpha, \bigvee_{\alpha \in \Gamma} A^U_\alpha]$.

(v) $A \cap B = [A^L \wedge B^L, A^U \wedge B^U]$.

(v)′ $\bigcap_{\alpha \in \Gamma} A_\alpha = [\bigwedge_{\alpha \in \Gamma} A^L_\alpha, \bigwedge_{\alpha \in \Gamma} A^U_\alpha]$.

Result 2.A [14, Theorem 1]. Let $A, B, C \in D(I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I)^X$. Then

(a) $\widetilde{0} \subset \mathrm{A} \subset \widetilde{1}$.

(b) $A \cup B = B \cup A$, $A \cap B = B \cap A$.

(c) $A \cup (B \cup C) = (A \cup B) \cup C$,
$A \cap (B \cap C) = (A \cap B) \cap C$.

(d) $A, B \subset A \cup B$, $A \cap B \subset A, B$.

(e) $A \cap (\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} (A \cap A_\alpha)$.

(f) $A \cup (\bigcap_{\alpha \in \Gamma} A_\alpha) = \bigcap_{\alpha \in \Gamma} (A \cup A_\alpha)$.

(g) $(\widetilde{0})^c = \widetilde{1}$, $(\widetilde{1})^c = \widetilde{0}$.

(h) $(A^c)^c = A$.

(i) $(\bigcup_{\alpha \in \Gamma} A_\alpha)^c = \bigcap_{\alpha \in \Gamma} A^c_\alpha$, $(\bigcap_{\alpha \in \Gamma} A_\alpha)^c = \bigcup_{\alpha \in \Gamma} A^c_\alpha$.

Definition 2.3 [8]. Let X be a set. Then a mapping $R = [R^L, R^U] : X \prod X \to D(I)$ is called an *interval-valued fuzzy relation*(*IVFR*) on X.

We will denote the set of all IVFRs on X as IVR(X).

Definition 2.4 [8]. Let $R \in$ IVR(X). Then the *inverse* of R, R^{-1} is defined by $R^{-1}(x, y) = R(y, x)$, for each $x, y \in X$.

Definition 2.5 [11]. Let X be a set and let $R, Q \in$ IVR(X). Then the composition of R and Q, $Q \circ R$, is defined as follows : For any $x, y \in X$,

$$(Q \circ R)^L(x, y) = \bigvee_{z \in X} [R^L(x, z) \wedge Q^L(z, y)]$$

and

$$(Q \circ R)^U(x, y) = \bigvee_{z \in X} [R^U(x, z) \wedge Q^U(z, y)].$$

Result 2.B [11, Proposition 3.4]. Let X be a set and let $R, R_1, R_2, R_3, Q_1, Q_2 \in \mathrm{IVR}(X)$. Then

(*a*) $(R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3)$.

(*b*) If $R_1 \subset R_2$ and $Q_1 \subset Q_2$, then $R_1 \circ Q_1 \subset R_2 \circ Q_2$.
In particular, if $Q_1 \subset Q_2$, then $R_1 \circ Q_1 \subset R_1 \circ Q_2$.

(*c*) $R_1(R_2 \cup R_3) = (R_1 \circ R_2) \cup (R_1 \circ R_3)$,
$R_1(R_2 \cap R_3) = (R_1 \circ R_2) \cap (R_1 \circ R_3)$.

(*d*) If $R_1 \subset R_2$, then $R_1^{-1} \subset R_2^{-1}$.

(*e*) $(R^{-1})^{-1} = R$, $(R_1 \circ R_2)^{-1} = R_2^{-1} \circ R_1^{-1}$.

(*f*) $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$, $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$.

Definition 2.6 [11]. An IVFR R on a set X is called an *interval-valued fuzzy equivalence relation*($IVFER$) on X if it satisfies the following conditions :

(1) it is *interval-valued fuzzy reflexiv*, i.e., $R(x, x) = [1, 1]$, for each $x \in X$,

(2) it is *interval-valued fuzzy symmetric*, i.e., $R^{-1} = R$,

(3) it is *interval-valued fuzzy transitive*, i.e., $R \circ R \subset R$.

We will denote the set of all IVFERs on X as $\mathrm{IVE}(X)$.

From Definition 2.6, we can easily see that the following hold.

Remark 2.7 (a) If R is an fuzzy equivalence relation on a set X, then $[R, R] \in \mathrm{IVE}(X)$.

(b) If $R \in \mathrm{IVE}(X)$, then R^L and R^U are fuzzy equivalence relation on X.

(c) Let R be an ordinary relation on a set X. Then R is an equivalence relation on X if and only if $[\chi_R, \chi_R] \in \mathrm{IVE}(X)$.

Result 2.C [11, Proposition 3.9]. Let X be a set and let $Q, R \in \mathrm{IVE}(X)$. If $Q \circ R = R \circ Q$, then $R \circ Q \in \mathrm{IVE}(X)$.

Let R be an IVFER on a set X and let $a \in X$. We define a mapping $Ra : X \to D(I)$ as follows : For each $a \in X$,

$$Ra(x) = R(a, x).$$

Then clearly $Ra \in D(I)^X$. In this case, Ra is called the *interval-valued fuzzy equivalence class* of R *containing* $a \in X$. The set $\{Ra : a \in X\}$ is called the *interval-valued fuzzy quotient set of* X *by* R and denoted by X/R.

Result 2.D [11, Proposition 3.10]. Let R be an IVFER on a set X. Then the following hold :

(*a*) $Ra = Rb$ if and only if $R(a, b) = [1, 1]$, for any $a, b \in X$.

(*b*) $R(a, b) = [0, 0]$ if and only if $Ra \cap Rb = \widetilde{0}$, for any $a, b \in X$.

(*c*) $\bigcup_{a \in X} Ra = \widetilde{1}$.

(*d*) There exits the surjection $\pi : X \to X/R$ defined by $\pi(x) = Rx$ for each $x \in X$.

Definition 2.8 [11]. Let X be a set, let $R \in \mathrm{IVR}(X)$ and let $\{R_\alpha\}_{\alpha \in \Gamma}$ be the family of all IVFERs on X containing R. Then $\bigcap_{\alpha \in \Gamma} R_\alpha$ is called the IVFER *generated by* R and denoted by R^e.

It is easily seen that R^e is the smallest IVFER containing R.

Definition 2.9 [11]. Let X be a set and let $R \in \mathrm{IVR}(X)$. Then the *interval-valued fuzzy transitive closure* of R, denoted R^∞, is defined as followings :

$$R^\infty = \bigcup_{n \in \mathbf{N}} R^n$$

,where $R^n = R \circ R \circ \cdots \circ R$(n factors).

Definition 2.10 [11]. We define two mappings $\triangle, \triangledown : X \to D(I)$ as follows : For any $x, y \in X$,

$$\triangle(x, y) = \begin{cases} [1, 1] & \text{if x = y}, \\ [0, 0] & \text{if fx} \neq \text{y}. \end{cases}$$

and

$$\triangledown(x, y) = [1, 1].$$

It is clear that $\triangle, \triangledown \in \mathrm{IVE}(X)$ and R is an interval-valued fuzzy reflexive relation on X if and only if $\triangle \subset R$.

Result 2.E [11, Proposition 4.7]. If R is an IVFR on a set X, then

$$R^e = [R \cup R^{-1} \cup \triangle]^\infty.$$

Definition 2.11 [17]. Let $(X, \cdot)$ be a groupoid and let $A, B \in D(I)^X$. Then the interval-valued fuzzy product of A and B, $A \circ B$ is defined as follows : For each $a \in X$,

$$(A \circ B)^L(x) = \begin{cases} \bigvee_{yz=x} & [A^L(y) \wedge B^L(z)] \text{ if } x = yz, \\ 0 & \text{if } x \text{ is not expressible as } x = yz, \end{cases}$$

and

$$(A \circ B)^U(x) = \begin{cases} \bigvee_{yz=x} & [A^U(y) \wedge B^U(z)] \text{ if } x = yz, \\ 0 & \text{if } x \text{ is not expressible as } x = yz. \end{cases}$$

Definition 2.12 [17]. Let $(X, \cdot)$ be a groupoid and let $A \in D(I)^X$. Then A is called an i*interval-valued fuzzy subgroupoid* (*IVGP*) of X if for any $x, y \in X$,

$$A^L \geq A^L(x) \wedge A^L(y)$$

and

$$A^U \geq A^U(x) \wedge A^U(y).$$

We will denote the set of all IVGPs of X as IVGP(X). Then it is clear that $\tilde{0}, \tilde{1} \in$ IVGP(X).

Definition 2.13 [17]. Let G be a group and let $A \in$ IVGP(G). Then A is an i*interval-valued fuzzy subgroup* (*IVG*) of G if for each $x \in G$,

$$A(x^{-1}) \geq A(x),$$

i.e.,

$$A^L(x^{-1}) \geq A^L(x) \text{ and } A^U(x^{-1}) \geq A^U(x).$$

We will denote the set of all IVGs of G as IVG(G).

Definition 2.14 [17]. Let G be a group and let $A \in$ IVG(G). Then A is said to be *normal* if $A(xy) = A(yx)$, for any $x, y \in G$.

We will denote the set of all interval-valued fuzzy normal subgroups of G as IVNG(G). In particular, we will denote the set $\{N \in \text{IVNG}(G) : N(e) = [1, 1]\}$ as IVN(G).

Result 2.F [17, Proposition 5.2]. Let G be a group and let $A \in D(I)^G$. If $B \in$ IVNG(G), then $A \circ B = B \circ A$.

Definition 2.15 [18]. Let G be a group, let $A \in$ IVG(G) and let $x \in G$. We define two mappings

$$Ax : G \to D(I)$$

and

$$xA : G \to D(I)$$

as follows, respectively : For each $g \in G$,

$$Ax(g) = A(gx^{-1}) \text{ and } xA(g) = A(x^{-1}g).$$

Then Ax[resp. xA] is called the *interval-valued fuzzy right*[resp. *left*] *coset* of G determined by x and A.

It is obvious that if $A \in$ IVNG(G), then the interval-valued fuzzy left coset coincides with the interval-valued fuzzy right coset of A on G. In this case, we will call *interval-valued fuzzy coset* instead of interval-valued fuzzy left coset or interval-valued fuzzy right coset.

3. Interval-Valued Fuzzy Congruences

Definition 3.1 [19]. A relation R on a groupoid S is said to be:

(1) *left compatible* if $(a, b) \in R$ implies $(xa, xb) \in R$, for any $a, b \in S$,

(2) *right compatible* if $(a, b) \in R$ implies $(ax, bx) \in R$, for any $a, b \in S$,

(3) *compatible* if $(a, b) \in R$ and $(s, d) \in R$ imply $(ab, cd) \in R$, for any $a, b, c, d \in S$,

(4) a *left*[resp. *right*] *congruence* on S if it is a left[resp. right] compatible equivalence relation.

(5) a *congruence* on S if it is both a left and a right congruence on S.

It is well-known [19, Proposition I.5.1] that a relation R on a groupoid S is congruence if and only if it is both a left and a right congruence on S. We will denote the set of all ordinary congruences on S as $C(S)$.

Now we will introduce the concept of interval-valued fuzzy compatible relation on a groupoid.

Definition 3.2 An IVFR R on a groupoid S is said to be :

(1) *interval-valued fuzzy left compatible* if for any $x, y, z \in G$,

$$R^L(x, y) \leq R^L(zx, zy) \text{ and } R^U(x, y) \leq R^U(zx, zy),$$

(2) *interval-valued fuzzy right compatible* if for any $x, y, z \in G$,

$$R^L(x, y) \leq R^L(xz, yz) \text{ and } R^U(x, y) \leq R^U(xz, yz),$$

(3) *interval-valued fuzzy compatible* if for any $x, y, z, t \in G$,

$$R^L(x, y) \wedge R^L(z, t) \leq R^L(xz, yz)$$

and

$$R^U(x, y \wedge R^U(z, t) \leq R^U(xz, yz).$$

Example 3.3 Let $S = e, a, b$ be the groupoid with multiplication table :

	e	a	b
e	e	a	b
a	a	b	a
b	b	b	a

(a) Let $R_1 : S \times S \to D(I)$ be the mapping defined as the matrix :

R_1	e	a	b
e	$[\lambda_{11}, \mu_{11}]$	$[\lambda_{12}, \mu_{12}]$	$[\lambda_{13}, \mu_{13}]$
a	$[\lambda_{21}, \mu_{21}]$	$[\lambda_{22}, \mu_{22}]$	$[\lambda_{23}, \mu_{23}]$
b	$[\lambda_{31}, \mu_{31}]$	$[\lambda_{32}, \mu_{32}]$	$[\lambda_{33}, \mu_{33}]$

where $[\lambda_{ij}, \mu_{ij}] \in D(I)$ such that $[\lambda_{1i}, \mu_{1i}](i = 1, 2, 3)$, $[\lambda_{21}, \mu_{21}]$ and $[\lambda_{31}, \mu_{31}]$ are arbitrary, and

$$[\lambda_{23}, \mu_{23}] = [\lambda_{32}, \mu_{32}], \quad [\lambda_{22}, \mu_{22}] = [\lambda_{33}, \mu_{33}],$$
$$[\lambda_{11}, \mu_{11}] \leq [\lambda_{22}, \mu_{22}],$$
$$[\lambda_{12}, \mu_{12}] \leq [\lambda_{23}, \mu_{23}] \wedge [\lambda_{22}, \mu_{22}],$$
$$[\lambda_{13}, \mu_{13}] \leq [\lambda_{23}, \mu_{23}] \wedge [\lambda_{22}, \mu_{22}],$$
$$[\lambda_{21}, \mu_{21}] \leq [\lambda_{23}, \mu_{23}] \wedge [\lambda_{22}, \mu_{22}],$$
$$[\lambda_{31}, \mu_{31}] \leq [\lambda_{23}, \mu_{23}] \wedge [\lambda_{22}, \mu_{22}].$$

Then we can see that R_1 is an interval-valued fuzzy left compatible relation on S.

(b) Let $R_2 : S \times S \to D(I)$ be the mapping defined as the matrix :

R_2	e	a	b
e	$[\lambda_{11}, \mu_{11}]$	$[\lambda_{12}, \mu_{12}]$	$[\lambda_{13}, \mu_{13}]$
a	$[\lambda_{21}, \mu_{21}]$	$[\lambda_{22}, \mu_{22}]$	$[\lambda_{23}, \mu_{23}]$
b	$[\lambda_{31}, \mu_{31}]$	$[\lambda_{32}, \mu_{32}]$	$[\lambda_{33}, \mu_{33}]$

where $[\lambda_{ij}, \mu_{ij}] \in D(I)$ such that $[\lambda_{ij}, \mu_{ij}](i, j = 1, 2, 3)$ is arbitrary and

$$[\lambda_{11}, \mu_{11}] \leq [\lambda_{21}, \mu_{21}], [\lambda_{12}, \mu_{12}] \leq [\lambda_{31}, \mu_{31}],$$
$$[\lambda_{13}, \mu_{13}] \leq [\lambda_{31}, \mu_{31}], [\lambda_{21}, \mu_{21}] \leq [\lambda_{31}, \mu_{31}],$$
$$[\lambda_{32}, \mu_{32}] \leq [\lambda_{22}, \mu_{22}],$$
$$[\lambda_{33}, \mu_{33}] \leq [\lambda_{23}, \mu_{23}] = [\lambda_{22}, \mu_{22}].$$

Then we can see that R_2 is an interval-valued fuzzy right compatible relation on S.

(c) Let $R_3 : S \times S \to D(I)$ be the mapping defined as the matrix :

R_3	e	a	b
e	$[\lambda_{11}, \mu_{11}]$	$[\lambda_{12}, \mu_{12}]$	$[\lambda_{13}, \mu_{13}]$
a	$[\lambda_{21}, \mu_{21}]$	$[\lambda_{22}, \mu_{22}]$	$[\lambda_{23}, \mu_{23}]$
b	$[\lambda_{31}, \mu_{31}]$	$[\lambda_{32}, \mu_{32}]$	$[\lambda_{33}, \mu_{33}]$

where $[\lambda_{ij}, \mu_{ij}] \in D(I)$ such that

$$\lambda_{11} \wedge \lambda_{12} \leq \lambda_{12}, \mu_{11} \wedge \mu_{12} \leq \mu_{12}, \lambda_{11} \wedge \lambda_{13} \leq \lambda_{13},$$
$$\mu_{11} \wedge \mu_{13} \leq \mu_{13}, \lambda_{12} \wedge \lambda_{13} \leq \lambda_{12}, \mu_{12} \wedge \mu_{13} \leq \mu_{12},$$
$$\lambda_{21} \wedge \lambda_{22} \leq \lambda_{32}, \mu_{21} \wedge \mu_{22} \leq \mu_{32}, \lambda_{21} \wedge \lambda_{23} \leq \lambda_{33},$$
$$\mu_{21} \wedge \mu_{23} \leq \mu_{33}, \lambda_{22} \wedge \lambda_{23} \leq \lambda_{32}, \mu_{22} \wedge \mu_{23} \leq \mu_{32},$$
$$\lambda_{31} \wedge \lambda_{32} \leq \lambda_{22}, \mu_{31} \wedge \mu_{32} \leq \mu_{22}, \lambda_{31} \wedge \lambda_{33} \leq \lambda_{23},$$
$$\mu_{31} \wedge \mu_{33} \leq \mu_{23}, \lambda_{32} \wedge \lambda_{33} \leq \lambda_{22}, \mu_{32} \wedge \mu_{33} \leq \mu_{22}.$$

Then we can see that R_3 is an interval-valued fuzzy compatible relation on S.

Lemma 3.4 Let R be a relation on a groupoid S. Then R is left compatible if and only if $[\chi_R, \chi_R]$ is interval-valued fuzzy left compatible.

Proof. $(\Rightarrow)$: Suppose R is left compatible. Let $a, b, x \in S$.

Case(1) Suppose $(a, b) \in R$. Then $\chi_R(a, b) = 1$. Since R is left compatible, $(xa, xb) \in R$, for each $x \in S$. Thus $\chi_R(xa, xb) = 1 = \chi_R(a, b)$.

Case(2) Suppose $\neg(a, b) \in R$. Then, for each $x \in S$, it holds that $\chi_R(a, b) = 0 \leq \chi_R(xa, xb)$. Thus, in either cases, $[\chi_R, \chi_R]$.

$(\Leftarrow)$: Suppose $[\chi_R, \chi_R]$ is interval-valued fuzzy compatible. Let $a, b, x \in S$ and $(a, b) \in R$. Then, by hypothesis, $\chi_R(xa, xb) \geq \chi_R(a, b) = 1$. Thus $\chi_R(xa, xb) = 1$. So $(xa, xb) \in R$. Hence R is left compatible.

Lemma 3.5 [The dual of Lemma 3.4]. Let R be a relation on a groupoid S. Then R is right compatible if and only if $[\chi_R, \chi_R]$ is interval-valued fuzzy right compatible.

Definition 3.6 An IVFER R on a groupoid S is called an :

(1) *interval-valued fuzzy left congruence* (*IVLC*) if it is interval-valued fuzzy left compatible,

(2) *interval-valued fuzzy right congruence* (*IVRC*) if it is interval-valued fuzzy right compatible,

(3) *interval-valued fuzzy congruence* (*IVC*) if it is interval-valued fuzzy compatible.

We will denote the set of all IVCs[resp. IVLCs and IVRCs] on S as IVC(S) [resp. IVLC(S) and IVRC(S)].

Example 3.7 Let $S = e, a, b$ be the groupoid defined in Example 3.3. Let $R_1 : S \times S \to D(I)$ be the mapping defined as the matrix :

R_1	e	a	b
e	$[1,1]$	$[0.4, 0.6]$	$[o.4, 0.6]$
a	$[0.4, 0.6]$	$[1,1]$	$[0.2, 0.7]$
b	$[0.4, 0.6]$	$[0.2, 0.7]$	$[1,1]$

Then it can easily be checked that $R \in$ IVE(S). Moreover we can see that $R \in$ IVC(S).

Proposition 3.8 Let S be a groupoid and let $R \in$ IVE(S). Then $R \in$ IVC(S) if and only if it is both an IVLC and an IVRC.

Proof. $(\Rightarrow)$: Suppose $R \in$ IVC(S) and let $x, y, z \in S$. Then

$R^L(x,y) = R^L(x,y) \wedge R^L(z,z) \leq R^L(xz, yz)$

and

$R^U(x,y) = R^U(x,y) \wedge R^U(z,z) \leq R^U(xz, yz)$.

Also,

$R^L(x,y) = R^L(z,z) \wedge R^L(x,y) \leq R^L(zx, zy)$

and

$R^U(x,y) = R^U(z,z) \wedge R^U(x,y) \leq R^U(zx, zy)$.

Thus R is both an IVLC and an IVRC.

$(\Leftarrow)$: Suppose R is both an IVLC and an IVRC. and let $x, y, z, t \in S$. Then

$$\begin{aligned} R^L(x,y) \wedge R^L(z,t) &= R^L(x,y) \wedge R^L(z,z) \\ &\quad \wedge R^L(y,y) \wedge R^L(z,t) \\ &\leq R^L(xz,yz) \wedge R^L(yz,yt) \\ &\leq R^L(xz,yt) \text{ [Since } R \circ R \subset R]. \end{aligned}$$

By the similar arguments, we have that

$$R^U(x,y) \wedge R^U(z,t) \leq R^U(xz, yt).$$

So R is interval-valued fuzzy compatible. Hence $R \in$ IVC(S).

The following is the immediate result of Remark 2.7(c), Lemmas 3.4 and 3.5, and Proposition 3.5.

Theorem 3.9 Let R be a relation on a groupoid S. Then $R \in$ C(S) if and only if $[\chi_R, \chi_R] \in$ IVC(S).

For any interval-valued fuzzy left[resp. right] compatible relation R, it is obvious that if G is a group, then $R(x,y) = R(tx,ty)$[resp. $R(x,y) = R(xt,yt)$], for any $x, y, t \in G$. Thus we have following result.

Lemma 3.10 Let R be an IVC on a group G. Then

$$R(xay, xby) = R(xa, xb) = R(ay, by) = R(a, b),$$

for any $a, b, x, y \in G$.

Example 3.11 Let V be the Klein 4-group with multiplication table :

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Let $R : V \times V \to D(I)$ be the mapping defined as the matrix :

R	e	a	b	c
e	$[1,1]$	$[0.3, 0.6]$	$[0.1, 0.9]$	$[0.3, 0.6]$
a	$[0.5, 0.6]$	$[1,1]$	$[0.3, 0.6]$	$[0.1, 0.9]$
b	$[0.1, 0.9]$	$[0.3, 0.6]$	$[1,1]$	$[0.3, 0.6]$
c	$[0.3, 0.6]$	$[0.1, 0.9]$	$[0.3, 0.6]$	$[1,1]$

Then we can see that $R \in$ IVC(V). Furthermore, it is easily checked that Lemma 3.10 holds : For any $s, t, x, y \in V$,

$$R(xsy, xty) = R(xs, xt) = R(sy, ty) = R(s, t).$$

The following is the immediate result of Proposition 3.8 and Lemma 3.10.

Theorem 3.12 Let R be an IVFR on a group G. Then $R \in$ IVC(G) if and only if it is interval-valued fuzzy left(right) compatible equivalence relation.

Lemma 3.13 Let P and Q be interval-valued fuzzy compatible relations on a groupoid S. Then $Q \circ P$ is also an interval-valued fuzzy compatible relation on S.

Proof. Let $a, b, x \in S$. Then

$$\begin{aligned}(Q \circ P)^L(ax, bx) &= \bigvee_{t \in S}[P^L(ax, t) \wedge Q^L(t, xb)] \\ &\geq P^L(xa, xc) \wedge Q^L(xc, xb) \text{ for each c} \in S \\ &\geq P^L(a, c) \wedge Q^L(c, b) \text{ for each c} \in S. \\ &\quad \text{[Since } P \text{ and } Q \text{ are compatible]}\end{aligned}$$

By the similar arguments, we have that

$$(Q \circ P)^U(ax, bx) \geq P^U(a, c) \wedge Q^U(c, b) \text{ for each c} \in S.$$

Thus

$$\begin{aligned}(Q \circ P)^L(ax, bx) &\geq \bigvee_{c \in S}[P^L(a, c) \wedge Q^L(c, b)] \\ &= (Q \circ P)(a, b)\end{aligned}$$

and

$$\begin{aligned}(Q \circ P)^U(ax, bx) &\geq \bigvee_{c \in S}[P^U(a, c) \wedge Q^U(c, b)] \\ &= (Q \circ P)(a, b).\end{aligned}$$

So $Q \circ P$ is interval-valued fuzzy right compatible. Similarly, we can see that $Q \circ P$ is interval-valued fuzzy left compatible. Hence $Q \circ P$ is interval-valued fuzzy compatible.

Theorem 3.14 Let P and Q be IVC on a groupoid S. Then the following are equivalent :

(a) $Q \circ P \in$ IVC(S).

(b) $Q \circ P \in$ IVE(S).

(c) $Q \circ P$ is interval-valued fuzzy symmetric.

(d) $Q \circ P = P \circ Q$.

Proof. It is obvious that $(a) \Rightarrow (b) \Rightarrow (c)$.

$(c) \Rightarrow (d)$: Suppose the condition (c) holds and let $a, b \in S$. Then

$$\begin{aligned}(Q \circ P)^L(a, b) &= \bigvee_{t \in S}[P^L(a, t) \wedge Q^L(t, b)] \\ &= \bigvee_{t \in S}[Q^L(b, t) \wedge P^L(t, a)] \\ &\text{[Since P and Q areinterval-valued fuzzy symmetric]} \\ &= (P \circ Q)^L(a, b).\end{aligned}$$

Similarly, we have that

$$(Q \circ P)^U(a, b) = (P \circ Q)^U(a, b).$$

Hence $Q \circ P = P \circ Q$.

$(d) \Rightarrow (a)$: Suppose the condition (d) holds. Then , by Result 2.C, $Q \circ P \in$ IVE(S). Since P and Q are interval-valued fuzzy compatible, by Lemma 3.13, $Q \circ P$ is interval-valued fuzzy compatible. So $Q \circ P \in$ IVC(S). This completes the proof.

Proposition 3.15 Let S be a groupoid and let $Q, P \in$ IVC(S). If $Q \circ P = P \circ Q$, then $P \circ Q \in$ IVC(S).

Proof. By Result 2.C, it is clear that $P \circ Q \in$ IVE(S). Let $x, y, t \in S$. Then, since P and Q are interval-valued fuzzy right compatible,

$$\begin{aligned}(P \circ Q)^L(x, y) &= \bigvee_{z \in S}[Q^L(x, z) \wedge P^L(z, y)] \\ &\leq \bigvee_{z \in S}[Q^L(xt, zt) \wedge P^L(zt, yt)] \\ &\leq \bigvee_{a \in S}[Q^L(xt, a) \wedge P^L(a, yt)] \\ &= (P \circ Q)^L(xt, yt).\end{aligned}$$

Similarly, we have that

$$(P \circ Q)^U(x, y) \leq (P \circ Q)^U(xt, yt).$$

By the similar arguments, we have that

$(P \circ Q)^L(x, y) \leq (P \circ Q)^L(tx, ty)$

and

$(P \circ Q)^U(x, y) \leq (P \circ Q)^U(tx, ty)$.

So $P \circ Q$ is interval-valued fuzzy left and right compatible. Hence $P \circ Q \in$ IVC(S).

Let R be an IVC on a groupoid S and let $a \in S$. Then $Ra \in D(I)^S$ is called an *interval-valued fuzzy congruence class of* R *containing* $a \in S$ and we will denote the set of all interval-valued fuzzy congruence classes of R as S/R.

Proposition 3.16 If R is an IVC on a groupoid S, then $Ra \circ Rb \subset Rab$, for any $a, b \in S$.

Proof. Let $x \in S$. If x is not expressible as $x = yz$, then clearly $(Ra \circ Rb)(x) = [0, 0]$. Thus $Ra \circ Rb \subset Rab$. Suppose x is expressible as $x = yz$. Then

$$\begin{aligned}(Ra \circ Rb)^L(x) &= \bigvee_{yz=x} [(Ra)^L(y) \wedge (Rb)^L(z)] \\ &= \bigvee_{yz=x} [R^L(a, y) \wedge Rb^L(b, z)] \\ &\leq \bigvee_{yz=x} [R^L(ab, yz)] \\ &\text{[Since R isinterval-valued fuzzy compatible]} \\ &= R^L(ab, x) = (Rab)^L(x).\end{aligned}$$

Similarly, we have that

$$(Ra \circ Rb)^U(x) \leq (Rab)^U(x).$$

Thus $Ra \circ Rb \subset Rab$. This completes the proof.

Proposition 3.17 Let G be a group with the identity e and let $R \in$ IVC(G). We define the mapping $A_R : G \to D(I)$ as follows : For each $a \in G$,

$$A_R(a) = R(a, e) = Re(a).$$

Then $A_R = Re \in$ IVNG(G).

Proof. From the definition of A_R, it is obvious that $A_R \in D(I)^G$. Let $a, b \in G$. Then

$$\begin{aligned}A^L_R(ab) &= R^L(ab, e) = R(a, b^{-1})\text{[By Lemma 3.10]} \\ &\geq (R \circ R)^L(a, b^{-1}\text{[Since R is transitive]} \\ &= \bigvee_{t \in G} [R^L(a, t) \wedge R^L(t, b^{-1})] \\ &\geq R^L(a, e) \wedge R^L(e, b^{-1} \\ &= R^L(a, e) \wedge R^L(b, e)\text{[By Lemma 3.10]} \\ &= A^L_R(a) \wedge A^L_R(b).\end{aligned}$$

Similarly, we have that

$$A^U_R(ab) \geq A^U_R(a) \wedge A^U_R(b).$$

On the other hand,

$$\begin{aligned}A_R(a^{-1}) &= [A^L_R(a^{-1}), A^U_R(a^{-1})] \\ &= [R^L(a^{-1}, e), R^U(a^{-1}, e)] \\ &= [R^L(e, a), R^U(e, a)]\text{[By Lemma 3.10]} \\ &= [R^L(a, e), R^U(a, e)]\text{[Since R is transitive]} \\ &= [A^L_R(a), A^U_R(a)] = A_R(a).\end{aligned}$$

Moreover,

$$A_R(e) = [A^L_R(e), A^U_R(e)] = [R^L(e, e), R^U(e, e)] = [1, 1].$$

So $A_R \in$ IVG(G) such that $A_R(e) = [1, 1]$.
Finally,

$$\begin{aligned}A_R(ab) &= [A^L_R(ab), A^U_R(ab)] \\ &= [R^L(ab, e), R^U(ab, e)] \\ &= [R^L(b(ab)b^{-1}, beb^{-1}), R^U(b(ab)b^{-1}, beb^{-1})] \\ &\quad\text{[By Lemma 3.10]} \\ &= [R^L(ba, e), R^U(ba, e)] \\ &= [A^L_R(ba), A^U_R(ba)] \\ &= A_R(ba).\end{aligned}$$

Hence $A_R \in$ IVNG(G). This completes the proof.

The following is the immediate result of Proposition 3.17 and Result 2.F. **Proposition 3.18** Let G be a group with the identity e. If $P, Q \in$ IVNG(G), then $Pe \circ Qe = Qe \circ Pe$.

Proposition 3.19 Let G be a group with the identity e. If $R \in$ IVC(G), then any interval-valued fuzzy congruence class Rx of $x \in G$ by R is an interval-valued fuzzy coset of Re. Conversely, each interval-valued fuzzy coset of Re is an interval-valued fuzzy congruence class by R.

Proof. Suppose $R \in$ IVC(G) and let $x.g \in G$. Then $Rx(g) = R(x, g)$. Since R is interval-valued fuzzy left compatible, by Lemma 3.10, $R(x, g) = R(e, x^{-1}g)$. Thus

$$Rx(g) = R(e, x^{-1}g) = Re(^{-1}g) = (xRe)(g).$$

So $Rx = xRe$. Hence Rx is an interval-valued fuzzy coset of Re.

Conversely, let A be any interval-valued fuzzy coset of Re. Then there exists an $x \in G$ such that $A = xRe$. Let $g \in G$.

Then

$$A(g) = (xRe)(g) = Re(x^{-1}g) = R(e, x^{-1}g).$$

Since R is interval-valued fuzzy left compatible,

$$R(e, x^{-1}g) = R(x, g) = Rx(g).$$

So $A = Rx$. Hence A is an interval-valued fuzzy congruence class of x by R.

Proposition 3.20 Let R be an IVC on a groupoid S. We define the binary operation $*$ on S/R as follows : For any $a, b \in S$,

$$Ra * Rb = Rab.$$

Then $*$ is well-defined.

Proof. Suppose $Ra = Rx$ and $Rb = Ry$, where $a, b, x, y \in S$. Then, by Result 2.D(a),

$$R(a, x) = R(b, y) = [1, 1].$$

Thus

$$\begin{aligned} R^L(ab, xy) &\geq \bigvee_{z\in S} [R^L(ab, z) \wedge R^L(z, xy)] \\ &\quad \text{[Since } R \text{ is transitive]} \\ &\geq R^L(ab, xb) \wedge R^L(xb, xy) \\ &\geq R^L(a, x) \wedge R^L(b, y) \\ &\quad \text{[Since } R \text{ is righy and left compatible]} \\ &= 1. \end{aligned}$$

Similarly, we have that

$$R^U(ab, xy) \geq 1.$$

Thus $R(ab, xy) = [1, 1]$. By Result 2.D(a), $Rab = Rxy$. So $Ra * Rb = Rx * Ry$. Hence $*$ is well-defined.

From Proposition 3.20 and the definition of semigroup, we obtain the following result.

Theorem 3.21 Let R be an IVC on a semigroup S. Then $(S/R, *)$ is a semigroup.

A semigroup S is called an *inverse semigroup* [7] if each $a \in S$ has a unique inverse, i.e., there exists a unique $a^{-1} \in S$ such that $aa^{-1}a = a$ and $a^{-1} = a^{-1}aa^{-1}$.

Corollary 3.21-1 Let R be an IVC on an inverse semigroup S. Then $(S/R, *)$ is an inverse semigroup. *Proof.* By Theorem 3.21, $(S/R, *)$ is a semigroup. Let $a \in S$. Since S is an inverse semigroup, there exists a unique $a^{-1} \in S$ such that $aa^{-1}a = a$ and $a^{-1} = a^{-1}aa^{-1}$. Moreover, it is clear that $(Ra)^{-1} = Ra^{-1}$. Then $(Ra)^{-1} * Ra * (Ra)^{-1} = Ra^{-1} * Ra * Ra^{-1} = Ra^{-1}aa^{-1} = Ra^{-1}$ and $Ra * (Ra)^{-1} * Ra = Ra * Ra^{-1} * Ra = Raa^{-1}a = Ra$.
So Ra^{-1} is an inverse of Ra for each $a \in S$.

An element a of a semigroup S is said to be *regular* if $a \in aSa$, i.e., there exists an $x \in S$ such that $a = axa$. The semigroup S is said to be *regular* if for each $a \in S$, a is a regular element. Corresponding to a regular element a, there exists at least one $\acute{a} \in S$ such that $a = a\acute{a}a \textit{ and } \acute{a} = \acute{a}a\acute{a}$. Such an $\acute{a}$ is called an *inverse* of a.

Corollary 3.21-2 Let R be an IVC on a regular semigroup S. Then $(S/R, *)$ is a regular semigroup.

Proof. By Theorem 3.21, $(S/R, *)$ is a semigroup. Let $a \in S$. Since S is a regular semigroup, there exists an $x \in S$ such that $a = axa$. It is obvious that $Rx \in S/R$. Moreover, $Ra * Rx * Ra = Raxa = Ra$. So Ra is an regular element of S/R. Hence S/R is a regular semigroup.

Corollary 3.21-3 Let R be an IVC on a group G. Then $(G/R, *)$ is a group.

Proof. By Theorem 3.21, $(G/R, *)$ is a semigroup. Let $x \in G$. Then

$$Rx * Re = Rxe = Rx = Rex = Re * Rx.$$

Thus Re is the identity in G/R with respect to $*$. Moreover,

$$Rx * Rx^{-1} = Rxx^{-1} = Re = Rx^{-1}x = Rx^{-1} * Rx.$$

So Rx^{-1} is the inverse of Rx with respect to $*$. Hence G/R is a group.

Proposition 3.22 Let G be a group and let $R \in$ IVC(G). We define the mapping $\pi : G/R \to D(I)$ as follows : For each $x \in G$,

$$\pi(Rx) = [(Rx)^L(e), Rx)^U(e)].$$

Then $\pi \in$ IVG(G/R).

Proof. From the definition of π, it is clear that $\pi = [\pi^L, \pi^U] \in D(I)^{G/R}$. Let $x, y \in G$. Then

$$\begin{aligned}\pi^L(Rx * Ry) = \pi^L(Rxy) = (Rxy)^L(e) = R^L(xy, e).\\ &\geq R^L(x,e) \wedge R^L(y,e)\\ &\quad [\text{Since } R \text{ is compatible}]\\ &= (Rx)^L(e) \wedge (Ry)^L(e)\\ &= \pi^L(Rx) \wedge \pi^L(Ry).\end{aligned}$$

Similarly, we have that

$$\pi^U(Rx * Ry) \geq \pi^U(Rx) \wedge \pi^U(Ry).$$

By the process of the proof of Corollary 3.21-1, $(Rx)^{-1} = Rx^{-1}$. Thus

$$\pi((Rx)^{-1}) = \pi(Rx^{-1}) = R(x^{-1}, e) = R(e, x) = \pi(Rx).$$

So $\pi((Rx)^{-1}) = \pi(Rx)$ for each $x \in G$. Hence $\pi \in$ IVG(G/R).

Proposition 3.23 If R is an IVC on an inverse semigroup S. Then $R(x^{-1}, y^{-1}) = R(x, y)$ for any $x, y \in S$. *Proof.* By Corollary 3.21-1, S/R is an inverse semigroup with $(Rx)^{-1} = Rx^{-1}$ for each $x \in S$. Let $x, y \in S$. Then

$$\begin{aligned}R(x^{-1}, y^{-1}) &= Rx^{1}(y^{-1}) = [Rx(y^{-1}]^{-1}\\ &= [Ry^{-1}(x)]^{-1} = [(Ry(x))^{-1}]^{-1}\\ &= Ry(x) = R(y,x) = R(x,y).\end{aligned}$$

Hence $R(x^{-1}, y^{-1}) = R(x, y)$.

The following is the immediate result of Proposition 3.22

Corollary 3.23 Let R be an IVC on a group G. Then

$$R(x^{-1}, y^{-1}) = R(x, y)$$

for any $x, y \in G$.

Proposition 3.24 Let R be an IVC on a semigroup S. Then

$$R^{-1}([1,1]) = \{(a,b) \in S \times S : R(a,b) = [1,1]\}$$

is a congruence on S. *Proof.* It is clear that $R^{-1}([1,1])$ is reflexive and symmetric. Let $(a,b), (b,c) \in R^{-1}([1,1])$. Then $R(a,b) = R(b,c) = [1,1]$. Thus

$$\begin{aligned}R^L(a,c) &\geq \bigvee_{x\in S}[R^L(a,x) \wedge R^L(x,c)]\\ &\quad [\text{Since } R \text{ is transitive}]\\ &\geq R^L(a,b) \wedge R^L(b,c) = 1.\end{aligned}$$

Similarly, we have that $R^U(a,c) \geq 1$. So $R(a,c) = [1,1]$, i.e., $(a,c) \in R^{-1}([1,1])$. Hence $R^{-1}([1,1])$ is an equivalence relation on S.

Now let $(a,b) \in R^{-1}([1,1])$ and let $x \in S$. Since R is an IVC on S,

$$R^L(ax, bx) \geq R^L(a,b) = 1 \text{ and } R^U(ax, bx) \geq R^U(a,b) = 1.$$

Then $R(ax, bx) = [1,1]$. Thus $(ax, bx) \in R^{-1}([1,1])$. Similarly, $(xa, xb) \in R^{-1}([1,1])$. So $R^{-1}([1,1])$ is compatible. Hence $R^{-1}([1,1])$ is a congruence on S.

Let S be a semigroup. Then S^1 denotes the monoid defined as follows :

$$S^1 = \begin{cases} S & \mathrm{if\,Shastheidentity1,}\\ S \cup \{1\} & \mathrm{otherwise.}\end{cases}$$

Definition 3.25 Let S be a semigroup and let $R \in$ IVR(S). Then we define a mapping $R^* : S \times S \to D(I)$ as follows : For any $c, d \in S$,

$$(R^*)^L(c,d) = \bigvee_{xay=c,\ xby=d,\ x,y\in S^1} R^L(a,b)$$

and

$$(R^*)^U(c,d) = \bigvee_{xay=c,\ xby=d,\ x,y\in S^1} R^U(a,b).$$

It is obvious that $R^* \in$ IVR(S).

Proposition 3.26 Let S be a semigroup and let $R, P, Q \in$ IVR(S). Then :

(*a*) $R \subset R^*$.

(*b*) $(R^*)^{-1} = (R^{-1})^*$.

(*c*) If $P \subset Q$, then $P^* \subset Q^*$.

(*d*) $(R^*)^* = R^*$.

(*e*) $(P \cup Q)^* = P^* \cup Q^*$.

(*f*) $R = R^*$ if and only if R is left and right compatible.

Proof. From Definition 3.25, the proofs of (*a*), (*b*) and (*c*) are clear.

(*d*) By (*a*) and (*c*), it is clear that $R^* \subset (R^*)^*$. Let $c, d \in S$. Then

$$((R^*)^*)^L(c,d) = \bigvee_{xay=c,\ xby=d,\ x,y\in S^1} (R^*)^L(a,b)$$
$$= \bigvee_{xay=c,\ xby=d,\ x,y\in S^1} \bigvee_{zpt=a,\ zqt=b,\ z,t\in S^1} R^L p,q)$$
$$\le \bigvee_{xzpty=c,\ xzqty=d,\ xz,ty\in S^1} R^L(p,q) = (R^*)^L(c,d).$$

By the similar arguments, we have that

$$((R^*)^*)^U(c,d) \le (R^*)^U(c,d).$$

Thus $(R^*)^* \subset R^*$. So $(R^*)^* = R^*$.

(*e*) By (*c*), $P^* \subset (P\cup Q)^*$ and $Q^* \subset (P\cup Q)^*$. Thus $P^*\cup Q^* \subset (P\cup Q)^*$. Let $c, d \in S$. Then

$$((P\cup Q)^*)^L(c,d)$$
$$= \bigvee_{xay=c,\ xby=d,\ x,y\in S^1} (P\cup Q)^L(a,b)$$
$$= \bigvee_{xay=c,\ xby=d,\ x,y\in S^1} [P^L(a,b)\wedge Q^L(a,b)]$$
$$\le (\bigvee_{xay=c,\ xby=d,\ x,y\in S^1} P^L(a,b))$$
$$\wedge (\bigvee_{xay=c,\ xby=d,\ x,y\in S^1} Q^L(a,b))$$
$$= (P^*)^L(a,b)\wedge (Q^*)^L(c,d).$$

Similarly, we have that

$$((P\cup Q)^*)^U(c,d) \le (P^*)^U(a,b)\wedge (Q^*)^U(c,d).$$

Thus $(P\cup Q)^* \subset P^*\cup Q^*$. So $(P\cup Q)^* = P^*\cup Q^*$.

(*f*) $(\Rightarrow)$: Suppose $R = R^*$ and let $c, d, e \in S$. Then

$$R^L(ec,ed) = (R^*)^L(ec,ed)$$
$$= \bigvee_{xay=ec,\ xby=ed,\ x,y\in S^1} R^L(a,b)$$
$$\ge R^L(c,d).$$

Similarly, we have that

$$R^U(ec,ed) \ge R^U(c,d).$$

By the similar arguments, we have that

$$R^L(ce,de) \ge R^L(c,d) \text{ and } R^U(ce,de) \ge R^U(c,d).$$

$(\Leftarrow)$: Suppose R is interval-valued fuzzy left and right compatible. Let $c, d \in S$. Then

$$(R^*)^L(c,d) = \bigvee_{xay=c,\ xby=d,\ x,y\in S^1} R^L(a,b)$$
$$\le \bigvee_{xay=c,\ xby=d,\ x,y\in S^1} R^L(xay,xby)$$
$$= R^L(c,d).$$

Similarly, we have that

$$(R^*)^U(c,d) \le R^U(c,d).$$

Thus $R^* \subset R$. So $R^* = R$. This completes the proof.

Proposition 3.27 If R is an IVFR on a semigroup S such that is interval-valued fuzzy left and right compatible, then so is R^∞. *Proof.* Let $a, b, c \in S$ and let $n \ge 1$. Then

$$(R^n)^L(a,b) = \bigvee_{z_1,\ \cdots,\ z_n\in S} [R^L(a,z_1)\wedge R^L(z_1,z_2)$$
$$\wedge\ \cdots\ \wedge R^L(z_{n-1},b)]$$
$$\le \bigvee_{z_1,\ \cdots,\ z_n\in S} [R^L(ac,z_1c)\wedge R^L(z_1c,z_2c)$$
$$\wedge\ \cdots\ \wedge R^L(z_{n-1}c,bc)]$$
$$= (R^n)^L(ac,bc).$$

Similarly, we have that

$$(R^n)^U(a,b) \le (R^n)^U(ac,bc).$$

By the similar arguments, we have that
$(R^n)^L(a,b) \le (R^n)^U(ca,cb)$
and
$(R^n)^U(a,b) \le (R^n)^U(ca,cb)$.
So R^n is interval-valued fuzzy left and right compatible for each $n \ge 1$. Hence R^∞ is interval-valued fuzzy left and right compatible.

Let $R \in$ IVR(S) and let $\{R\alpha\}_{\alpha\in\ \Gamma}$ be the family of all IVCs on a semigroup S containing R. Then the IVFR $\widehat{R}$ defined by $\widehat{R} = \bigcap_{\alpha\in\ \Gamma} R_\alpha$ is clearly the least IVC on S. In this case, $\widehat{R}$ is called the IVC on S *generated by* R.

Theorem 3.28 If R is an IVFR on a semigroup S, then $\widehat{R} = (R^*)^e$. *Proof.* By Definition 2.8, $(R^*)^e \in$ IVE(S) such that $R^* \subset (R^*)^e$. Then, by Proposition 3.26(a), $R \subset (R^*)^e$. Also, by (a) and (b) of Proposition 3.26 $R^* \cup (R^*)^{-1} \cup \triangle = (R \cup R^{-1} \cup \triangle)^*$. Thus, by Proposition 3.26(f) and Result 2.E, $R^* \cup (R^*)^{-1} \cup \triangle$ is left and right compatible. So, by Proposition 3.27, $(R^*)^e = [R^* \cup (R^*)^{-1} \cup \triangle]^\infty$ is left and right compatible. Hence, by Proposition 3.8, $(R^*)^e \in$ IVC(S). Now suppose $Q \in$ IVC(S) such that $R \subset Q$. Then, by (c) and (d) of Proposition 3.26, $R^* \subset Q^* = Q$. Thus $(R^*)^e \subset Q$. So $\widehat{R} = (R^*)^e$. This completes the proof.

4. Homomorphisms

Let $f : S \longrightarrow T$ be a semigroup homomorphism. Then it is well-known that the relation

$$\text{Ker}(f) = \{(a, b) \in S \times S : f(a) = f(b)\}$$

is a congruence on S.

The following is the immediate result of Theorem 3.9.

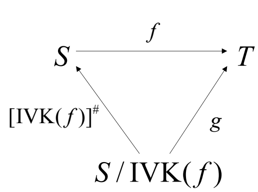
Proposition 4.1 Let $f : S \longrightarrow T$ be a semigroup homomorphism. Then $R = [\chi_{\text{Ker}(f)}, \chi_{\text{Ker}(f)}] \in$ IVC(S).

In this case, R is called the *interval-valued fuzzy kernel* of f and denoted by IVK(f). In fact, for any $a, b \in S$,

$$\text{IVK}(f)(a, b) = \begin{cases} [1, 1] & \text{if f(a) = f(b)}, \\ [0, 0] & \text{if f(a)} \neq \text{f(b)}. \end{cases}$$

Theorem 4.2 (a) Let R be an interval-valued fuzzy congruence on a semigroup S. Then the mapping $\pi : S \to S/R$ defined same as in Result 2.D(d) is an epimorphism.

(b) If $f : S \to T$ is a semigroup homomorphism, then there is a monomorphism $g : S/\text{IVK}(f) \to T$ such that the diagram



commutes, where $[\text{IVK}(f)]^\sharp$ denotes the natural mapping. *Proof.* (a) Let $a, b \in S$. Then, by the definition of $R^\sharp$ and Theorem 3.21,

$$\pi(ab) = Rab = Ra * Rb = \pi(a) * \pi(b).$$

So π is a homomorphism. By Result 2.D(d), π is surjective. Hence π is an epimorphism.

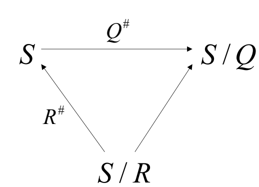
(b) We define $g : S/\text{IVK}(f) \to T$ by $g([\text{IFK}(f)]a) = f(a)$ for each $a \in S$. Suppose $[\text{IVK}(f)]a = [\text{IVK}(f)]b$ for any $a, b \in S$. Since $\text{IVK}(f)(a, b) = [1, 1]$, i.e. $\chi_{\text{IVK}(f)}(a, b) = 1$. Thus $(a, b) \in$ Ker(f). So $(a, b) \in Ker(f)$. So $g([\text{IVK}(f)]a) = f(a) = f(b) = g([\text{IVK}(f)]b)$. Hence g is well-defined.

Suppose $f(a) = f(b)$. Then $\text{IVK}(f)(a, b) = [1, 1]$. Thus, by Result 2.D($a$), $[\text{IVK}(f)]a = [\text{IVK}(f)]b$. So g is injective. Now let $a, b \in S$, Then

$$\begin{aligned} g([\text{IVK}(f)]a * [\text{IVK}(f)]b) &= g([\text{IVK}(f)]ab) \\ &= f(ab) \\ &= f(a)f(b) \\ &= g([\text{IVK}(f)]a)g([\text{IVK}(f)]b). \end{aligned}$$

So g is a homomorphism. Let $a \in S$. Then $g([\text{IVK}(f)]^\sharp(a)) = g([\text{IVK}(f)]a) = f(a)$. So $g \circ [\text{IVK}(f)]^\sharp = f$. This completes the proof.

Theorem 4.3 Let R and Q be IVCs on a semigroup such that $R \subset Q$. Then there exists a unique semigroup S homomorphism $g : S/R \to S/Q$ such that the diagram



commutes and $(S/R)/\text{IVK}(g)$ is isomorphic to S/Q, where $R^\sharp$ and $Q^\sharp$ denote the natural mappings, respectively. *Proof.* Define $g : S/R \longrightarrow S/Q$ by $g(Ra) = Qa$ for each $a \in S$. Suppose $Ra = Rb$. Then, by Result 2.D(a), $R(a, b) = [1, 1]$. Since $R \subset Q$,

$$1 = R^L(a, b) \leq Q^L(a, b) \text{ and } 1 = R^U(a, b) \leq Q^U(a, b).$$

Then $Q(a, b) = [1, 1]$. Thus $Qa = Qb$, i.e., $g(Ra) = g(Rb)$. So g is well- defined.

Let $a, b \in$S. Then

$$g(Ra * Rb) = g(Rab) = Qab = Qa * Qb = g(Ra) * g(Rb).$$

So g is a semigroup homomorphism. The remainders of the proofs are easy. This completes the proof.

5. Conclusion

Hur et al. [11] studied interval-valued fuzzy relations in the sense of a lattice. Cheong and Hur [13], Hur et al. [14], and Kim et al. [15] investigated interval-valued fuzzy ideals/(generalized) bi-ideas and quasi-ideals in a semigroup, respectively.

In this paper, we mainly study interval-valued fuzzy congruences on a semigroup. In particular, we obtain the result that $\hat{R} = (R^*)^e$ for the IVC $\hat{R}$ on S generated by R for each IVFR R on a semigroup S (See Theorem 3.28). Finally, for any IVCs R and Q on a semigroup S such that $R \subset Q$, there exists a unique semigroup homomorphism $g : S/K \to S/Q$ such that $(S?R)$/IVK(g) is isomorphic to S/Q (See Theorem 4.3).

In the future, we will investigate interval-valued fuzzy congruences on a semiring.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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